Fermilab

Stochastic Extraction at Fermilab

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Recently, S. Van der Meer suggested that the quality of beam extraction could be improved by use of a procedure which he describes as "stochastic extraction".[1] Stochastic extraction, as he describes it, is a modification of resonant extraction in which particles are moved to the extraction resonance by random kicks from a noisy RF system. In this note we discuss the equations describing stochastic extraction, and describe a program which numerically integrates these equations. This program is applied to Fermilab main ring parameters, and will be used to interpret the results of experimental tests of stochastic extraction.

I. The Van der Meer Equations

The concept of stochastic extraction is schematically illustrated in Figure 1. There is an initial distribution of particles in tune $\psi = \frac{dN}{dQ}$, which is produced in acceleration. We assume that the tune deviation of an individual particle is found from its momentum deviation by its chromaticity, $\xi\, Q_o$

$$\frac{\Delta Q}{Q_o} = \xi \frac{\Delta p}{p_o} \tag{1}$$

where Q_o, p_o are the central tune and momentum of the initial particle distribution. We also assume that tune is a function only of the momentum.

A flat noise spectrum with idealized square ends at upper and lower frequencies corresponding to the tunes Q_2, Q_1 is used to change the momenta (and tunes) of particles in the distribution. This noise spectrum spreads across a resonant frequency, a harmonic of the revolution frequency, given by $f_o = \frac{h}{T}$, where f_o is the RF resonant frequency, h is the harmonic number, and T is the revolution period.

The bandwidth W of the noise spectrum is given by

$$W = \left| h \cdot \frac{df}{d\left(\frac{\delta p}{p}\right)} \cdot \frac{d\left(\frac{\delta p}{p}\right)}{d\left(\frac{\delta Q}{Q}\right)} \cdot \frac{(Q_2 - Q_1)}{Q} \right| \tag{2}$$

where f is the revolution frequency. This can be rewritten as

$$W = h \cdot \frac{|\eta|}{T} \cdot \frac{1}{|\xi|} \cdot \frac{\Delta Q}{Q} . \tag{3}$$

The RF noise is determined by its frequency $\frac{h}{T}$, bandwidth W, and its rms voltage V_n.

The effect of the noise voltage is to increase the energy spread of the beam, and thus the momentum and tune spread, according to the formula:

$$\frac{d\langle\Delta E\rangle^2}{dt} = \frac{(\Delta E_k)^2}{t_k} = \frac{e^2 \, V_n^{\,2}}{T} \; \frac{1}{WT} \tag{4}$$

ΔE_k is the mean energy change per kick, t_k is the time between kicks, and V_n is the rms voltage kick given to a particle on each turn. The factor (1/WT) appears since the noise spectrum covers only one harmonic of the revolution frequency and kicks within the time period (1/W) are correlated so that the appropriate voltage per kick in ΔE_k is (V_n/WT) and the mean time period between kicks is 1/W.

The particle distribution function ψ should obey the following diffusion equation:

$$\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial x}\left(D\ \frac{\partial\psi}{\partial x}\right) \tag{5}$$

where t = time

$x = Q-Q_1,\ x_o = Q_o-Q_1$

$\psi = \frac{dN}{dx} = \frac{dN}{dQ}$, the particle density,

and $D = \frac{1}{2}\frac{d\langle(x-x_o)^2\rangle}{dt}$ is the diffusion coefficient.

The diffusion coefficient can be found from equation (4) and the direct relationship between δE and x (equation 1) and is given by the equation:

$$D = \frac{\xi^2 Q^2}{2\beta^2\ p^2}\ \frac{d\langle(\delta E)^2\rangle}{dt} = \frac{1}{2W}\left(\frac{\xi Q V_n}{\beta\rho T}\right)^2 \quad . \tag{6}$$

The diffusion equation (5) can be integrated numerically and results of numerical integration will be discussed in Section III. In Section II, we first discuss some features of the solutions of equation 5 which can be developed analytically.

