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A Fast Way for Calculating Longitudinal Wakefields for High Q Resonances

Cheng-Yang Tan
James Steimel

Beams Division/Tevatron

ABSTRACT: We have come up with a way for calculating longitudinal wakefields for high Q resonances by mapping the wake functions to a two dimension vector space. Then in this space, a transformation which is basically a scale change and a rotation, allows us to calculate the new wakefield by knowing only one previous wakefield and one previous particle passage through the cavity. We will also compare this method to the brute force method which needs to know all the passages of the previous particles through the cavity.

INTRODUCTION

The inspiration for this paper came when we wondered whether it is possible to do our simulation in a smarter way when it comes to wakefield calculations. The brain dead way of calculating the wakefield requires us to remember the longitudinal positions of all the particles which have gone through the rf cavity. Then from all these previous longitudinal positions, the wakefield is calculated for the particle which is just about to enter the cavity. This is really inefficient, what we really like to do is to be able to calculate the new wakefield by only knowing the previous wakefield. Intuitively, we thought that this can be done — and in fact, we did do this — however, it was not too obvious when we started.

We will illustrate what we want to do by creating a fictitious universe where the wake function is

$$\mathcal{W}'_m(z) = \begin{cases} 0 & \text{if } z > 0 \\ \mathcal{W} & \text{if } z = 0 \\ 2\mathcal{W}e^{\lambda z} & \text{if } z < 0 \end{cases} \tag{1}$$

where λ is the decay length. For simplicity, we will assume that every particle which goes through the cavity has the same charge in this paper. Referring to Figure 1, the wake w_0 left by particle 0 is $w_0 = \mathcal{W}$ as it just leaves the cavity. When particle 1 is about to enter the cavity, the wake is

$$w_0^+ = 2\mathcal{W}e^{\lambda z_{10}} \tag{2}$$

where $z_{ij} = z_j - z_i$. When it just leaves

$$w_1 = 2\mathcal{W}e^{\lambda z_{10}} + 2\mathcal{W} \tag{3}$$

For particle 2, the wake just before it enters the cavity is

$$w_1^+ = 2\mathcal{W}e^{\lambda(z_{21}+z_{10})} + 2\mathcal{W}e^{\lambda z_{21}} \tag{4}$$

and similarly, for particle 3 before it enters the cavity is

$$w_2^+ = 2\mathcal{W}e^{\lambda(z_{32}+z_{21}+z_{10})} + 2\mathcal{W}e^{\lambda(z_{32}+z_{21})} + 2\mathcal{W}e^{\lambda z_{32}} \tag{5}$$

and *ad infinitum* for every particle before it enters the cavity. Clearly we can write

$$\begin{aligned} w_0^+ &= 2\mathcal{W}e^{\lambda z_{10}} \\ w_1^+ &= e^{\lambda z_{21}}\left(w_0^+ + 2\mathcal{W}\right) \\ w_2^+ &= e^{\lambda z_{32}}\left(w_1^+ + 2\mathcal{W}\right) \end{aligned} \tag{6}$$

which is really nice because for each particle going through the cavity, we need remember only the last w^+ and the position of the last particle through the cavity, i.e. w_1^+ is written in terms of w_0^+ and z_{21}, w_2^+ is in terms of w_1^+ and z_{32} etc. The reason why we have such a nice relationship in the fictitious universe is because exponentials have the property

$$\begin{aligned} e^x e^y &= e^y e^x \\ e^x e^y &= e^{x+y} \end{aligned} \tag{7}$$

Next, let us come back to the real universe and see whether we have the same nice relationship. For a high Q resonator with resonant frequency ω_R, the longitudinal wake function $W_m'(z)$ from equation (2.84) of Chao is

$$W_m'(z) = \begin{cases} 0 & \text{if } z > 0 \\ \alpha R_s & \text{if } z = 0 \\ 2\alpha R_s e^{\alpha z/c}\left(\cos\frac{\bar{\omega}z}{c} + \frac{\alpha}{\bar{\omega}}\sin\frac{\bar{\omega}z}{c}\right) & \text{if } z < 0 \end{cases} \tag{8}$$

