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A MODEL TO DESCRIBE DIFFUSION ENHANCEMENT BY THE BEAM-BEAM INTERACTION

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1. Introduction

In a previous paper[1], two of us investigated the effects of a non-linear "beam-beam" force on random diffusion in a numerical simulation of one-dimensional (1-D) motion. Our investigation found that diffusion can be enhanced substantially by the nonlinear force if the non-linear force parameter ("beam-beam" tune shift) be large enough to include a major resonance within its width.

In this paper we study this diffusion enhancement and develop analytical and numerical methods to estimate the magnitude and variation of this enhancement, and find substantial agreement between simulation results and our theoretical model that describes diffusion enhancement. In future papers we will extend this understanding to 2-D forces with multiple resonances and include other effects to develop a realistic model of the beam-beam interaction in $\bar{p}p$ storage rings.

2. The Effects of Resonances on Particle Motion - Theoretical Formulation

In this section we follow standard methods to describe particle motion in a resonance region. The particle Hamiltonian is expanded in a double Fourier series with respect to phase and time and only lowest order resonant terms are retained. The resulting approximate Hamiltonian is used to obtain expressions for resonance locations and widths. These expressions are used in other sections for explicit calculations and comparisons with numerical simulations.

We start with the equation of motion:

$$x'' + k(s)x = -\frac{4\pi\ \Delta\nu}{\beta_o}\left(\frac{1-e^{\frac{-x^2}{2\sigma^2}}}{\frac{x^2}{2\sigma^2}}\right) x\ \delta_p(s) \tag{1}$$

where the independent variable is s (distance around ring), and the dependent variable is x. The factors $\Delta\nu$ and β_o are the "linear tune shift" and the unperturbed β-function value at the interaction region, $\delta_p(s)$ is a periodic δ-function and the nonlinear expression

$$\left(\frac{1-e^{\frac{x^2}{2\sigma^2}}}{\frac{x^2}{2\sigma^2}}\right) x$$

is the 1-D truncation of the beam-beam space charge force due to a round gaussian beam of rms radius σ. To allow more general forms of this nonlinear function we rewrite equation 1 as

$$x'' + k(\delta)x = -\frac{4\pi\ \Delta\nu}{\beta_o}\ f(x)\ \delta_p(s)$$

or (2)

$$x'' + k(\delta)x \equiv -A\,\frac{\partial U(x)}{\partial x}\,\delta_p(s)$$

with $U(x) \equiv \int f(x)dx$ a potential function and $A \equiv 4\pi\ \Delta\nu/\beta_o$.

To apply the single resonance approximation we must apply three successive canonical transformations to the Hamiltonian derived from equation (2):

$$H = \frac{1}{2}(p^2+k(s)x^2) + A\ U(x)\ \delta_p(s).$$

The transformations are a Courant-Synder transformation, an action angle transformation, and a slow variable transformation. As noted by Smith[2] the three transformations can be combined into a single one with the following generating function

$$S(x,\psi,s) = -\frac{x^2}{2\beta(s)}\left[\tan\phi(s,\psi) - \frac{\beta'(\delta)}{2}\right] \tag{5}$$

with

$$\phi \equiv \psi + \nu_p \frac{s}{R} + \int_0^s \left(\frac{1}{\beta(s)} - \frac{\nu}{R}\right) ds \tag{6}$$

β is the Courant-Snyder β-function[3], $2\pi R$ is the circumference of the accelerator ring, ν_p is some resonant tune value (see below), ϕ is a betatron phase. From the generating function we find:

$$x = \sqrt{2I\beta}\cos\phi$$

$$p = -\sqrt{\frac{2I}{\beta}}\left[\sin\phi - \frac{\beta'}{2}\cos\phi\right]. \tag{7}$$

The new Hamiltonian is:

$$H(I,\psi,\theta) = (\nu-\nu_o)I + \delta_p(\theta)\ A\ U(\sqrt{2I\beta_o}\cos\phi) \tag{8}$$

where $\theta \equiv \frac{s}{R}$ and we have changed our independent variable from s to θ. (We have followed procedures similar to those used by Chao[4] in these transformations.)

The beam-beam perturbation $A\ \delta_p(\theta)\ U(\sqrt{2I\beta_o\cos\phi})$ is expanded as a double Fourier series in θ and ϕ using:

$$\delta_p(\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{in\theta} \tag{9}$$

and $U(\sqrt{2I\beta_o}\cos\phi) = U_o(I) + 2\sum_{m=1}^{\infty} U_m(I)\cos m\phi$

$$= \sum_{-\infty}^{\infty} U_m(I)\ e^{im\phi}$$

where $$U_o(I) = \frac{1}{2\pi}\int U(\sqrt{2I\beta_o}\cos\phi)\, d\phi$$

$$U_{-m}(I) = U_m(I) = \frac{1}{2\pi}\int U(\sqrt{2I\beta_o}\cos\phi)\cos m\phi d\phi. \tag{10}$$

We now use the "smooth approximation" which means ignoring fast changing parameters. In

$$\phi = \psi + \nu_p \underline{\theta} + \int_o^{R\theta} ds' \left(\frac{1}{\beta(s')} - \frac{\nu}{R}\right)$$

