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Formulas and Scaling Laws for Thresholds of Coherent Instabilities of Storage Ring Beams

L.C. Teng

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The traditional forms of the formulas giving the thresholds of the longitudinal (subscript ℓ) and the transverse (subscript t) instabilities of a charged particle beam travelling inside the vacuum pipe of a storage ring are:

Longitudinal

$$\frac{|Z_\ell|}{n} < \frac{E/e}{I} |\eta| \beta^2 \left(\frac{\Delta p}{p}\right)^2 \tag{1}$$

where

Z_ℓ = longitudinal impedance of the whole beam pipe (in unit Ω)

n = mode number = number of instability waves per turn

E, I = energy and current of beam

$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_t^2} = \frac{p}{\omega_o} \frac{d\omega_o}{dp}$ = revolution frequency dispersion factor

$$\left(\begin{array}{l} \omega_o = \text{revolution frequency} \\ \gamma_t = \text{transition energy} \end{array}\right)$$

$\beta = \frac{v}{c} = \frac{\text{beam velocity}}{\text{velocity of light}}$,

e = charge of particle

$\frac{\Delta p}{p}$ = FWHM of momentum spread in beam .

This formula is derived for a continuous (coasting) beam but can also be applied to a bunched beam if I is interpreted as the peak current in the bunch and Z_ℓ is modified to contain only the relevant frequency components.

<u>Transverse</u>

$$|Z_t| < \pi \frac{E/e}{I} \frac{\beta^3}{\langle\beta_t\rangle} \left|(n-\nu)\eta-\xi\right| \left(\frac{\Delta p}{p}\right) \tag{2}$$

where

Z_t = transverse impedance of the whole beam pipe (in unit Ω/m)

$\langle\beta_t\rangle$ = amplitude-function of the transverse (betatron) oscillation averaged over the whole ring

ν = tune (wave number) of the transverse oscillation

$\xi = \dfrac{d\nu}{d\left(\frac{\Delta p}{p}\right)}$ = chromaticity .

Again, when applied to bunched beams I should be interpreted as the peak current during the bunch and Z_t should contain only the relevant frequency components.

The longitudinal and transverse impedances of a circular beam pipe (ignoring the space charge impedances) are related by

$$Z_t = \frac{2R}{b^2\beta} \frac{Z_\ell}{n} \tag{3}$$

where b is the radius of the pipe and $2\pi R$ is the circumference of the ring.

A. Alternative Forms

Since these coherent instabilities do not involve reasonances with the revolution frequency, explicit reference to the circumference of the ring should be avoidable. Furthermore, it should be possible to replace the impedances of the whole ring by the average impedances per unit length of beam pipe. This can indeed be done. Equation (1) can be rewritten as

$$\left|Z_\ell'\right| < \frac{E/e}{I}\,\frac{\beta}{2\pi c}\,\frac{\Delta p}{p}\,\Delta(n\omega_o) \tag{4}$$

where

$Z_\ell' \equiv \frac{Z_\ell}{2\pi R}$ = average longitudinal impedance per unit length of beam pipe

$n\omega_o$ = frequency of the n^{th} mode of the longitudinal instability picked up at fixed azimuthal location = frequency sensed by the beam pipe. [To be exact this should be $(n-\nu_s)\omega_o$ where ν_s = synchrotron (longitudinal) oscillation wave number. But since generally $\nu_s \ll 1$ it can be neglected.]

Similarly, Equation (2) can be written as

$$\left|Z_t'\right| < \frac{1}{2}\,\frac{E/e}{I}\,\frac{\beta^2}{\langle\beta_t\rangle c}\,\Delta\left[(n-\nu)\omega_o\right] \tag{5}$$

where

$Z_t' \equiv \frac{Z_t}{2\pi R}$ = average transverse impedance per unit length of beam pipe

$(n-\nu)\omega_o$ = frequency of the n^{th} mode of the transverse instability picked up at fixed azimuthal location = frequency sensed by the beam pipe

and Equation (3) can be written as

$$Z_t'(\omega) = \frac{2c}{b^2}\left(\frac{Z_\ell'(\omega)}{\omega}\right) \tag{6}$$

where $\omega = n\omega_o$ = frequency at which the impedances are measured.