where R_s is the shunt impedance of the cavity, $\alpha = \omega_R/2Q$ and $\bar{\omega} = \sqrt{\omega_R^2 - \alpha^2}$. Again referring to Figure 1, we can go through our previous analysis of particles 0 to 3 and see that

$$\begin{aligned} w_0^+ &= 2\alpha R_s e^{\alpha z_{10}/c}\left[\cos\frac{\bar{\omega}z_{10}}{c} + \frac{\alpha}{\bar{\omega}}\sin\frac{\bar{\omega}z_{10}}{c}\right] \\ w_1^+ &= 2\alpha R_s e^{\alpha(z_{21}+z_{10})/c}\left[\cos\frac{\bar{\omega}}{c}(z_{21}+z_{10}) + \frac{\alpha}{\bar{\omega}}\sin\frac{\bar{\omega}}{c}(z_{21}+z_{10})\right] + \\ &\qquad 2\alpha R_s e^{\alpha z_{21}/c}\left[\cos\frac{\bar{\omega}z_{21}}{c} + \frac{\alpha}{\bar{\omega}}\sin\frac{\bar{\omega}z_{21}}{c}\right] \\ w_2^+ &= 2\alpha R_s e^{\alpha(z_{32}+z_{21}+z_{10})/c}\left[\cos\frac{\bar{\omega}}{c}(z_{32}+z_{21}+z_{10}) + \frac{\alpha}{\bar{\omega}}\sin\frac{\bar{\omega}}{c}(z_{32}+z_{21}+z_{10})\right] + \\ &\qquad 2\alpha R_s e^{\alpha(z_{32}+z_{21})/c}\left[\cos\frac{\bar{\omega}}{c}(z_{32}+z_{21}) + \frac{\alpha}{\bar{\omega}}\sin\frac{\bar{\omega}}{c}(z_{32}+z_{21})\right] + \\ &\qquad\quad 2\alpha R_s e^{\alpha z_{32}/c}\left[\cos\frac{\bar{\omega}z_{32}}{c} + \frac{\alpha}{\bar{\omega}}\sin\frac{\bar{\omega}z_{32}}{c}\right] \end{aligned} \tag{9}$$

Looking at w_0^+, w_1^+, and w_2^+, there is not any obvious relationship between them like that of the fictitious universe. The cause of the problems are the cosine and sine terms in (8) which spoil the properties of the exponentials which means that we cannot write w_1^+ in terms of w_0^+ and z_{10} etc. This is the reason why we must remember the z position of every particle which have entered the cavity before the current particle in order to calculate the wakefield on the current particle — we will call this the brute force method. Of course, the point of this paper is that we have found a way reproduce the efficiency of (6).

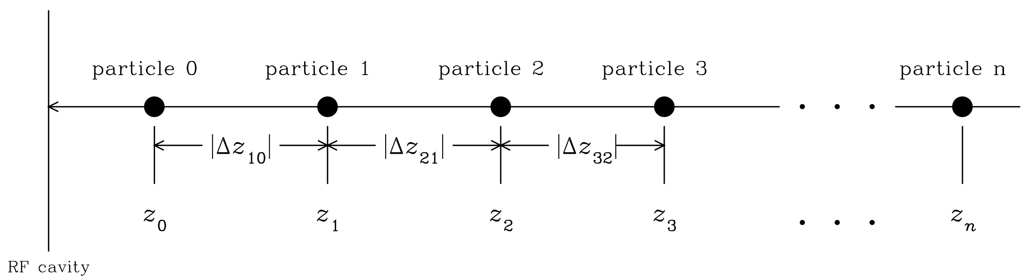

Figure 1 This is the example which we will use throughout the paper. The particles 0, 1, . . . are all going into a rf cavity.