ψ is the slowly changing variables. We set ν_p to a resonant tune $\frac{n_o}{m_o}$. Note that the integral over s' is a fast oscillation and, in fact, all terms in the Fourier expansion except those with $-\nu_p = \frac{n}{m}$ are fast oscillations. We keep only slowly varying terms to obtain

$$A\,\delta_p(\theta)\, U(\sqrt{2I\beta_o}\cos\phi)$$

$$\cong A\left(\frac{U_o}{2\pi}(I) + \frac{2}{2\pi}\sum_{k=1}^{\infty} U_{km_o}(I)\cos km_o\psi\right). \tag{11}$$

For the moment we keep only the lowest order resonant term k = 1. The resulting Hamiltonian is:

$$H \cong (\nu-\nu_p)\, I + \frac{A}{2\pi} U_o(I) + \frac{2A}{2\pi} U_{m_o}(I)\cos m_o\psi. \tag{12}$$

From the equations of motion we find fixed points where $\psi' = 0$ and $I' = 0$

$$\psi' = \frac{\partial H}{\partial I} = \nu-\nu_p + \frac{A}{2\pi} U_o' + \frac{2A}{2\pi} U_{m_o}'\cos m_o\psi$$

$$I' = \frac{-\partial H}{\partial \psi} = \frac{2A}{2\pi} U_{m_o} m_o \sin m_o\psi \tag{13}$$

we require

$$\psi = 0, \frac{\pi}{m_o}, \ldots, \frac{2(m_o-1)\pi}{m_o}$$

and

$$\nu-\nu_p + \frac{A}{2\pi} U_o' \pm \frac{2A}{2\pi} U_m' = 0$$

where the ± sign comes from the even or odd multiplier of π appearing in the argument of the cosine. If we assume U_m' is small, the fixed points are located at $I = I_p$, the solution of the equation

$$\nu-\nu_p + \frac{A}{2\pi} U_o'(I) = 0. \tag{14}$$

We can expand U_o around I_p

$$U_o = U_o(I_p) + U_o'(I_p)\ (I-I_p) + \frac{U_o''}{2}(I_p)\ (I-I_p)^2.$$

The Hamiltonian in the region around I_p can be written as:

$$H \cong \frac{1}{2}\frac{A}{2\pi} U_o''\ (I-I_p)^2 + \frac{2A}{2\pi} U_{m_o}(I_p)\cos m_o\psi + H_o \tag{15}$$

which is now recognizable as the Hamiltonian for a pendulum. A resonance width can be found from the boundary of the separatrix. It is:

$$\Delta^2 = W^2_{max} \cong \frac{8\ U_{m_o}}{U_o''} \cong (I-I_p)^2_{max}\ . \tag{16}$$

3. A Model to Describe Diffusion Enhancement by the Beam-Beam Interaction

In this section we describe the model which we use to explore diffusion enhancement. In the previous section we noted that resonances with characteristic locations I_o and characteristic widths Δ can appear in particle motion. In Figure 1A we show phase space plots - trajectories of individual particle positions measured at the interaction point - for a case with tune ν = .16, and $\Delta\nu = .04$ (so that ν_o of reference 1 is .20). We clearly see resonant regions characteristic of sixth order resonance. Figures 1B, 1C, and 1D show other resonances for other tunes and tune shifts.

Particle motion is significantly distorted by the resonance so that if a particle amplitude reaches the threshold amplitude $I_o-\Delta$ the resonance also moves the particle to regions with amplitude $(I_o+\Delta)$ from which a random kick easily moves the particle amplitude to the nonresonance region past $I = I_o+\Delta$. The effect of the resonance, when averaged over an ensemble of particles, is to add an effective change in amplitude I from $I_o-\Delta$ to $I_o+\Delta$ to each particle which reaches the lower resonant amplitude.

The diffusion process, which is a random velocity kick, increases rms particle amplitudes in a Brownian manner. The resonance places the amplitude $I_p-\Delta$ and $I_p+\Delta$ adjacent so that the Brownian motion pushes rms amplitudes past the resonance.

We have defined our emittance as

$$\varepsilon = 3 \left< \frac{x^2}{\beta_o} + x'^2 \beta_o \right> \cong 6\left< I \right>.$$

Our model for diffusion gives us

$$\dot{\varepsilon} = \dot{\varepsilon}_o + 12 \frac{\dot{N}}{N}(t)\Delta. \tag{17}$$

The factor $\dot{\varepsilon}_o = \frac{3\beta}{T}(\Delta x')^2$ is the Brownian diffusion from the random velocity kicks of amplitude $\Delta x'$ of frequency $(1/T)$ and which appears in the absence of resonant behavior. The resonance with width 2Δ provides the enhancement term. The factor $\dot{N}$ is the rate at which particles reach the amplitude $(I_p-\Delta) = I_T$ by the Brownian motion and the factor N normalizes the expression by dividing by the total number of particles.

