DT 3/1/89 ①

Amplitude & Phase Errors in Linac 143

Phase oscillation equations with random amplitude error δE_0 and phase error $\delta\phi$ at each gap.

① $$\frac{d\Delta\phi}{dn} = 2\pi\beta_s\left(\frac{1}{\beta}-\frac{1}{\beta_s}\right) = -\frac{2\pi}{\beta_s^2\gamma_s^3}(\gamma-\gamma_s) \equiv -a\Delta\gamma$$

This is the kinematic equation which is not affected by the errors. It simply gives the phase advance from one gap to the next (n = gap no.) when the energy deviation from synchronous value is $\Delta\gamma$.

② $$\frac{d\Delta\gamma}{dn} = \frac{eE_0T_0\beta_s\lambda}{mc^2}\left[\left(1+\frac{\delta E_0}{E_0}\right)\sin(\phi_s+\Delta\phi+\delta\phi) - \sin\phi_s\right]$$

$$= \frac{eV_0\cos\phi_s}{mc^2}\Delta\phi + \frac{eV_0\cos\phi_s}{mc^2}\left(\delta\phi + \tan\phi_s\frac{\delta E_0}{E_0}\right)$$

$$\equiv b\Delta\phi + b\left(\delta\phi + \tan\phi_s\frac{\delta E_0}{E_0}\right)$$

where $V_0 \equiv E_0T_0\beta_s\lambda$ = peak voltage across gap. This is the dynamic equation giving the kick received in a gap. It is therefore affected by the errors.

Without error ($\delta\phi = \frac{\delta E_0}{E_0} = 0$) we have

$$\begin{cases} \dfrac{d\Delta\phi}{dn} = -a\Delta\gamma \\[2ex] \dfrac{d\Delta\gamma}{dn} = b\Delta\phi \end{cases} \quad \text{gives} \quad \frac{(\Delta\phi)^2}{a} + \frac{(\Delta\gamma)^2}{b} = R^2 = \text{const.}$$

(phase-point trajectory)

With errors the phase trajectory becomes

$$\frac{d}{dn}\left[\frac{(\Delta\phi)^2}{a}+\frac{(\Delta\gamma)^2}{b}\right]=2\Delta\gamma\left(\delta\phi+\tan\phi_s\frac{\delta E_0}{E_0}\right)$$

or

$$\frac{(\Delta\phi)^2}{a}+\frac{(\Delta\gamma)^2}{b}=R^2+\underbrace{\int 2\Delta\gamma\left(\delta\phi+\tan\phi_s\frac{\delta E_0}{E_0}\right)dn}_{\equiv\,\delta(R^2)=2R\,\delta R}$$

Hence

$$\boxed{\delta R=\frac{1}{R}\sum_n \Delta\gamma\left(\delta\phi+\tan\phi_s\frac{\delta E_0}{E_0}\right)}$$

(a) First order $\sum_n = 0$ because $\delta\phi$ & $\frac{\delta E_0}{E_0}$ are assumed to be random.

(b) Hence we get only the mean-squared value

$$\sum_n\left\langle(\Delta\gamma)^2\left[\delta\phi^2+2\tan\phi_s\,\delta\phi\frac{\delta E_0}{E_0}+\tan^2\phi_s\left(\frac{\delta E_0}{E_0}\right)^2\right]\right\rangle$$

$$=n\underbrace{\langle\Delta\gamma^2\rangle}_{\ll\,\frac{\Delta\gamma_{max}^2}{2}\ \text{(if } \Delta\gamma \text{ oscillates sinusoidally)}}\left[\langle\delta\phi^2\rangle+\tan^2\phi_s\left\langle\left(\frac{\delta E_0}{E_0}\right)^2\right\rangle\right]$$

This gives

$$\delta R=\underbrace{\frac{\Delta\gamma_{max}}{R}\sqrt{\frac{n}{2}}}_{\ll\,\sqrt{\frac{nb}{2}}\ \text{because } R=\frac{\Delta\gamma_{max}}{\sqrt{b}}}\sqrt{\delta\phi_{rms}^2+\tan^2\phi_s\left(\frac{\delta E_0}{E_0}\right)_{rms}^2}$$

$$=\sqrt{\frac{neV_0\sin\phi_s}{2mc^2}}\sqrt{\cot\phi_s\,\delta\phi_{rms}^2+\tan\phi_s\left(\frac{\delta E_0}{E_0}\right)_{rms}^2}$$

Take

$$neV_0 \sin\phi_s = neE_0T_0\beta_s\lambda\sin\phi_s$$

$$= \text{total energy gain} = 284 \text{ MeV}$$

(Since this is independent of n the resulting δR applies to either cell to cell, or sector to sector or module to module random errors.)

$$\phi_s = 60^\circ \quad \left(\tan\phi_s = \sqrt{3} \quad \cot\phi_s = \frac{1}{\sqrt{3}}\right)$$

$$\delta\phi_{rms} = 0.02\ (1.1^\circ), \quad \left(\frac{\delta E_0}{E_0}\right)_{rms} = 0.02\ (2\%)$$

This gives

$$\boxed{\delta R = 0.012}$$

We compare this to R which is related to the emittance

$$R^2 = \frac{(\Delta\phi_{max}\Delta\gamma_{max})}{\sqrt{ab}} = \frac{1}{\Omega}\frac{\omega_{rf}}{mc^2}\underbrace{(\Delta t_{max}\Delta\mathcal{E}_{max})}_{\epsilon\ (\text{in } \pi\ \text{eV sec})}$$

Take average values

$$\beta^2\gamma^3 \sim 1 \quad (304.7 \text{ MeV})$$

$$eV_0\sin\phi_s = \frac{400-116}{420}\text{ MeV} = 0.676 \text{ MeV (cell)}$$

$$\phi_s = 60^\circ \quad \omega_{rf} = 2\pi \times 8.05\times10^8 \text{ sec}^{-1}$$

$$\epsilon = 2\times10^{-5}\ (\pi\ \text{eV sec})$$

We get

$$\Omega = \sqrt{\frac{2\pi}{\beta^2\gamma^3}\frac{eV_0\sin\phi_s}{mc^2}\cot\phi_s} = 0.071/\text{sec}^{-1}$$

$$R^2 = \frac{2\pi \times 8.05\times10^8}{0.0511\times 0.938\times10^9} \times 2\times10^{-5} = 0.0021$$

$$\boxed{R = 0.046} \qquad \boxed{\frac{\delta R}{R} = 0.26}$$ (error random cell to cell)

If error is random from module to module R is quite a bit smaller because now.

$$eV_0 \sin\phi_s = \frac{400-116}{7}\,\text{MeV} = 40.57\,\text{MeV}$$ (module)

We get

$$\Omega = 0.396\ \text{sec}^{-1}$$

and

$$\boxed{R = 0.0165} \qquad \boxed{\frac{\delta R}{R} = 0.73}$$ (error random module to mod.)