If we look closely at the wake function $W'_m(z)$

$$W'_m(z) = \begin{cases} 0 & \text{if } z > 0 \\ \alpha R_s & \text{if } z = 0 \\ 2\alpha R_s e^{\alpha z/c}\left(\cos\frac{\bar{\omega} z}{c} + \frac{\alpha}{\bar{\omega}}\sin\frac{\bar{\omega} z}{c}\right) & \text{if } z < 0 \end{cases} \tag{10}$$

and stare at it long enough, we see that when the third line of (10) is re-written as

$$w'(z) \equiv W'_m(z)/2\alpha R_s = e^{\alpha z/c}\left(\cos\frac{\bar{\omega} z}{c} + \frac{\alpha}{\bar{\omega}}\sin\frac{\bar{\omega} z}{c}\right) \qquad \text{if } z < 0 \tag{11}$$

the rhs looks suspiciously like a vector projection onto the v axis of a plane defined by an orthogonal axis $(\hat{\boldsymbol{v}}, \hat{\boldsymbol{v}}')$. Our suspicions will be vindicated with the calculations shown next.

First let us define

$$\boldsymbol{V} = \begin{pmatrix} 1 \\ \frac{\alpha}{\bar{\omega}} \end{pmatrix} \tag{12}$$

to represent $\boldsymbol{V} = \hat{\boldsymbol{v}} + \frac{\alpha}{\bar{\omega}}\hat{\boldsymbol{v}}'$ and $\boldsymbol{T}$ to be a transformation which is a scale change and a rotation about the origin in this plane

$$\boldsymbol{T}(z) = e^{\alpha z/c}\begin{pmatrix} \cos\frac{\bar{\omega} z}{c} & \sin\frac{\bar{\omega} z}{c} \\ -\sin\frac{\bar{\omega} z}{c} & \cos\frac{\bar{\omega} z}{c} \end{pmatrix} \tag{13}$$

Therefore, if we write

$$\begin{aligned} \boldsymbol{w}' &= \boldsymbol{T}(z)\cdot\boldsymbol{V} \\ &= e^{\alpha z/c}\begin{pmatrix} \cos\frac{\bar{\omega} z}{c} & \sin\frac{\bar{\omega} z}{c} \\ -\sin\frac{\bar{\omega} z}{c} & \cos\frac{\bar{\omega} z}{c} \end{pmatrix}\begin{pmatrix} 1 \\ \frac{\alpha}{\bar{\omega}} \end{pmatrix} \end{aligned} \tag{14}$$

then clearly the $\hat{\boldsymbol{v}}$ component of $\boldsymbol{w}'$ which is the projection of $\boldsymbol{w}'$ onto the $\hat{\boldsymbol{v}}$ is (11). Since the $\hat{\boldsymbol{v}}$ component of any vector in the (v, v') plane is a wake function, we will call these vectors the *wake vectors*. Furthermore, $\boldsymbol{T}$ satisfies the properties

$$\begin{aligned} \boldsymbol{T}(a)\boldsymbol{T}(b) &= \boldsymbol{T}(b)\boldsymbol{T}(a) \\ \boldsymbol{T}(a)\boldsymbol{T}(b) &= \boldsymbol{T}(a+b) \end{aligned} \tag{15}$$

which is exactly the type of transformation which we want as was discussed in the *Introduction.* This is the key for getting to the fast wakefield calculation method.

The fast wakefield calculation can be summarized with the following iterative formula

$$\begin{aligned}
\boldsymbol{V}_k^+ &= \boldsymbol{T}(\Delta z_{k+1,k}) \cdot \boldsymbol{V}_k && \text{particle } (k+1) \text{ just before entering the cavity} \\
\boldsymbol{V}_k^- &= \boldsymbol{V}_k^+ + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} && \text{particle } (k+1) \text{ is in the cavity} \\
\boldsymbol{V}_{k+1} &= \boldsymbol{V}_k^+ + \begin{pmatrix} 1 \\ \frac{\alpha}{\bar{\omega}} \end{pmatrix} && \text{particle } (k+1) \text{ just leaves the cavity}
\end{aligned} \tag{16}$$

where $k \in \{0, 1, \ldots\}$, $\Delta z_{k+1,k} = z_k - z_{k+1}$, and $\boldsymbol{V}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This means that for every iteration, only two parameters $\boldsymbol{V}_k$ and $\Delta z_{k+1,k}$ are needed to calculate $\boldsymbol{V}_{k+1}$. This is what differentiates the fast method from the brute force method which requires the knowledge of *every* z_0, z_1, ...,z_k for calculating the wakefield on particle $k+1$.

