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Fermilab

Effects of Misalignments of the Solenoid of the Colliding-Beams Detector on the Orbit of the Beams

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The 4x4 coupled transfer matrix of a weak solenoid with length taken out so that it acts at one point is (see Appendix)

$$N = \left(\begin{array}{c|c} I & \theta I \\ \hline -\theta I & I \end{array}\right) \tag{1}$$

where

$$\begin{cases} I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{2x2 unit matrix} \\ \theta = \dfrac{1}{2}\dfrac{B\ell}{(B\rho)} \qquad (B\rho) = \text{rigidity of beam} \\ B,\ell = \text{field and length of solenoid.} \end{cases}$$

The transfer matrix from the collision point all the way around the ring is

$$M = \left(\begin{array}{c|c} Mx & 0 \\ \hline 0 & My \end{array}\right)$$

$$= \left(\begin{array}{c|c} I\cos 2\pi\nu_x + J_x \sin 2\pi\nu_x & 0 \\ \hline 0 & I\cos 2\pi\nu_y + J_y \sin 2\pi\nu_y \end{array}\right) \tag{2}$$

where

$$\begin{cases} J_x = \begin{pmatrix} \alpha_x & \beta_x \\ -\gamma_x & -\alpha_x \end{pmatrix}, \qquad J_y = \begin{pmatrix} \alpha_y & \beta_y \\ -\gamma_y & -\alpha_y \end{pmatrix} \\ \nu_x, \nu_y = \text{x and y betatron tunes.} \end{cases}$$

Let the beam 4-vector immediately after the solenoid be

$$Z \equiv \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}$$

and the misalignment of the solenoid be given by the 4-vector

$$\varepsilon = \begin{pmatrix} \frac{X}{Y} \end{pmatrix} = \begin{pmatrix} \xi \\ \xi' \\ \eta \\ \eta' \end{pmatrix} .$$

We then obtain the following relation in going around the ring.

$$N(MZ-\varepsilon)+\varepsilon = Z$$

or

$$(NM-I)Z = (N-I)\varepsilon$$

or

$$Z = (NM-I)^{-1}\,(N-I)\varepsilon. \tag{3}$$

Since N is approximately the unit matrix we can write

$$(NM-I)^{-1} \cong (M-I)^{-1} = \left(\begin{array}{c|c} \dfrac{M_x^{-1}-I}{4\sin^2\pi\nu_x} & 0 \\ \hline 0 & \dfrac{M_y^{-1}-I}{4\sin^2\pi\nu_y} \end{array}\right)$$

$$= \left(\begin{array}{c|c} -\dfrac{1}{2\sin\pi\nu_x}\,(I\sin\pi\nu_x+J_x\cos\pi\nu_x) & 0 \\ \hline 0 & -\dfrac{1}{2\sin\pi\nu_x}(I\sin\pi\nu_y+J_y\cos\pi\nu_y) \end{array}\right).$$

Equation (3) then gives

$$
Z = \left(\begin{array}{c|c} 0 & \theta \dfrac{M_x^{-1}-I}{4\sin^2\pi\nu_x} \\ \hline -\theta \dfrac{M_y^{-1}-I}{4\sin^2\pi\nu_y} & 0 \end{array}\right) \left(\frac{X}{Y}\right)
$$

$$
= \left(\begin{array}{c} \theta \dfrac{M_x^{-1}-I}{4\sin^2\pi\nu_x} Y \\ \hline -\theta \dfrac{M_y^{-1}-I}{4\sin^2\pi\nu_y} X \end{array}\right) \tag{4}
$$

The amplitude invariants W_x and W_y are given by

$$
\left\{\begin{aligned} W_x &= \left(\frac{\theta}{4\sin^2\pi\nu_x}\right)^2 \tilde{Y}\widetilde{(M_x^{-1}-1)}\tilde{S}J_x(M_x^{-1}-1)Y \\ &= \frac{\theta^2}{4\sin^2\pi\nu_x}(\tilde{Y}\tilde{S}J_xY) \\ &= \frac{\theta^2}{4\sin^2\pi\nu_x}(\gamma_x\eta^2+2\alpha_x\eta\eta'+\beta_x\eta'^2) \\ W_y &= \frac{\theta^2}{4\sin^2\pi\nu_y}(\gamma_y\xi^2+2\alpha_y\xi\xi'+\beta_y\xi'^2) \end{aligned}\right. \tag{5}
$$

where

$$
S \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \text{unit simpletic matrix}
$$

and ~ denotes transposition.