The factor $\dot{N}/N$ can be estimated by considering the change in the particle distribution function due to diffusion. The initial distribution function is gaussian:

$$f(\dot{x},x') \,\alpha\, e^{\frac{-x^2}{2\sigma^2}} \cdot e^{\frac{-x'^2}{2\sigma'^2}} \cdot dxdx'$$

or in action-angle variables:

$$f(I,\phi) = \frac{1}{2\pi I_o} e^{\frac{-I}{I_o}} \, dId\phi$$

with $I_o = \sigma^2/\beta$ and I_o is a function of time because of the diffusion.

To estimate the particle flux through a threshold value of I, $I = I_T$ we calculate the change in the number of particles with $I<I_T$ obtaining

$$\left.\frac{\dot{N}}{N}\right|_{I=I_T} \cong -\frac{d}{dt}\int_o^{I_T} \frac{e^{\frac{-I}{I_o}}}{I_o} dI$$

$$= \frac{\dot{I}_o}{I_o^2} I_T e^{\frac{-I_T}{I_o}} .$$

We can recognize $\frac{\dot{I}_o}{I_o}$ as $\frac{\dot{\varepsilon}_o}{\varepsilon_o}$ and rewrite equation 17 as

$$\dot{\varepsilon} \cong \dot{\varepsilon}_o \left(1 + \frac{12}{\varepsilon_o} \frac{I_T}{I_o} e^{\frac{-I_T}{I_o}} \Delta \right) \tag{18}$$

This equation shows that the diffusion enhancement is "multiplicative" (proportional to the diffusion) as previously shown numerically in Reference 1.

In the next section we will quantitatively compare the predictions of equation (18) with results of numerical simulations.

4. Analysis of Sample Cases:

In this section we investigate diffusion enhancement by an analysis of sample cases. In figures 1A, IB, 1C, and 1D we show phase space plots with sample particle trajectories. Resonances are clearly visible and their locations and widths agree quite well with calculations based on the previous section.

In Table I we summarize some of these calculations, which include calculations of resonance location and width using equations 15-17. The potential function

$$U(x) = \int_0^x 2\sigma^2 \frac{1-e^{-\frac{x^2}{2\sigma^2}}}{x} dx$$

was integrated numerically and the Fourier component functions ($U_0(I)$, $U_2(I)$, etc.) obtained by the integrations in equation 10. Some of these functions are shown graphically in figures 2. The resonance widths and locations are calculated using these functions. In all these cases we have kept the initial parameters β, σ^2, ε_0 at values given by $\beta = 2$ m, $\sigma^2 = .00667$ $(\mathrm{mm})^2$, $\varepsilon_0 = .02$ mm-mr.

We have also calculated diffusion enhancement x_E using the model of the previous section. In this calculation, we have noted that the very wide resonances of fourth and sixth order show substantial deviations from the simple linearized theory at their inside edges (small-I) and therefore for the evaluation of I_T we have used the phase-space plots directly (figures 1A-1D) rather than the simple formula $I_T = I_p - \Delta$ which would give an unreasonable negative value. Except for this adjustment, equations 14, 16 and 18 have been used directly. We compare these calculated values of x_E with values from numerical simulations similar to those of Reference 1 and find excellent agreement. The deviations are consistent with expected deviations from the linearizations in section 2.

5. Discussion

In Reference 1 it was reported that diffusion enhancement appears to remain constant as the tune shift $\Delta\nu$ increases with tune at zero amplitude ν_0 constant when $\Delta\nu$ contains a major resonance. We now recognize this as a partial cancellation between competing processes: As the resonant tune moves closer to the origin (as $\Delta\nu$ increases) $\frac{\dot{N}}{N}$ increases but the resonant width Δ decreases. In fact the diffusion enhancement does vary, following equation 18.

Also it was reported that the enhanced diffusion remains linear. This is only approximately true; the enhanced diffusion decreases when the resonance is "saturated"; that is, most of the particles have crossed the resonant tune, or at a time T such that

$$T \cdot \frac{\dot{N}}{N}\bigg|_{I=I_T} \gtrsim 1.$$

The expanding distribution can also show an increased enhancement if the spreading distribution reaches a resonance at large I.

In Figure 3 we show emittance as a function of time for $\nu_o = 30$, $\Delta\nu = .08$. There is initially a large enhancement due to the $\frac{1}{4}$ resonance. After ~200,000 turns the diffusion decreases as the resonance is saturated. Then at ~600,000 turns diffusion is enhanced again as particles reach the $\frac{4}{18}$ resonance at large amplitude.

In a second case $\left(\nu_o = .2, \Delta\nu = .10,\right)$ our calculations show that both the $\frac{1}{6}$ and $\frac{1}{8}$ resonances contribute substantially to diffusion enhancement. This indicates that the presence of more than one major resonance does affect diffusion. In the future we will explore the effects of overlapping or nearly overlapping resonances in 1-D and 2-D cases.

Some of the detailed features of the enhancement are dependent on our choice of potential function, which has the large amplitude behavior $U(x) \propto \ell n x$. This makes resonances for large values of I much broader than for other potentials. Also our distribution function, originally gaussian, has much fewer particles at large amplitudes than actual beam bunches. The analytical tools we have developed in this note allow us to explore the effects of other potentials and other distributions which more closely approximate physical situations quite easily.