II. Features of the Diffusion Equation

We first assume that the diffusion coefficient D is constant in time and also constant in x between the tunes $x = x_1$ and $x = x_2$, with $D = 0$ for $x < x_1$ and $x > x_2$ (a step function dependence). As noted in reference 1, equation 5 can then be solved in terms of Fourier components, imposing appropriate boundary conditions.

The boundary conditions are:

1. $\frac{\partial\psi}{\partial x} = 0$ at $x = x_1 \equiv 0$ $(Q = Q_1)$. This says that the flux of particles $(\phi = -D\ \frac{\partial\psi}{\partial x})$ through the lower limit of the noise bandwidth $(Q = Q_1)$ is zero. Particle motion is confined within the limits $Q_1<Q<Q_2$.

2. $\psi = 0$ at $x = x_R$ $(Q = Q_R)$. This means that particles

are removed from the distribution ("extracted") whenever their tunes reach the extraction resonant tunes. We assume that particles do not cross the resonance without being extracted.

The solution of equation 5 under these assumptions is:

$$\psi(x,t) = \sum_{n=1,3,5,\cdots} A_n \cos\left(\frac{\pi n x}{2x_R}\right) e^{-\left(\frac{\pi n}{2x_R}\right)^2 Dt} . \tag{7}$$

The coefficients A_n are found from the initial distribution by Fourier decomposition:

$$A_n = \frac{2}{x_R} \int_0^{x_R} \psi(x,0) \cos\left(\frac{\pi n x}{2x_R}\right) dx . \tag{8}$$

Because the decay rate of higher order harmonics is greater (decay α n^2), after some time only the lowest harmonic remains ($n = 1$) and we find:

$$\psi \cong A_1 \cos\left(\frac{\pi x}{2 \cdot x_R}\right)^{-\frac{\pi^2}{4x^2} Dt} \tag{9}$$

Both the distribution function ψ and the extraction flux:

$$\phi(x_R) = -D \frac{\partial\psi(x_R)}{\partial x}$$

decay exponentially with time.

The evolution of a typical distribution with time is displayed graphically in Figures (2:A-D). An initially gaussian distribution (Figure 2A) is stochastically "heated" following the diffusion equation and spreads across the bandwidth (Figure 2B). If extraction did not occur, the particles would spread over the bandwidth, forming a square distribution (This heating method is used at CERN to generate proton beams with large momentum spread for cooling experiments.[2]). Extraction at $x = x_R$ makes the distribution asymmetric (Figure 2C), eventually forming a cosine distribution with a maximum at $x = 0$ and a zero at $x = x_R$ (Figure 2D).

This process can be separated into three stages: (1) an initial heating stage, in which the beam spreads over the bandwidth, (2) an intermediate stage in which extraction begins and reaches a maximum rate as higher harmonics are removed from the beam, and (3) a final stage in which the distribution is cosine shaped and the extraction rate decreases exponentially. In Figure 3 we show extraction rate as a function of time for a typical case with constant diffusion, calculated using VNDRMR.

It is more desirable to obtain a constant extraction rate and this can be obtained if D is varied with time, as discussed in reference 1. If D(t) is not constant equation (6) is no longer an exact solution, and numerical integration is necessary (see section III).

However, some qualitative features remain. For large t, the distribution still forms a cosine shape:

$$\psi(t) \simeq \psi_{max}(t) \cos \frac{\pi x}{2x_R} . \tag{10}$$

The extraction rate is

$$\phi = -D \frac{\partial \psi}{\partial x} (x_R) \cong D(t) \frac{\pi}{2} \frac{\psi_{max}}{x_R}(t) \tag{11}$$

and the total number of particles remaining is:

$$N(t) = \int_0^{x_R} \psi dx \cong \frac{2x_R}{\pi} \psi_{max}(t) . \tag{12}$$

If extraction proceeds at a constant rate ϕ_o for a total time t_s, from $t = 0$, we must have:

$$N(t) = (t_s - t)\phi_o. \tag{12}$$

Combining (10), (11) and (12), we obtain:

$$D(t) \cong \left(\frac{2x_R}{\pi}\right)^2 \frac{1}{t_s - t} \quad \text{as } t \to t_s$$

so that D must increase as $1/t-t_s$. D is increased by increasing the rms voltage $\bar{V}_n$ ($D \propto V_n^2$) and for any RF source there is a maximum obtainable voltage V_{max}. We must, therefore, reach a maximum value of D before $t = t_s$ at a time t_{max}, given by:

$$D_{max} = \frac{1}{2W}\left(\frac{\xi Q V_{max}}{B_p T}\right)^2 \cong \left(\frac{2\ x_R}{\pi}\right)^2 \frac{1}{t_s - t_{max}} .$$

