Fermi National Accelerator Laboratory

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Toroidal Resonant Impedances in RHIC*

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I. INTRODUCTION

In a toroidal beam pipe, a wave with a particular azimuthal variation travels with different speeds depending on the distance from the center of the toroidal ring. For example, if the beam travels with velocity βc at a toroidal radius R, the electromagnetic wave traveling with the beam will have a velocity $r\beta c/R$ at a radius r. If this velocity reaches c, this electromagnetic wave can also propagate. This wave will then interact back with the beam and a resonance occurs.[1–4] The beam will see an impedance. The condition for this to occur is therefore[5]

$$\frac{R_{+}\beta}{R} \gtrsim 1 \ , \tag{1.1}$$

where R_{+} is the radius of the outer edge of the beam pipe.

RHIC has a mean ring radius of $\bar{R} = 610.18$ m and a pipe radius of $b = 3.645$ cm. Therefore for a beam at the center of the beam pipe, such resonance will occur when the beam velocity $\beta \gtrsim (1 + b/R)^{-1} = 0.999940$. At injection, protons and gold ions have $\beta = 0.99947$ and 0.99680 respectively, so no toroidal resonances will be excited. However, during collisions, protons and gold ions reach kinetic energies of 250.7 GeV/amu and 100.0 GeV/amu respectively, corresponding to $\beta = 0.999993$ and 0.999957. Therefore toroidal resonances will be exited.

These resonances are positioned at azimuthal harmonics of the order $n_{co}^{3/2} \sim 8.1 \times 10^{6}$ corresponding to frequencies of ~ 620 GHz, where $n_{co} \sim O(\bar{R}/b) \sim 4.0 \times 10^{4}$ is the cutoff harmonic of the beam pipe. For a perfectly conducting pipe wall, a beam at a particular radius r from the center of the toroidal ring may excite one infinitely sharp resonance at one azimuthal harmonic n_r which is an integer. The resonance at the next harmonic $n_{r'} = n_r + 1$ will be excited by the beam particles at radius r' which is very close to r. However, for a beam pipe with wall resistivity, each of these resonances will have a azimuthal harmonic width of Δn which is of the order of 100. In order words, the beam at r will excite about 100 adjacent resonances which overlap each other. What the beam particle sees will be a broad resonance which, in principle, can drive a "microwave" growth. For this reason, the study of toroidal resonances is meaningful and important.

II. TOROIDAL RESONANCES

Consider a beam pipe of rectangular cross section with width $2b$ and height h. The beam is at the center of the beam pipe. The toroidal resonances can be divided into two series: the TM modes with *vertical* magnetic field vanishes and the TE modes with *vertical* electric field vanishes. The resonant harmonics $n = n_{ik}^{\rm TM}$ or $n_{ik}^{\rm TE}$ are given by solving

$$\text{TM}: \qquad Z_n(q_i R_+) = 0 \; , \tag{2.1}$$

or

$$\text{TE}: \qquad \tilde{Z}_n'(q_i R_+) = 0 \; , \tag{2.2}$$

simultaneously with

$$q_i^2 = \frac{n^2\beta^2}{R^2} - \frac{\pi^2(2k-1)^2}{h^2} \; . \tag{2.3}$$

In above,

$$Z_n(q_i r) = Y_n(q_i R_-)J_n(q_i r) - J_n(q_i R_-)Y_n(q_i r) \; , \tag{2.4}$$

$$\tilde{Z}_n(q_i r) = Y_n'(q_i R_-)J_n(q_i r) - J_n'(q_i R_-)Y_n(q_i r) \; , \tag{2.5}$$

where J_n and Y_n are Bessel function and Neumann function of order n, and $R_\pm = \bar{R} \pm b$ is the radius of the outer (inner) edge of the beam pipe, while i and k denote the radial and vertical mode numbers characterizing the resonances.

The resonant harmonics n_{ik} are in general very much bigger than the cutoff harmonic which is of the order $\bar{R}/b$. Then, to a high degree of accuracy, the solution to Eqs. (2.1) and (2.3) or Eqs. (2.2) and (2.3) can be obtained simply by solving instead

$$2^{1/3} n_{ik}^{2/3} \left[\frac{b}{\bar{R}} - \frac{1}{2\gamma^2} - \frac{\bar{R}^2\pi^2(2k-1)^2}{2n_{ik}^2 h^2} \right] = \begin{cases} y_i & \text{TM} \\ y_i' & \text{TE} \; , \end{cases} \tag{2.6}$$

where $-y_i$ and $-y_i'$ are respectively the ith zeroes of the Airy function $\text{Ai}(-y)$ and its derivative $\text{Ai}'(-y)$. Some lowest zeroes are $y_1 = 2.3381$, $y_2 = 4.0879$, $\cdots$, and $y_1' = 1.0188$, $y_2' = 3.2482$, $\cdots$. In many cases (usually not the lowest mode), the third term on the left side of Eq. (2.4) can be neglected so that n_{ik} can be solved very easily.

If we take $b = h/2 = 3.5$ cm, $R = 610$ m, and $\gamma = 268.2$, we obtain for the lowest resonance, which is a TE mode, $n_{11}^{\rm TE} = 4.33 \times 10^6$ corresponding to a frequency of $f_{11}^{\rm TE} = 339$ GHz.