Conclusion

In this note we have developed an analytical model which adequately describes diffusion enhancement in 1-D motion by a non-linear (beam-beam) force, in the case where this enhancement is dominated by a single or a few non-overlapping resonances. We hope to extend this analysis to 2-D motion with overlapping resonances.

References

1. D. Neuffer and A. Ruggiero, "Enhancement of Diffusion by a Nonlinear Force", FN-325, April 1980
2. L. Smith, private communications (1975)
3. E.D. Courant and H.S. Snyder, Ann. of Physics 3, 1 (1958)
4. A.W. Chao in "Nonlinear Dynamics and the Beam-Beam Interaction", AIP Conf. Proc. 57, p. 42 (1979)
5. A. Ruggiero, $\bar{p}$-note 62 (1980)

Table I

ν	$\Delta\nu$	ν_o	ν_p	I_p	Δ	$\left.\frac{\dot{N}}{N}\right\vert_{I_T}$	x_E (cal.)	x_E (sim.)
tune	beam-beam tune shift	tune at zero amplitude	resonant tune	resonance amplitude	resonance width	(t=0)	diffusion enhancement	
.16	.04	.20	1/6	.0325	.026	.043	2.7	2.3
.24	.06	.30	1/4	.0325	.046	.12	9.1	7.5
.22	.08	.30	1/4	.012	.013	.14	3.7	4.0
			4/18	.224	.0083	10^{-8}		
.245	.015	.26	1/4	.014	.016	.14	4.4	4.7
.20	.10	.20	1/6	.004	.003	.14	1.7	2.0
			1/8	.020	.006	.0025		
			2/18	.051	.006	4×10^{-6}		

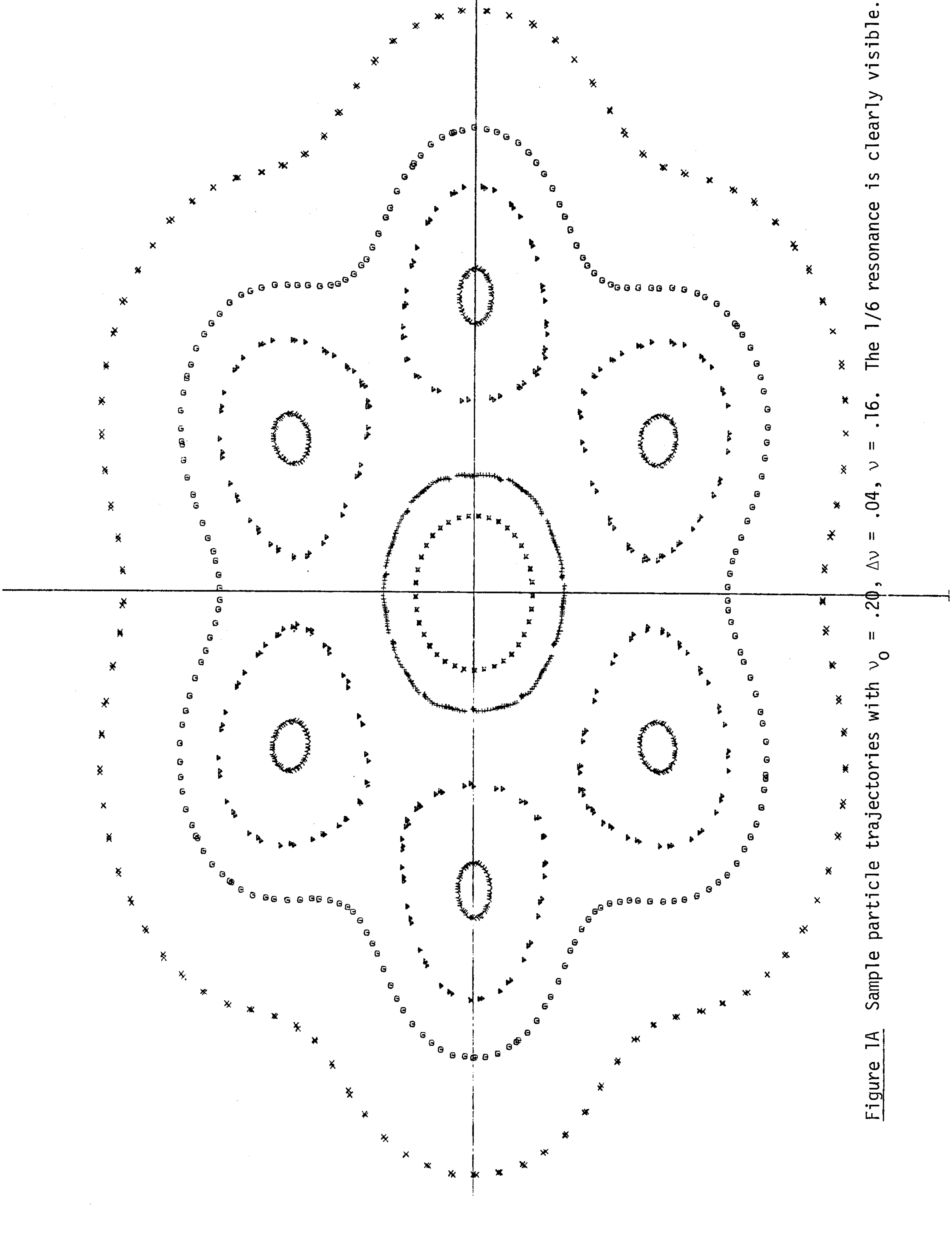

Figure 1A Sample particle trajectories with $\nu_0 = .20$, $\Delta\nu = .04$, $\nu = .16$. The 1/6 resonance is clearly visible.