At $t = t_{max}$ a fraction F of the initial cosine distribution remains, with F given by:

$$F \cong \left(\frac{2x_R}{\pi}\right)^2 \frac{1}{t_s\ D_{max}} .$$

It is desirable that F be small (for more complete stochastic extraction).

III. Numerical Integration of the Diffusion Equation

A program named VNDRMR has been written to integrate equation (5) numerically and this program is described in this section.

In VNDRMR, an array X of evenly spaced points is used to determine a coordinate system in x, with $X(1) = x_o$ a lower bound, which is chosen smaller than x_1, and $X(n+1) = X(n) + \Delta x$, where Δx is some sufficiently small increment, and $X(n_{max}) = X(N)$ is some upper bound greater than x_R or x_2. In the program the function $\psi(x)$ is represented as an array SI(N) with the same dimension as X, and this array is calculated as a function of time.

An initial distribution function $\psi(x)$, which, for instance, may be a gaussian with width x_W, is used to generate an initial array SI(N), which is then changed in time following equation 5

$$\psi(x,t+\Delta t) = \psi(x,t) + \frac{\partial\psi}{\partial t}(s,t)\ \Delta t$$

with
$$\frac{\partial\psi(x,t)}{\partial t} = \frac{\partial}{\partial_x}\left(D(x,t)\ \frac{\partial\psi}{\partial x}\ (x,t)\right).$$

In VNDRMR, the derivatives are evaluated using the second order formula:

$$\frac{\partial f}{\partial x} = \frac{1}{2\delta x}\left(f(x+\delta x)-f(x-\delta x)\right).$$

The interval Δt must be chosen such that

$$D_{max}(t)\Delta t \lesssim (\Delta x)^2$$

or the integration diverges.

For extraction, we require $\psi(x_R) = 0$, that is $\psi = 0$ at resonant tune. After each integration step, $\psi(x_R)$ is set to zero. This removes an amount $\frac{-D\ \partial\psi(x_R,t)\ \Delta t}{\partial x}$ from the distribution and simulates extraction. In practice the resonant $\psi(x) = 0$ is given a finite width about x_R, to set a few grid points X(I) within the resonance.

In the above equations, x is dimensionless and only time t provides dimensions. Results from one calculation can be scaled to obtain results applicable to a number of situations, using equations 5 and 6.

In Figure 3 we show extraction rate as a function of time for D constant. This rate increases rapidly to some maximum value ϕ_{max} and then decreases exponentially. To obtain an acceptably steady extraction rate ϕ_o we vary $D(x,t)$ as follows:

(1) D is initially constant with the value set so that $\phi_{max}(D) = \phi_o$.

(2) Extraction proceeds until $\phi(t) = \phi_o$. From that time on $D(t)$ is increased so that the extraction rate $\phi(t) = -D(t) \frac{\partial\psi}{\partial x} (x_R,t)$ is constant.

In Figure 4, we display the results of this extraction procedure with the same initial conditions as the case in Figure 3. $V_n(t)(\alpha \sqrt{D})$ is also shown in Figure 4. $V_n(t)$ has a characteristic behavior corresponding to the three stages of section II:

(A) V_n is initially held constant until the desired maximum flux ϕ_o is obtained. The next stage is:

(B) An intermediate stage in which the increase in $V_n(t)$ is very close to a linear ramp in time. In this stage the distribution loses its higher harmonic content until

(C) a cosine shape is obtained. In this region we must have $D \alpha 1/(t-t_s)$ or $V_n \alpha 1/\sqrt{t-t_s}$. In practice the increase in V_n must stop when $V_n(t) = V_{max}$, after which the extraction rate decreases exponentially.