Now wall resistivity is introduced as a perturbation. If we assume that the resonances are far apart (which is not true), the resonances become broadened with figures of merit

$$Q_{ik}^{\rm TM} \approx \frac{h}{2\delta_{ik}} \qquad \text{and} \qquad Q_{ik}^{\rm TE} \approx \frac{b}{\delta_{ik}} \frac{\mathcal{N}_{ik}^{\rm TE}}{\left|\tilde{Z}_{n_{ik}}\right|_{\rm beam}^2} \; , \tag{2.7}$$

where δ_{ik} is the skin depth into the pipe wall and $\mathcal{N}_{ik}^{\rm TE}$ is a normalization factor defined as

$$\mathcal{N}_{ik}^{\rm TE} = \frac{\bar{R}}{2b} \left[\left(\frac{R_+^2}{\bar{R}^2} - \frac{n_{ik}^2}{q_i^2 \bar{R}^2} \right) \tilde{Z}_{n_{ik}}^2(q_i R_+) - \left(\frac{R_-^2}{\bar{R}^2} - \frac{n_{ik}^2}{q_i^2 \bar{R}^2} \right) \tilde{Z}_{n_{ik}}^2(q_i R_-) \right] \; . \tag{2.8}$$

The shunt impedances per unit harmonic are given by

$$\frac{Z_{\rm sh}}{n} \approx \begin{cases} \dfrac{4\pi^3 Z_0 Q_{ik}}{n_{ik}^4} \dfrac{(2k-1)^2 R^4}{hb^3} \dfrac{|Z_{n_{ik}}|^2_{\rm beam}}{\mathcal{N}_{ik}^{\rm TM}} & \text{TM}\ , \\[2ex] \dfrac{4\pi Z_0 Q_{ik}}{n_{ik}^4} \dfrac{R^4}{hb^3} \dfrac{\left|d\tilde{Z}_{n_{ik}}/dx\right|^2_{\rm beam}}{\mathcal{N}_{ik}^{\rm TE}} & \text{TE}\ . \end{cases} \tag{2.9}$$

In above, $Z_0 \approx 377\ \Omega$, $\tilde{Z}_{n_{ik}}(q_i r)$ is considered a function of x defined by $r = \bar{R} + bx$, and the normalization constant for the TM mode is

$$\mathcal{N}_{ik}^{\rm TM} = \frac{\bar{R}}{2b} \left[\frac{R_+^2}{\bar{R}^2} Z'^2_{n_{ik}}(q_i R_+) - \frac{R_-^2}{\bar{R}^2} Z'^2_{n_{ik}}(q_i R_-) \right] \ , \tag{2.10}$$

The lowest TM and TE toroidal resonances for stainless steel wall conductivity $\sigma = 2.0 \times 10^6$ (Ω-m)$^{-1}$ at $\gamma = 268.2$ and 123.0 are listed in Table I. The former corresponds to the top energy of protons while the latter corresponds to the top energy of copper ions. Due to the incompetence of the computer program library, we have not been able to compute the resonances at $\gamma = 108.4$ that corresponds to the top energy of gold ions. This is because the velocity $\beta = 0.999957$ is too close to the resonance requirement of $\beta \gtrsim 0.999940$ given by Eq. (1.1). However, it is clear that at this γ, the resonant frequency will be much higher and the impedance much smaller.

The resonance at harmonic next to $n_{ik}^{\rm TE}$ (or $n_{ik}^{\rm TE}+1$) will be excited by the beam particles traveling at radius $\bar{R} + \Delta R$, where ΔR is given by

$$\Delta R \approx \frac{2}{3} \left[\frac{b}{n_{ik}} + \frac{\bar{R}^3}{n_{ik}^3} \frac{\pi^2 (2k-1)^2}{h^2} \right] \ , \tag{2.11}$$

which gives 1.63×10^{-9} m for the lowest TE mode. This implies that for a beam of transverse width 1 mm, about 6×10^5 adjacent resonances will be excited. The lowest resonance has a width $\Delta n_{11}^{\rm TE} = n_{11}^{\rm TE}/Q_{11}^{\rm TE} = 71.2$. In other words, the particles at each toroidal radius will excite about $\Delta n_{11}^{\rm TE} = 71.2$ adjacent resonances. Since they overlap, the result is a broad band with an effective impedance per harmonic seen by the beam

$$\left.\frac{Z_{\rm sh}}{n}\right|_{\rm eff} \sim \left(\frac{Z_{\rm sh}}{n} \right) \Delta n_{ik} = \frac{Z_{\rm sh}}{Q_{ik}} \ , \tag{2.12}$$

which turns out to be independent of the wall resistivity. Since all these resonances overlap by so much, Eqs. (2.7) and (2.9) may not be correct at all. However, we believe that Eq. (2.12) should give us a reasonable estimate. These effective impedances per harmonic are listed in the last row of Table I. We see that the largest effective Z/n is only about 2 Ω which may be too small to excite any "microwave" growth.

	$\gamma = 286.2$		$\gamma = 123.0$	
	TE	TM	TE	TM
n_{ik}	4.33×10^6	8.36×10^6	8.67×10^6	2.21×10^6
f_{ik}	3.39×10^2 GHz	6.54×10^2 GHz	6.73×10^2 GHz	1.73×10^3 GHz
Q_{ik}	6.08×10^4	7.95×10^4	5.42×10^4	1.29×10^5
$\dfrac{Z_{\rm sh}}{n}$	$2.69 \times 10^{-2}\ \Omega$	$8.63 \times 10^{-3}\ \Omega$	$5.57 \times 10^{-4}\ \Omega$	$1.56 \times 10^{-7}\ \Omega$
$\left.\dfrac{Z_{\rm sh}}{n}\right\|_{\rm eff}$	$1.92\ \Omega$	$0.91\ \Omega$	$0.085\ \Omega$	$2.66 \times 10^{-5}\ \Omega$

Table I: Impedances and positions of the lowest TE and TM modes

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5. Most of the formulas in this paper are derived in Ref. 4.