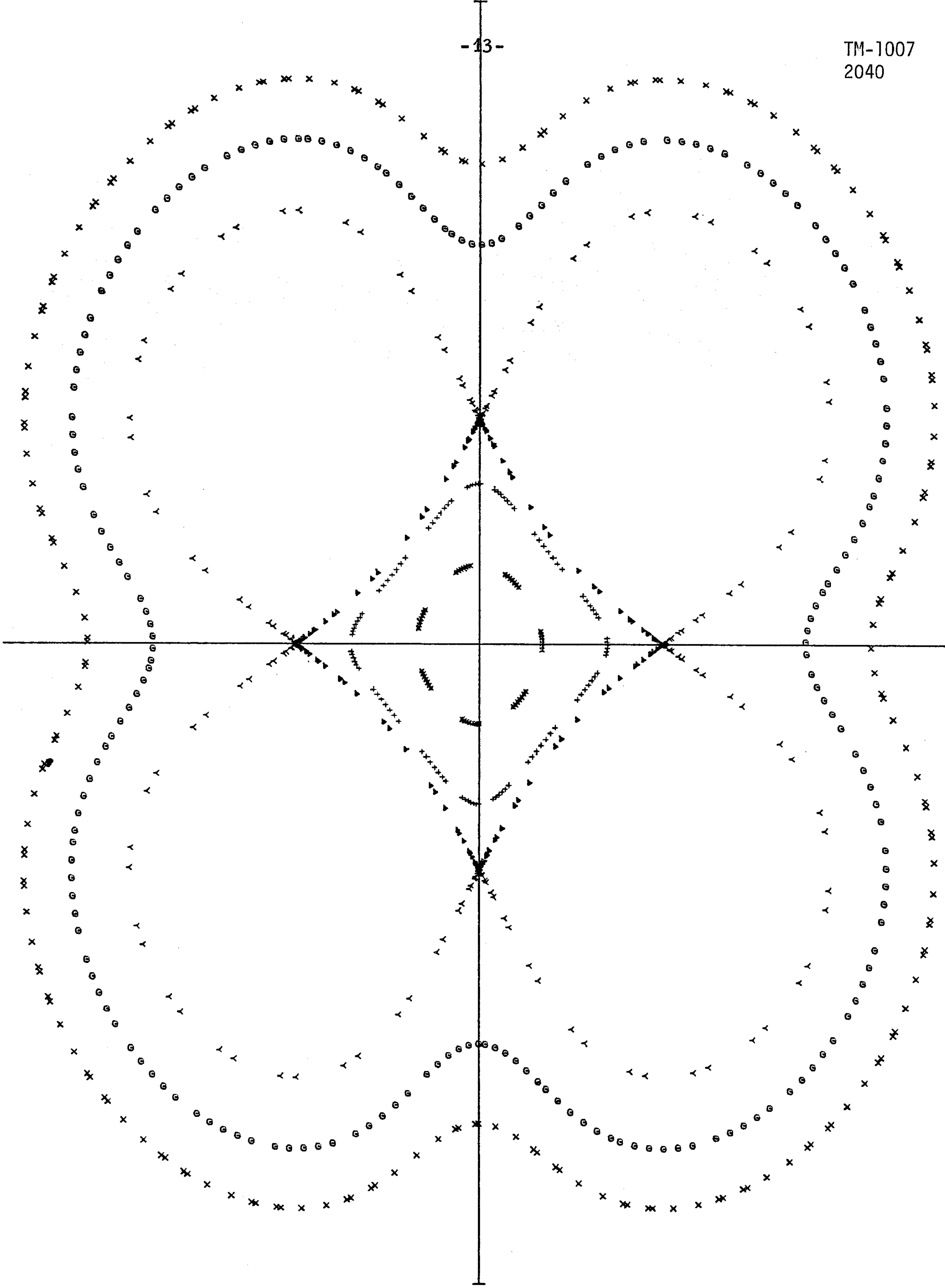

Figure 1B Particle trajectories with $\nu_0 = .30$, $\Delta\nu = .06$, $\nu = .24$.