IV. Parameters for Stochastic Extraction

In this section we exhibit sample parameters for a case suggested as a test of stochastic extraction at Fermilab, a case with 200 GeV/c momentum and a spill time of 1 sec. These parameters may be used during a low energy studies period. The parameters are:

(1) tune $Z = 19.465$ (extraction at $Q_R = 19.5$) - Beam can

accelerate stably at this tune.

(2) velocity $\beta = 1$

(3) tune width of noise $\Delta Q = .08$

(4) chromaticity $|\xi|\ Q = 39$

(5) period $T = 21\ \mu s$

(6) off-momentum factor $|\eta| = .00316$

(7) momentum $p = 200$ GeV/c

(8) RF harmonic $h = 1113$ (normal acceleration harmonic)

(9) spill time $t_s = 1$ sec.

These parameters imply (from equation 3) that the bandwidth is W = 340 Hz. This case is calculated in Figure 2,3,4 and we find that the desired constant spill rate can be approximately obtained by setting

$$V_n(t = 0 \text{ to } t = 0.25) = 100 \text{ kV}$$

and then adding a linear ramp so that $V_n(t>0.25 \text{ sec}) = 100 + 267(t-0.25)$kV. After $t \cong 1.0$ sec the extraction rate decreases with about 10% of the initial beam remaining. It will be important to check these expectations with experiments.

For practical main ring extraction we must increase the momentum to 400 GeV/c and the spill time to ~2.5 sec although we may set the other parameters at the same values shown above. For more complete extraction, we would no longer provide a simple linear ramp to V_n, but instead let V_n vary in the manner shown in Figure 4. V_{max} could have a value of ~2 MV. Scaling from the previous case, we would have $V(t=0) = 126$ kV and find that when $V(t) \to V_{max}$ that $F = 7\text{x}10^{-3}$ of the initial particles would remain, to be removed by some other method of extraction. For an actual system the parameters would be optimized following experiments such as those suggested above.

V. More Comments on Stochastic Extraction

In the discussion above, we have assumed that particle motion is dominated by the diffusion equation 5, that extraction occurs when a particle crosses the resonant tune, and that an ideal RF noise source can be applied with the properties of a constant amplitude within a square-edged bandwidth. This picture may not be completely accurate.

This treatment ignores individual particle motions. If the particle motion per kick is large, particles may move outside the bandwidth limits Q_1, Q_2 or cross the resonance without being extracted. We can gain some feeling for the size of individual motions by calculation. In one turn a particle tune may change by

$$\Delta Q = \frac{|\xi|Q|eV_n|}{cp}$$

which for the second case above (V_n goes from 126 kV to 2 MV) gives as a range $\Delta Q = 1.23\text{x}10^{-5}$ to $1.95\text{x}10^{-4}$, which is fairly small. Since kicks are correlated over the bandwidth W, particle tunes will change, over correlated kicks, by

$$\Delta Q = \frac{|\xi|Q|eV_n|}{cp} \quad \frac{1}{WT}$$

which, for case 2 above, gives $\Delta Q = 1.72\text{x}10^{-3}$ to $2.73\text{x}10^{-2}$ and the correlation is over 148 turns.

This is fairly large, and may give significant individual particle motion effects, particularly at the higher voltages. Particles may cross the resonance without being extracted. However, this may not be harmful since the noise should move them back across the resonance from the other side. This problem should be studied by calculation or experiment.

A related question is whether individual bunches in extraction of bunched beam will heat similarly or whether there will be substantial bunch to bunch jitter in extraction. It is not clear whether a noise voltage source can be provided which would heat each bunch equally. However, the large increase in the longitudinal phase space caused by the stochastic "heating" will probably move particles out of RF buckets. Stochastic extraction is more suitable for unbunched beam.

In the diffusion equation we have assumed an ideal constant amplitude noise source with a perfectly square bandwidth. Small deviations from this ideal form will not affect extraction.

In the case of section VI we have $W \cong 340$ Hz. It is possible that this bandwidth is too small to remove undesirable ripple in extraction. Substantial experimental tests are necessary to test "stochastic extraction".