Only the radius of the beam pipe and not the total length appears in Equations (4), (5) and (6). Together with the fact that only impedances per unit length are involved this makes them purely local equations. The only implicit reference to the ring circumference is through the mode number n. But since $n\omega_o$ and $(n-\nu)\omega_o$ are simply the frequencies of the instabilities sensed by the beam pipe we see that the appearance of n is quite incidental to the physical meanings stated by these equations. Indeed they have the same forms whether the beam and pipe form a closed ring or are straight but infinitely long. In this sense these formulas apply equally well to a linac as to a circular machine.

B. The Impedances

There are 4 types of contribution to the impedance.

1. Space charge term — This term depends on the energy and the dimensions of the beam and is non-zero even when the beam pipe is removed.

<u>Longitudinal</u>

$$Z_\ell = \frac{i}{2c} \frac{Z_o R}{\beta^2 \gamma^2} \omega g_\ell$$

or

$$\frac{Z_\ell}{n} = \frac{i}{2} \frac{Z_o}{\beta\gamma^2} g_\ell$$

or

$$\frac{Z_\ell'}{\omega} = \frac{i}{4\pi c} \frac{Z_o}{\beta^2 \gamma^2} g_\ell \tag{7}$$

where

$Z_o = 377\,\Omega$ = impedance of vacuum

$g_\ell = 1 + 2\,\ell n \frac{b}{a}$ = geometrical factor (a,b = radii of beam and pipe both assumed cylindrical)

<u>Transverse</u>

$$Z_t = i \frac{Z_o R}{\beta^2 \gamma^2} g_t$$

or

$$Z_t' = \frac{i}{2\pi} \frac{Z_o}{\beta^2\gamma^2} g_t \tag{8}$$

where

$$g_t = \frac{1}{a^2} - \frac{1}{b^2} = \text{geometrical factor} .$$

2. Resistive wall term - This term depends on the conductivity σ and the permeability μ of the wall of the beam pipe through the skin depth. It is rich in low frequencies and is generally small.

<u>Longitudinal</u>

$$Z_\ell = \frac{1-i}{2c} \frac{Z_o R}{b} \omega\delta$$

or

$$\frac{Z_\ell}{n} = \frac{1-i}{2} \frac{Z_o R}{b} \delta$$

or

$$\frac{Z_\ell'}{\omega} = \frac{1-i}{4\pi c} \frac{Z_o}{b} \delta \tag{9}$$

where

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \text{skin depth} .$$

<u>Transverse</u>

$$Z_t = (1-i) \frac{Z_o R}{b^3} \delta$$

or

$$Z_t' = \frac{1-i}{2\pi} \frac{Z_o}{b^3} \delta . \tag{10}$$

3. Non-resonant broadband term - This term is the contribution from discontinuities in the pipe, bellows, electrodes etc. It has the frequency dependence of a parallel $R(=R_s)$, L, C circuit with a resonant

frequency roughly equal to the cut-off frequency of the beam pipe. This term is mainly responsible for high frequency instabilities within each single bunch.

Longitudinal

$$Z_\ell = R_s \left(\frac{\omega}{\omega_r}\right) \left[\frac{\omega}{\omega_r} + iQ\left(1 - \frac{\omega^2}{\omega_r^2}\right)\right]^{-1} \equiv R_s \frac{\omega}{\omega_r} \; f(\omega)$$

or

$$\frac{Z_\ell}{n} = R_s \left(\frac{c\beta}{R\omega_r}\right) f(\omega), \qquad f(\omega) \cong \left(\frac{\omega}{\omega_r} - i\right) \quad \text{for } \frac{\omega}{\omega_r} << 1$$

or

$$\frac{Z_\ell'}{\omega} = \frac{R_s'}{\omega_r} f(\omega) \tag{11}$$

where

R_s = total shunt resistance of beam pipe

$R_s' = \frac{R_s}{2\pi R}$ = shunt resistance per unit length

$Q \cong \frac{R_s}{\omega L}$ = "Q" of the resonant circuit $\cong 1$

$\omega_r \equiv \frac{1}{\sqrt{LC}}$ = resonant frequency of circuit $\cong \frac{c}{b}$ = cut-off frequency .