For a numerical estimate we have

$$
\theta = \frac{1}{2}\,\frac{(15\ \text{kG})\text{x}(5\ \text{m})}{10^5/3\ \text{kGm}} = 1.1\text{x}10^{-3}
$$

and since the x and y parameters at the collision point are about the same

$$\nu_x \simeq \nu_y = 19.6$$

$$\alpha_x \simeq \alpha_y = 0$$

$$\beta_x \simeq \beta_y = 1 \text{ m}$$

$$\gamma_x \simeq \gamma_y = 1 \text{ m}^{-1} .$$

Taking some unreasonably large misalignment errors of

$$\begin{cases} \xi \simeq \eta = 2 \text{ in} = 0.05 \text{ m} & \text{(displacement)} \\ \xi' \simeq \eta' = 50 \text{ mrad} = 0.05 \text{ rad} & \text{(tilt)} \end{cases} \tag{6}$$

we get

$$W_x \simeq W_y = 1.75 \times 10^{-9} \text{m}$$

where the contributions due to displacement and tilt are equal. At $\beta = 100$ m, near to β_{max}, the orbit excursions are

$$\delta x \simeq \delta y = \sqrt{1.75 \times 10^{-9} \times 100 \text{ m}} = 0.4 \text{ mm}$$

which is not good but is nevertheless tolerable. Of course the alignment errors can never be as bad as those given in Equation (6). This result together with that from TM-1119 (Compensation of the Solenoid Field in the Colliding-Beams Detector) can be summarized as follows:

"Because the solenoid is very weak for 1 TeV beams, the alignment tolerances as far as the beams are concerned can be negligibly loose (The tolerances will be determined rather by accuracies for the measurement and analysis of particle tracks.) and there is no need to compensate for its orbital effects."

Appendix

We give here a derivation of the transfer matrix for a weak solenoid, Equation (1). A cylindrically symmetric magnetic field is given by the scalar potential

$$\Phi = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \left[A(z)\right]^{(2n)} r^{2n} \tag{7}$$

where r, ϕ, z are the cylindrical coordinates and the superscript (2n) denotes the 2n-th derivative. This gives

$$\begin{cases} B_r = \dfrac{\partial\Phi}{\partial r} = \sum \dfrac{(-1)^n 2n}{2^{2n}(n!)^2} A^{(2n)} r^{2n-1} \\ B_\phi = \dfrac{\partial\Phi}{r\partial\phi} = 0 \\ B_z = \dfrac{\partial\Phi}{\partial z} = \sum \dfrac{(-1)^n}{2^{2n}(n!)^2} A^{(2n+1)} r^{2n} \end{cases} \tag{8}$$

from which we see that

$$A'(z) = B_z(r=0) = \text{longitudinal field on axis.}$$

In the interior of the solenoid

$$B_z = B = \text{uniform.}$$

Across the fringe field at the end we have

$$\int B_r dz = -\frac{r}{2}\int A'' dz = \begin{cases} -\dfrac{B}{2}r & \text{(entry)} \\ \dfrac{B}{2}r & \text{(exit)} \end{cases} .$$

Since this fringe field is "thin", its orbital effects are

$$\begin{cases} \Delta x = \Delta y = 0 \\ \Delta x' = \pm ky \qquad \Delta y' = \mp kx \end{cases}$$

where $k \equiv \frac{1}{2}\frac{B}{(B\rho)}$. In matrix form this can be written as

$$Z_{after} = \left(\begin{array}{c|c} I & \pm K \\ \hline \mp K & I \end{array}\right) Z_{before} \qquad \text{where } K \equiv \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} \tag{9}$$

and the upper (lower) signs are for entry into (exit from) the solenoid.

In a cylindrically symmetric field the second order Lagrangian for the transverse motion is

$$L = \frac{1}{2}(x'^2+y'^2)+\kappa(xy'-yx') \tag{10}$$

where $\kappa(z) \equiv \frac{1}{2}\frac{B_z(r=0)}{(B\rho)}$. The equations of motion are

$$\begin{cases} x'' = 2\kappa y'+\kappa' y \\ y'' = -2\kappa x'-\kappa' x \ . \end{cases} \tag{11}$$

Transforming to the helical coordinates u and v by

$$\begin{cases} x = u\cos\theta+v\sin\theta \\ y = -u\sin\theta+v\cos\theta \end{cases} \qquad \text{with } \theta \equiv \int\kappa dz \tag{12}$$

or in matrix notation

$$Z = \left(\begin{array}{c|c} C & S \\ \hline -S & C \end{array}\right) U \tag{13}$$

where

$$C \equiv \begin{pmatrix} \cos\theta & 0 \\ -\kappa\sin\theta & \cos\theta \end{pmatrix} \qquad U \equiv \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}$$

$$S \equiv \begin{pmatrix} \sin\theta & 0 \\ \kappa\cos\theta & \sin\theta \end{pmatrix}$$

we get

$$\begin{cases} u''+\kappa^2 u = 0 \\ v''+\kappa^2 v = 0 \ . \end{cases} \tag{14}$$