To understand how to use (16), we will refer again to Figure 1 as an example. Just before particle 0 enters the cavity, $\boldsymbol{V}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore $\boldsymbol{V}_0^+ = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which means that when particle 0 just leaves, the wake vector left by particle 0 is $\boldsymbol{V}_1 = \begin{pmatrix} 1 \\ \frac{\alpha}{\bar{\omega}} \end{pmatrix}$. Next, when particle 1 is just about to enter the cavity

$$\boldsymbol{V}_1^+ = \boldsymbol{T}(\Delta z_{10}) \cdot \boldsymbol{V}_1 \tag{17}$$

and when it leaves

$$\boldsymbol{V}_2 = \boldsymbol{T}(\Delta z_{10}) \cdot \boldsymbol{V}_1 + \begin{pmatrix} 1 \\ \frac{\alpha}{\bar{\omega}} \end{pmatrix} \tag{18}$$

And thus when particle 2 is just about to enter the cavity

$$\begin{aligned}
\boldsymbol{V}_2^+ &= \boldsymbol{T}(\Delta z_{21}) \left[\boldsymbol{T}(\Delta z_{10}) \cdot \boldsymbol{V}_1 + \begin{pmatrix} 1 \\ \frac{\alpha}{\bar{\omega}} \end{pmatrix} \right] \\
&= \boldsymbol{T}(\Delta z_{21} + \Delta z_{10}) \cdot \boldsymbol{V}_1 + \boldsymbol{T}(\Delta z_{21}) \begin{pmatrix} 1 \\ \frac{\alpha}{\bar{\omega}} \end{pmatrix}
\end{aligned} \tag{19}$$

Expanding (19) with (13),

$$\begin{aligned}
\boldsymbol{V}_2^+ = \Bigg[& e^{\alpha(\Delta z_{21} + \Delta z_{10})/c} \begin{pmatrix} \cos\frac{\bar{\omega}}{c}(\Delta z_{21} + \Delta z_{10}) & \sin\frac{\bar{\omega}}{c}(\Delta z_{21} + \Delta z_{10}) \\ -\sin\frac{\bar{\omega}}{c}(\Delta z_{21} + \Delta z_{10}) & \cos\frac{\bar{\omega}}{c}(\Delta z_{21} + \Delta z_{10}) \end{pmatrix} + \\
& e^{\alpha \Delta z_{21}/c} \begin{pmatrix} \cos\frac{\bar{\omega}}{c}\Delta z_{21} & \sin\frac{\bar{\omega}}{c}\Delta z_{21} \\ -\sin\frac{\bar{\omega}}{c}\Delta z_{21} & \cos\frac{\bar{\omega}}{c}\Delta z_{21} \end{pmatrix} \Bigg] \begin{pmatrix} 1 \\ \frac{\alpha}{\bar{\omega}} \end{pmatrix}
\end{aligned} \tag{20}$$

and extracting the $\hat{\boldsymbol{v}}$ component of $\boldsymbol{V}_2^+$, we have

$$\begin{aligned}[\boldsymbol{V}_2^+]_{\hat{\boldsymbol{v}}} &= e^{\alpha(\Delta z_{21}+\Delta z_{10})/c}\left[\cos\frac{\bar{\omega}}{c}(\Delta z_{21}+\Delta z_{10})+\frac{\alpha}{\bar{\omega}}\sin\frac{\bar{\omega}}{c}(\Delta z_{21}+\Delta z_{10})\right]+\\ &\qquad e^{\alpha\Delta z_{21}/c}\left[\cos\frac{\bar{\omega}}{c}\Delta z_{21}+\frac{\alpha}{\bar{\omega}}\sin\frac{\bar{\omega}}{c}\Delta z_{21}\right] \\ &= w'\Big(\Delta z_{21}+\Delta z_{10}\Big)+w'\Big(\Delta z_{21}\Big)\end{aligned} \tag{21}$$

which is exactly the solution if we had used (11) to calculate the wake function for particle 2 just before it enters the cavity.