Transverse

$$Z_t = \frac{2R}{b^2\beta}\left(\frac{Z_\ell}{n}\right) = R_s\left(\frac{2c}{b^2\omega_r}\right) f(\omega)$$

or

$$Z_t' = R_s' \frac{2c}{b^2\omega_r} f(\omega) = \frac{2c}{b^2}\left(\frac{Z_\ell'}{\omega}\right) . \tag{12}$$

4. High-Q resonant term - This term arises from resonances of specific configurations or devices in the beam pipe and hence can not be expressed in a standard form. It is generally rich in low frequencies and gives rise

to interbunch or single bunch multi-turn instabilities. These instabilities can be cured by feedback. Therefore this term can usually be neglected.

From the above discussion we see that if one neglects the space charge terms one gets the relation

$$Z_t = \frac{2R}{b^2\beta}\left(\frac{Z_\ell}{n}\right)$$

or

$$Z'_t = \frac{2c}{b^2}\left(\frac{Z'_\ell}{\omega}\right). \tag{13}$$

Since $b^2 g_t = \frac{b^2}{a^2} - 1$ is generally much larger than $g_\ell = 1 + 2\,\ln\frac{b}{a}$ the above relation gives a lower limit for Z_t, thus a separate threshold for $\frac{Z_\ell}{n}$ through the threshold value of Z_t for transverse instability.

C. Scaling Laws

Here we assume that each beam bunch contains N particles, has a longitudinal emittance ε_ℓ and is bunched by an rf with peak voltage V and wave length $\lambda \equiv \frac{2\pi R}{h}$ where h is the harmonic number. Generally ε_ℓ is much smaller than the (stationary) rf bucket area. Thus, we can write for the full dimensions of the beam bunch

$$\begin{cases} \text{Length} = \Delta\ell = \dfrac{\lambda}{\pi}\left(\dfrac{\varepsilon_\ell}{\pi}\,\dfrac{1}{a}\right)^{1/2} \\ \text{Momentum spread} = \dfrac{\Delta p}{p} = 2\dfrac{mc}{p}\left(\dfrac{\varepsilon_\ell}{\pi}\,a\right)^{1/2} \end{cases} \tag{14}$$

where a is given by

$$a^2 = \frac{1}{(2\pi)^2}\,\frac{\lambda}{R}\,\frac{eV}{mc^2}\,\frac{\gamma}{|n|}\,. \tag{15}$$

For the peak current during the beam bunch we have

$$I = \frac{ec\beta}{\Delta\ell}N\,. \tag{16}$$

Equations (1), (2), (3), (14), (15) and (16) give altogether the following scaling laws.

$$\left(\frac{|Z_\ell|}{n}\right)_{th} \propto \frac{1}{I}\,\frac{\varepsilon_\ell \lambda^{\frac{1}{2}}}{R^{\frac{1}{2}}}\,\frac{V^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}\nu} \propto \frac{1}{N}\,\frac{\varepsilon_\ell^{3/2}\,\lambda^{5/4}}{R^{\frac{1}{4}}}\,\frac{V^{\frac{1}{4}}}{\gamma^{3/4}\nu^{3/2}} \tag{17}$$

$$\left(|Z_t|\right)_{th} \propto \frac{\Delta\nu}{I}\,\frac{1}{R}\,\gamma\,\nu \propto \frac{\Delta\nu}{N}\,\frac{\varepsilon_\ell^{\frac{1}{2}}\,\lambda^{3/4}}{R^{3/4}}\,\frac{\gamma^{3/4}\nu^{1/2}}{V^{\frac{1}{4}}} \tag{18}$$

and

$$b^2 = 2R\,\frac{\left(|Z_\ell|/n\right)_{th}}{\left(|Z_t|\right)_{th}} \propto \frac{1}{\Delta\nu}\,\varepsilon_\ell \lambda^{\frac{1}{2}} R^{3/2}\,\frac{V^{\frac{1}{2}}}{\gamma^{3/2}\nu^2} \tag{19}$$

where the subscript th denotes threshold value and where we have used the following approximations

$$\beta \cong 1, \qquad \gamma_t \propto \nu$$

$$-\Delta\left[(n-\nu)\omega_o\right] \cong \omega_o\,\Delta\nu \quad .$$

Although the tune spread $\Delta\nu$ is shown explicitly in these formulas, it is limited by resonances to a small and fixed value, and can therefore be considered as constant. The dependencies in $\left(|Z_\ell|/n\right)_{th}$ and $\left(|Z_t|\right)_{th}$ can be summarized in the following table of exponents.