References

1. S. Van der Meer, CERN/PS/AA 78-6, 1978
2. S. Van der Meer, CERN/PS/AA 79-27, 1979

Figure 1. Stochastic extraction coordinates

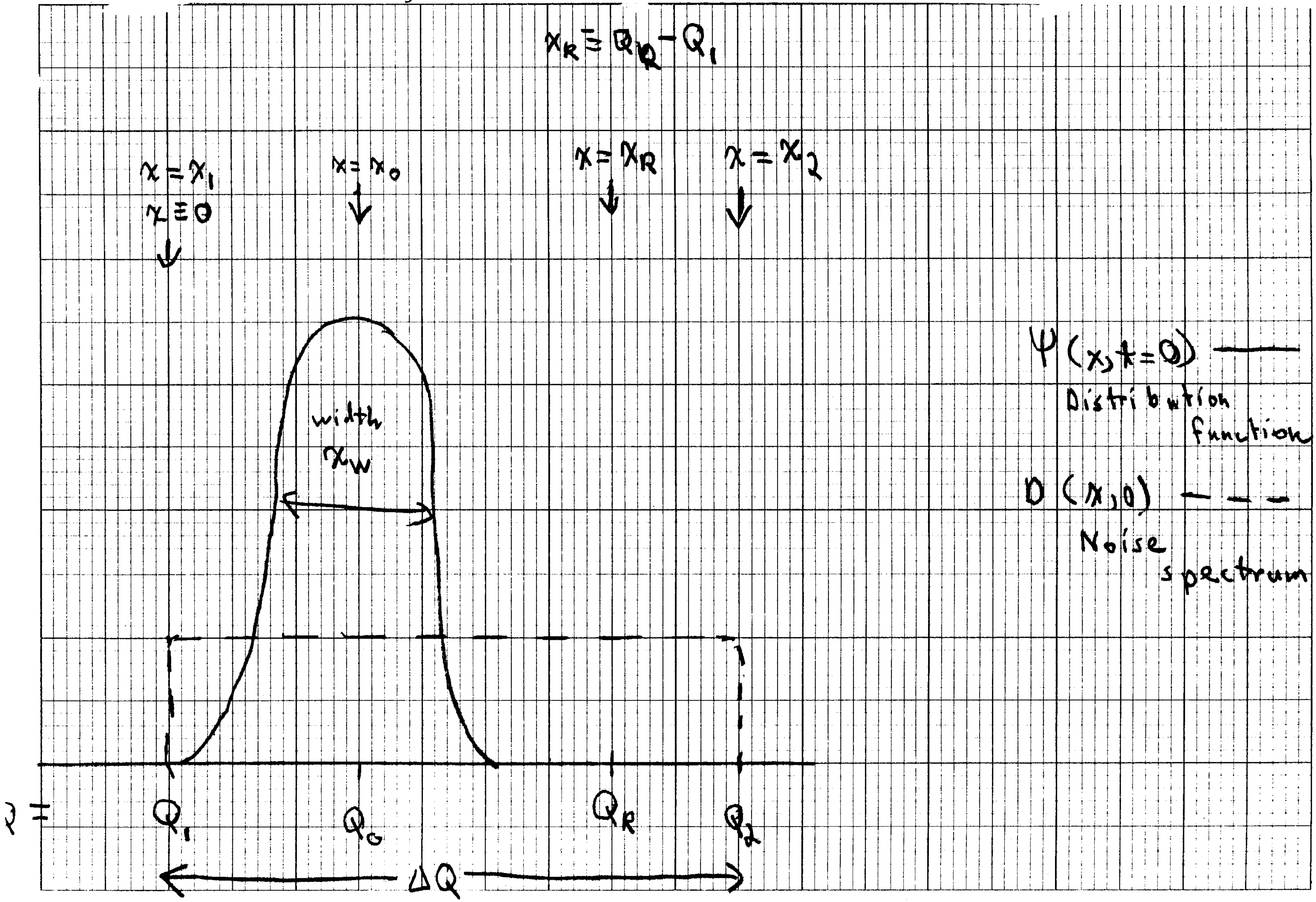

Figure 2. Stochastic extraction of a gaussian distribution with constant diffusion A: t=0, B: t=0.5, C: t=0.20, D: t=1.0. The vertical scales are changed as the extraction proceeds.