Proof

We will use mathematical induction to prove that the iterative formula (16) works. Using the previous example, the formula is true for $k = 1$. Assume that the formula is true for $k = n$, i.e.

$$\begin{aligned}[\boldsymbol{V}_n^+]_{\hat{\boldsymbol{v}}} &= \Big[\boldsymbol{T}(\Delta z_{n+1,n})\cdot\boldsymbol{V}_n\Big]_{\hat{\boldsymbol{v}}} \\ &= \sum_{i=0}^{n} w'\left(\sum_{j=i}^{n}\Delta z_{j+1,j}\right)\end{aligned} \tag{22}$$

then

$$\begin{aligned}[\boldsymbol{V}_{n+1}^+]_{\hat{\boldsymbol{v}}} &= \Big[\boldsymbol{T}(\Delta z_{n+2,n+1})\cdot\boldsymbol{V}_{n+1}\Big]_{\hat{\boldsymbol{v}}} \\ &= \left[\boldsymbol{T}(\Delta z_{n+2,n+1})\cdot\left\{\boldsymbol{V}_n^+ + \begin{pmatrix}1\\ \frac{\alpha}{\bar{\omega}}\end{pmatrix}\right\}\right]_{\hat{\boldsymbol{v}}} \\ &= \Big[\boldsymbol{T}(\Delta z_{n+2,n+1}+\Delta z_{n+1,n})\cdot\boldsymbol{V}_n\Big]_{\hat{\boldsymbol{v}}} + \left[\boldsymbol{T}(\Delta z_{n+2,n+1})\cdot\begin{pmatrix}1\\ \frac{\alpha}{\bar{\omega}}\end{pmatrix}\right]_{\hat{\boldsymbol{v}}}\end{aligned} \tag{23}$$

where we have used (15) and (22). In order to continue, we have to expand (22)

$$\begin{aligned}\Big[\boldsymbol{T}(\Delta z_{n+1,n})\cdot\boldsymbol{V}_n\Big]_{\hat{\boldsymbol{v}}} &= w'(\Delta z_{n+1,n})+w'(\Delta z_{n+1,n}+\Delta z_{n,n-1})+\ldots \\ &\qquad + w'(\Delta z_{n+1,n}+\Delta z_{n,n-1}+\ldots+\Delta z_{10})\end{aligned} \tag{24}$$

and thus

$$\begin{aligned}\big[\boldsymbol{T}(\Delta z_{n+2,n+1}+\Delta z_{n+1,n})\cdot\boldsymbol{V}_n\big]_{\hat{\boldsymbol{v}}} &= w'(\Delta z_{n+2,n+1}+\Delta z_{n+1,n})+ \\ &\qquad w'(\Delta z_{n+2,n+1}+\Delta z_{n+1,n}+\Delta z_{n,n-1})+\ldots \\ &\qquad + w'(\Delta z_{n+2,n+1}+\Delta z_{n+1,n}+\ldots+\Delta z_{10})\end{aligned} \tag{25}$$

which is the first term in the last line of (23). It is also trivial to show that the second term of the same line is

$$\left[\boldsymbol{T}(\Delta z_{n+2,n+1}) \cdot \begin{pmatrix} 1 \\ \frac{\alpha}{\bar{\omega}} \end{pmatrix}\right]_{\hat{\boldsymbol{v}}} = w'(\Delta z_{n+2,n+1}) \tag{26}$$

Summing (25) and (26) gives us the required result

$$[\boldsymbol{V}^{+}_{n+1}]_{\hat{\boldsymbol{v}}} = \sum_{i=0}^{n+1} w' \left(\sum_{j=i}^{n+1} \Delta z_{j+1,j} \right) \tag{27}$$

Therefore, our iterative formula is true by mathematical induction.

CONCLUSION

Although we have proved our formula for the case when the charge of each particle is the same, it is trivial to extend it to the case of different charges by modifying (16) by multiplying the charge to $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \frac{\alpha}{\bar{\omega}} \end{pmatrix}$. By incorporating this new method to our wakefield calculations, we have seen a huge improvement in speed between the fast and brute force methods.

REFERENCES

[1] *Physics of Collective Beam Instabilities in High Energy Accelerators*, A.W. Chao, Wiley Series in Beam Physics, 1993.