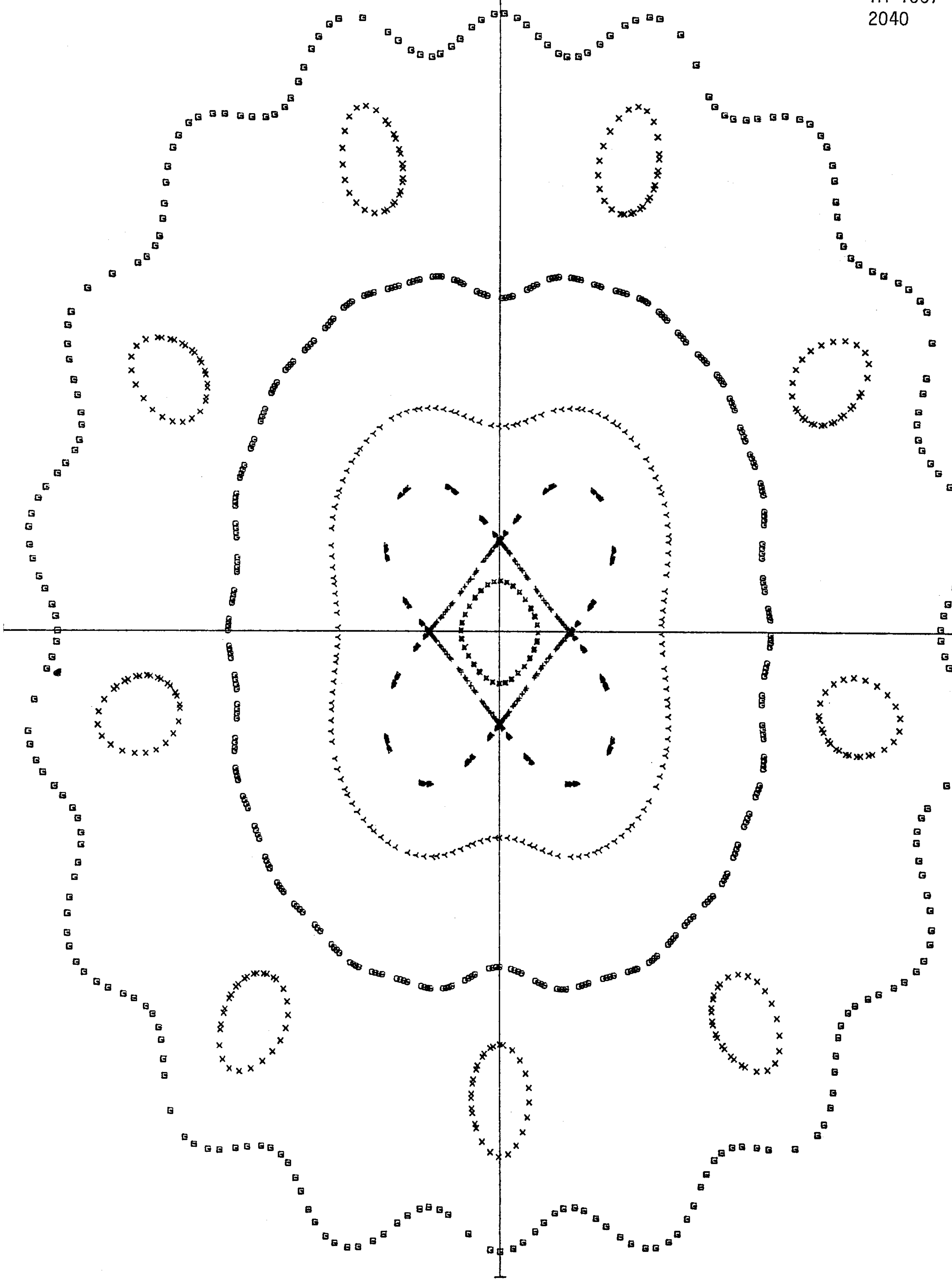

Figure 1C Particle trajectories with $\nu_0 = 0.30$, $\Delta\nu = .08$, $\nu = .22$.