	$\left(\lvert Z_\ell\rvert/n\right)_{th}$		$\left(\lvert Z_t\rvert\right)_{th}$	
	I fixed	N fixed	I fixed	N fixed
ε_ℓ	1	3/2	0	1/2
λ	1/2	5/4	0	3/4
R	-1/2	-1/4	-1	-3/4
γ	-1/2	-3/4	1	3/4
ν	-1	-3/2	1	1/2
V	1/2	1/4	0	-1/4

To make the beam more stable i.e. to increase $\left(|Z_\ell|/n\right)_{th}$ and $\left(|Z_t|\right)_{th}$ this table shows that:

1. For ε_ℓ, λ and R the exponents have the same signs for all cases. Hence we conclude:

The longitudinal emittance ε_ℓ should be large.

The ring circumference $2\pi R$ should be small.

The rf wave length λ should be large.

(Namely, the harmonic number $h = \frac{2\pi R}{\lambda}$ should be small. For the ranges of energy and rf frequency of interest, λ and R are essentially independent.)

All three are actually fairly obvious statements.

2. The exponents of γ and ν in $\left(|Z_\ell|/n\right)_{th}$ and $\left(|Z_t|\right)_{th}$ have opposite signs. Therefore, no simple statement can be made regarding the choice of their values.

3. For the fixed-current (I) case one may want to make the rf voltage V large to improve the longitudinal stability. However, for the more realistic fixed-number (N) case the effects of V on the longitudinal and the transverse instabilities are equal and opposite.

To reduce the magnet aperture or equivalently the radius b of the beam pipe Equation (19) states that one should reduce ε_ℓ, λ, R and V; and increase γ and ν.

Combining the effects on the beam pipe radius and those on the threshold impedances we are left with only one unequivocal requirement, namely that the ring circumference should be small.

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Choice of Tune and Aperture of a Circular Collider

L. C. Teng

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At low energies the considerations in the choice of the focusing strength (choice of ν) are the beam size and the orbit distortion due to field errors which, together, generate a geometrical requirement on the size of the beam pipe and the good field aperture. Indeed, the strong focusing principle was invented to reduce the necessary magnet aperture, thereby the cost of the magnets. But, at high energies this is no longer true. The beam size is, generally, negligibly small and the orbit distortion is corrected to arbitrary desired accuracy. With properly designed trim dipole system the correction is straightforward. Other types of geometrical demands on the aperture arise from beam manipulations such as stacking and resonant extraction. These requirements tend to be local and can usually be satisfied by local lattice insertions (high or low-β, high or zero-dispersion etc.)

The excitation of higher order resonances by magnet field errors is small and negligible beyond the octupole. However, in colliders the excitation by beam-beam forces is large and resonances up to the 7^{th} order must be avoided. (This severely limits the allowable tune spread $\Delta\nu$, hence the available Landau damping of beam instabilities.) But this excitation depends only on the orbit functions at the collision point and not on the overall focusing strength.

In a high energy collider for which there is no geometrical demand on aperture we are left only with the electromagnetic consideration in choosing the focusing strength and the aperture, namely that of the coherent instabilities of the beam. The beam current induces a voltage through an "impedance" of the beam pipe. This voltage acts back on the beam as positive feedback and making it unstable. The instability is damped by a spread in the natural frequencies of individual particles in the beam which causes the instability to lose coherence. The larger is the frequency spread (generally generated by a momentum spread) and the smaller is the "impedance" the more stable is the beam.

The condition for longitudinal stability is, at high energies

$$\frac{|Z_\ell|}{n} < \frac{E/e}{I} \frac{1}{\nu^2} \left(\frac{\Delta p}{p}\right)^2$$

where

Z_ℓ = longitudinal impedance

n = mode number = number of instability waves around the ring

E and I = energy and current of beam

ν = tune

$\frac{\Delta p}{p}$ = full momentum spread.