In the interior $\kappa = k =$ constant and the solution for u and v can be written as

$$U_\ell = \left(\begin{array}{c|c} T & 0 \\ \hline 0 & T \end{array}\right) U_o \quad \text{where } T \equiv \begin{pmatrix} \cos\theta & \frac{1}{k}\sin\theta \\ -k\sin\theta & \cos\theta \end{pmatrix} \tag{15}$$

and, now, $\theta = k\ell$. The interior transfer matrix for Z is, therefore, given by

$$\begin{aligned} Z_\ell &= \left(\begin{array}{c|c} C & S \\ \hline -S & C \end{array}\right)_\ell U_\ell = \left(\begin{array}{c|c} C & S \\ \hline -S & C \end{array}\right)_\ell \left(\begin{array}{c|c} T & 0 \\ \hline 0 & T \end{array}\right) U_o \\ &= \left(\begin{array}{c|c} C & S \\ \hline -S & C \end{array}\right)_\ell \left(\begin{array}{c|c} T & 0 \\ \hline 0 & T \end{array}\right) \left(\begin{array}{c|c} C & S \\ \hline -S & C \end{array}\right)_o Z_o \\ &= \left(\begin{array}{c|c} C & S \\ \hline -S & C \end{array}\right) \left(\begin{array}{c|c} T & 0 \\ \hline 0 & T \end{array}\right) \left(\begin{array}{c|c} I & -K \\ \hline K & I \end{array}\right) Z_o \end{aligned} \tag{16}$$

where K is as defined in Equation (9) and where in the last expression the subscript ℓ on the first matrix was dropped as being understood.

To cross the whole solenoid from outside to outside we need to multiply the interior transfer matrix fore and aft by the appropriate fringe-field crossing matrices Equation (9). This gives

$$\begin{aligned} Z_\ell^{out} &= \left(\begin{array}{c|c} I & -K \\ \hline K & I \end{array}\right) \left(\begin{array}{c|c} C & S \\ \hline -S & C \end{array}\right) \left(\begin{array}{c|c} T & 0 \\ \hline 0 & T \end{array}\right) \left(\begin{array}{c|c} I & -K \\ \hline K & I \end{array}\right) \left(\begin{array}{c|c} I & K \\ \hline -K & I \end{array}\right) Z_o^{out} \\ &= \left(\begin{array}{c|c} I\cos\theta & I\sin\theta \\ \hline -I\sin\theta & I\cos\theta \end{array}\right) \left(\begin{array}{c|c} T & 0 \\ \hline 0 & T \end{array}\right) Z_o^{out} \\ &= \left(\begin{array}{c|c} T\cos\theta & T\sin\theta \\ \hline -T\sin\theta & T\cos\theta \end{array}\right) Z_o^{out} \end{aligned} \tag{17}$$

which is the same as what one would obtain by putting the explicit $k = 0$ in C and S (and K) in Equation (16) because the coordinate transformations are, now, made outside the solenoid where instead of $\kappa = k$ we have $\kappa = 0$.

To take the length ℓ out of the transfer matrix so that it acts at one point we should replace T by

$$L^{-1}TL^{-1} = \begin{pmatrix} 1 & -\frac{\ell}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \frac{\ell}{\theta}\sin\theta \\ -\frac{\theta}{\ell}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & -\frac{\ell}{2} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta + \frac{\theta}{2}\sin\theta & \frac{\ell}{\theta}\left(\sin\theta - \theta\cos\theta - \frac{\theta^2}{4}\sin\theta\right) \\ -\frac{\theta}{\ell}\sin\theta & \cos\theta + \frac{\theta}{2}\sin\theta \end{pmatrix}$$

$$\cong \begin{pmatrix} 1 & \frac{\ell}{12}\theta^2 \\ -\frac{\theta^2}{\ell} & 1 \end{pmatrix} \tag{18}$$

where the last expression is accurate to θ^2 terms. If θ^2 terms are also dropped this becomes the unit matrix and we have from Equation (17)

$$Z_\ell^{out} = \left(\begin{array}{c|c} I & \theta I \\ \hline -\theta I & I \end{array}\right) Z_o^{out} = NZ_o^{out}$$

as in Equation (1).