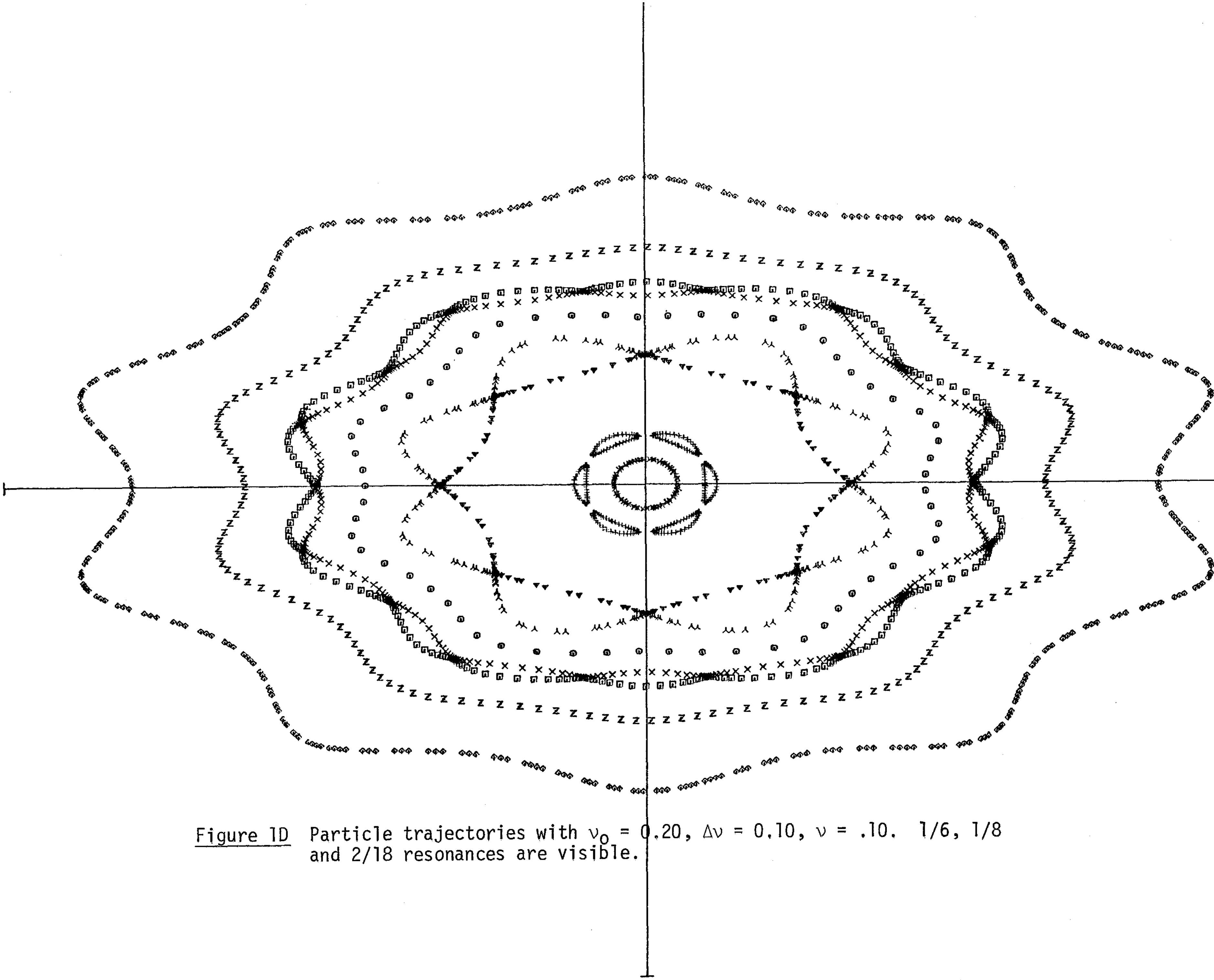

Figure 1D Particle trajectories with $\nu_o = 0.20$, $\Delta\nu = 0.10$, $\nu = .10$. 1/6, 1/8 and 2/18 resonances are visible.