Solving for ν we get

$$\nu < \left(\frac{E/e}{I}\right)^{\frac{1}{2}} \frac{\Delta p}{p} \left(\frac{|Z_\ell|}{n}\right)^{-\frac{1}{2}} \quad . \tag{1A}$$

The condition for transverse stability is

$$|Z_t| < \pi \frac{E/e}{I} \frac{\nu}{R} \Delta\nu$$

or

$$\nu > \frac{1}{\pi} \frac{I}{E/e} \frac{R}{\Delta\nu} |Z_t| \tag{2A}$$

where

$$Z_t = \text{transverse impedance}$$
$$R = \text{radius of ring}$$
$$\Delta\nu = \text{tune spread in beam.}$$

For a circular beam pipe the transverse impedance is related to the longitudinal impedance through the pipe dimensions by the inequality

$$\left|Z_t\right| \gtrsim \frac{2R}{b^2} \frac{\left|Z_\ell\right|}{n} \tag{3A}$$

where

$$b = \text{beam pipe radius.}$$

Equations (2A) and (3A) together give

$$\nu > \frac{2}{\pi} \frac{I}{E/e} \left(\frac{R}{b}\right)^2 \frac{1}{\Delta\nu} \frac{\left|Z_\ell\right|}{n} . \tag{4A}$$

Thus, ν is hemmed in by

$$B \frac{\left|Z_\ell\right|}{n} < \nu < A \left(\frac{\left|Z_\ell\right|}{n}\right)^{-\frac{1}{2}} \tag{5A}$$

with

$$\left\{\begin{array}{ll} A = \left(\frac{E/e}{I}\right)^{\frac{1}{2}} \frac{\Delta p}{p} & \propto \left(\frac{E}{I}\right)^{\frac{1}{2}} \frac{\Delta p}{p} \\ B \equiv \frac{2}{\pi} \frac{I}{E/e} \left(\frac{R}{b}\right)^2 \frac{1}{\Delta\nu} & \propto \frac{I}{E} \left(\frac{R}{b}\right)^2 \end{array}\right. \tag{6A}$$

where the tune spread $\Delta\nu$ is limited by resonances to a fixed value of ~0.01.

A. Scaling

A wider range of acceptable ν value would allow larger values of $\left|Z_\ell\right|/n$. Hence we would like A to be large and B to be small. To increase A and decrease B we should

1. Increase $\frac{E}{I}$. This extends the acceptable ν-range at both ends.

This also indicates that the tightest constraint occurs at injection when E is smallest. Reducing I helps, but the luminosity suffers.

2. Increase $\frac{\Delta p}{p}$. This raises the upper limit of the ν-range, but requires either blowing up the longitudinal emittance or a huge increase in rf voltage (as the 4th power of $\frac{\Delta p}{p}$). Neither alternative is very attractive.

3. Increase $\frac{b}{R}$. Because of the squared dependence this is very effective in lowering the lower limit of the ν-range. Since the stored energy in the magnet ring is proportional to $B^2 \times (b^2 R) \sim b^2/R$, to minimize the increase in stored energy it is more desirable to reduce R than to increase b.

B. Numerical example

For a 20 TeV collider assuming

$E = 1$ TeV (injection energy)

$I = 1$ A (2×10^{10} protons in a 1 m long bunch)

$\frac{\Delta p}{p} = 2\times10^{-4}$ (a large value)

$R = 9$ km (using 10 T dipoles)

$b = 0.0254$ m (1" radius aperture)

$\Delta\nu = 0.01$ (limited by resonances)

we get

$$(8\ \Omega^{-1})\ \frac{|Z_\ell|}{n} < \nu < (200\ \Omega^{\frac{1}{2}}) \left(\frac{|Z_\ell|}{n}\right)^{-\frac{1}{2}}$$

which gives a range of about 40 to 90 even with $|Z_\ell|/n = 5\,\Omega$ and shows that $\nu = 60$ is a good choice. Going to R = 36 km (2.5 T dipoles) raises the lower limit by a factor 16 and gives

$$(128\ \Omega^{-1})\ \frac{|Z_\ell|}{n} < \nu < (200\ \Omega^{\frac{1}{2}}) \left(\frac{|Z_\ell|}{n}\right)^{-\frac{1}{2}} .$$

The ν-range narrows to zero at $|Z_\ell|/n \cong 1.35\,\Omega$. The value $|Z_\ell|/n \cong 1\,\Omega$ is very difficult to achieve and the value $\nu \cong 170$ is not very desirable.