Figure 2A The potential function $U_0(I)$ and Derivative $U_0'(I)$

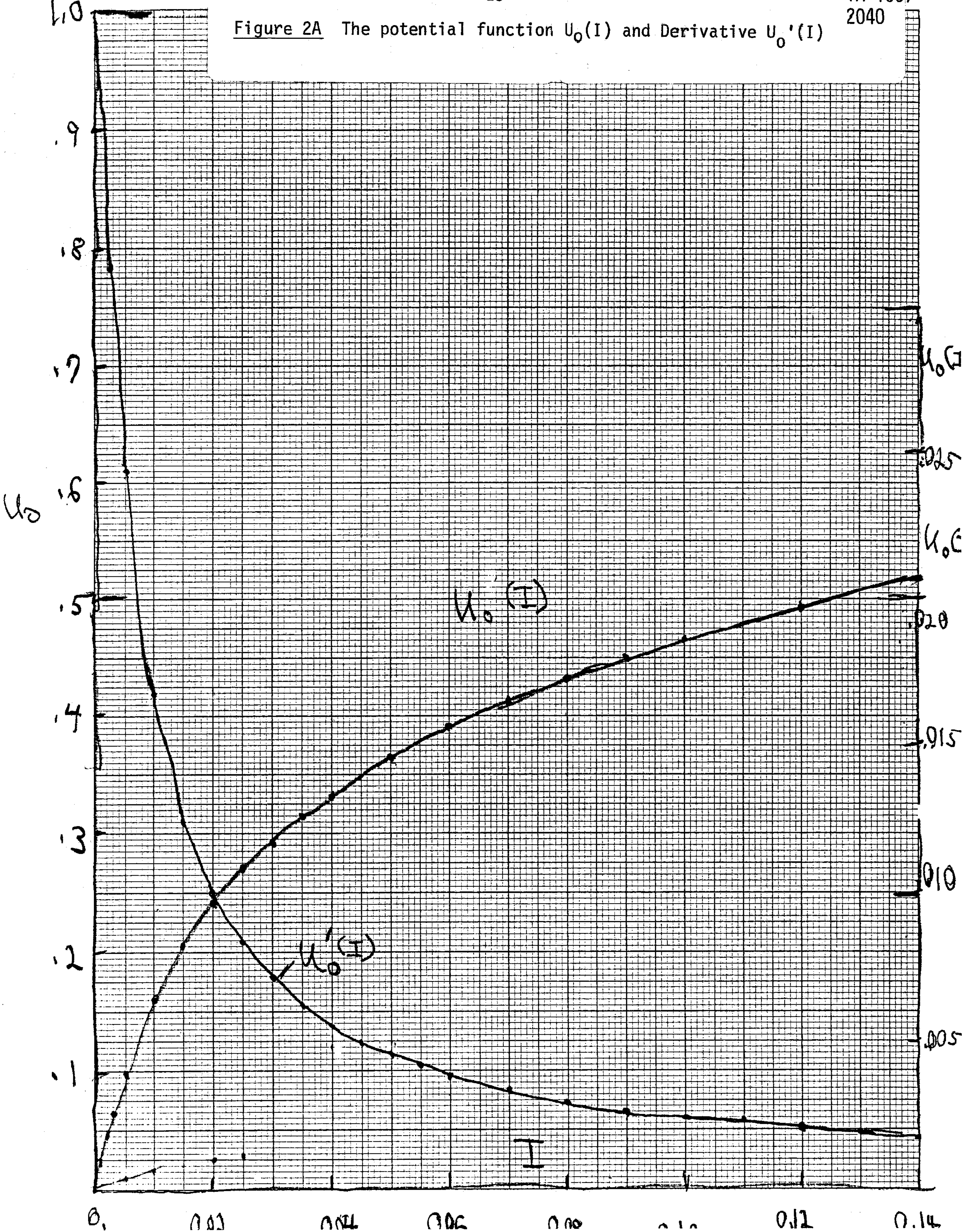

Figure 2B The potential functions U_0, U_2, U_4, U_6 and U_8 calculated from equation (10) using the beam-beam force of equations 1 and 2.

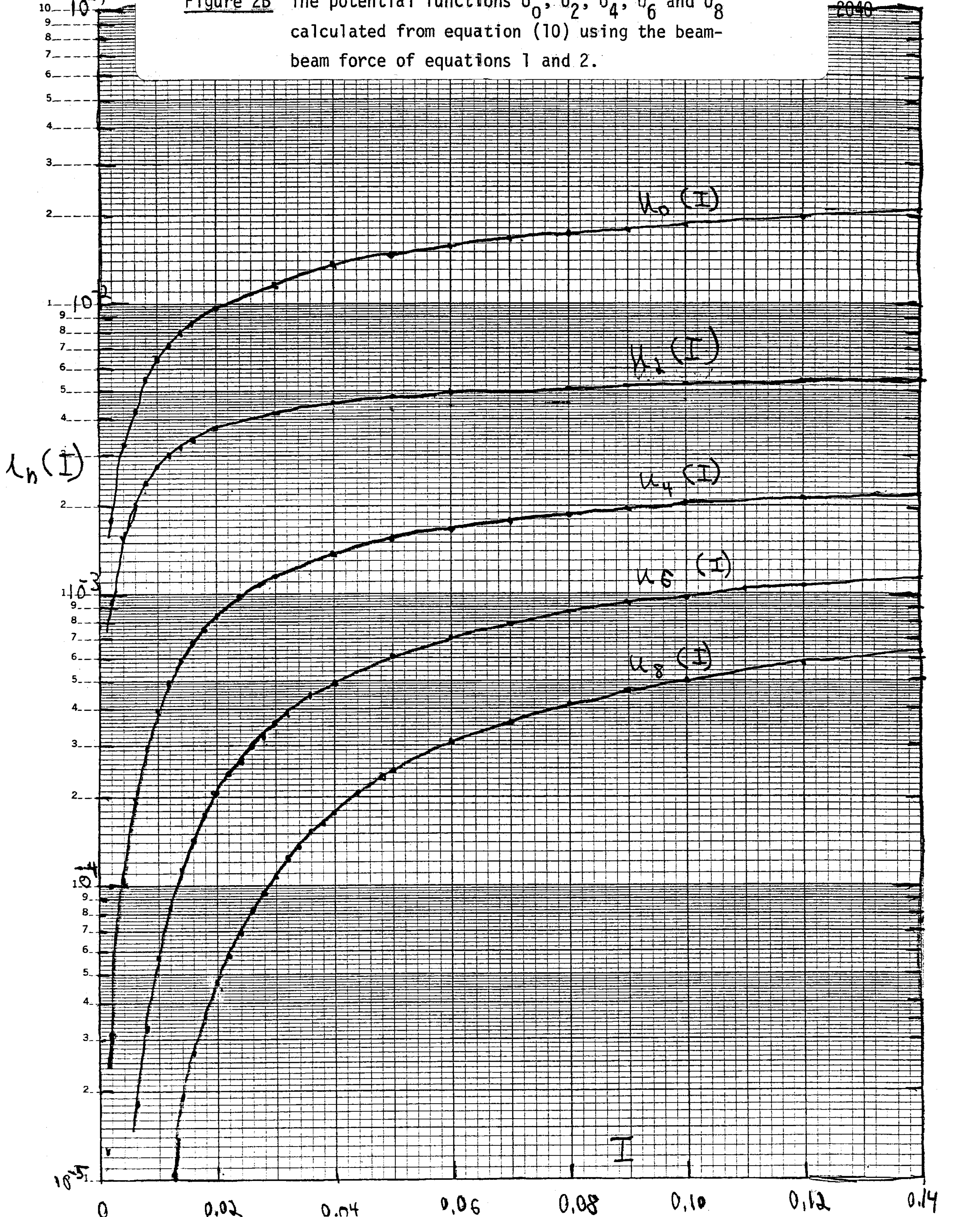

Figure 3 Emittance as a function of time with $\nu_0 = .30$, $\Delta\nu = .08$ ($\nu = .22$) with a large diffusion ($D_0 = .032/100{,}000$ turns)