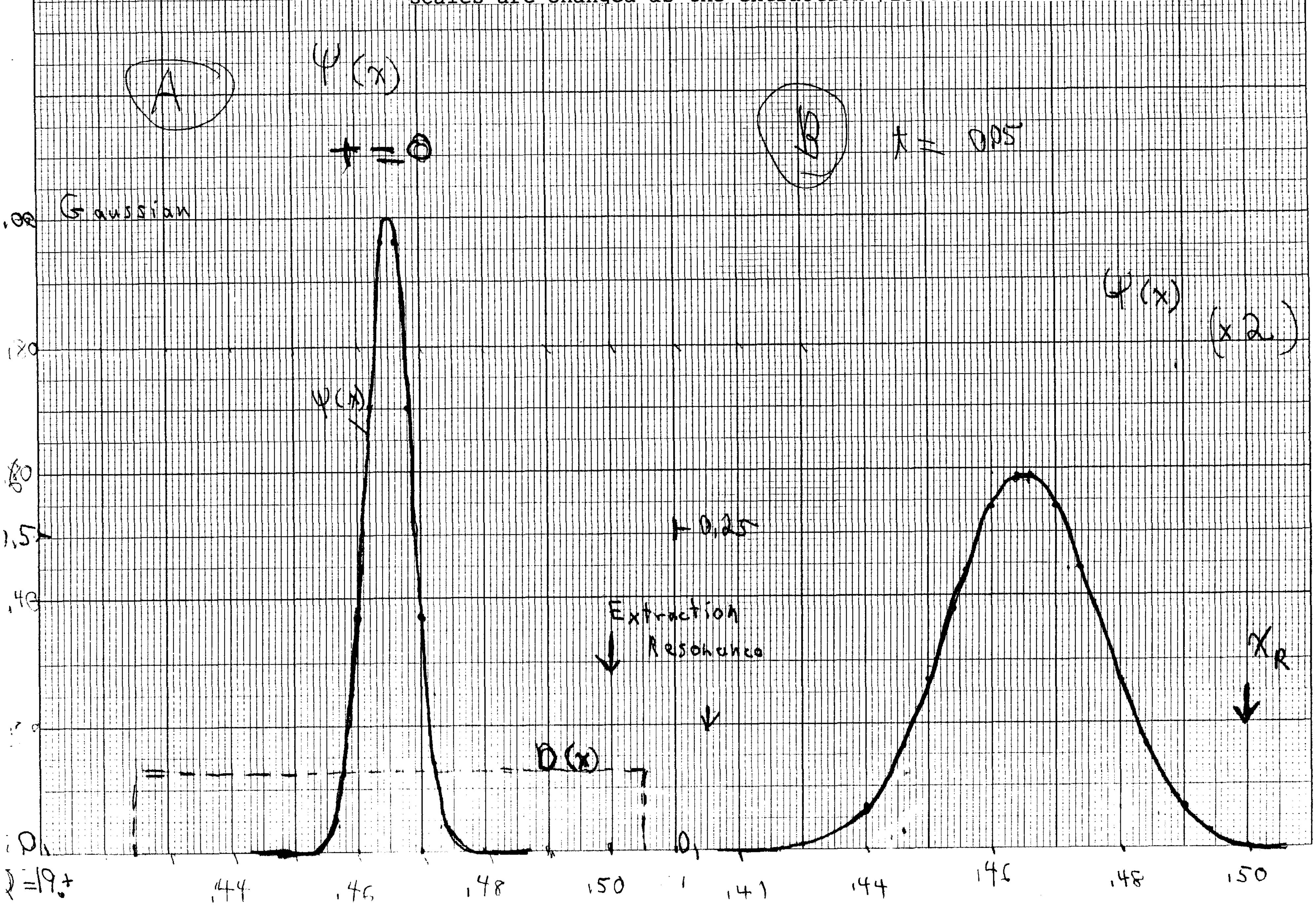

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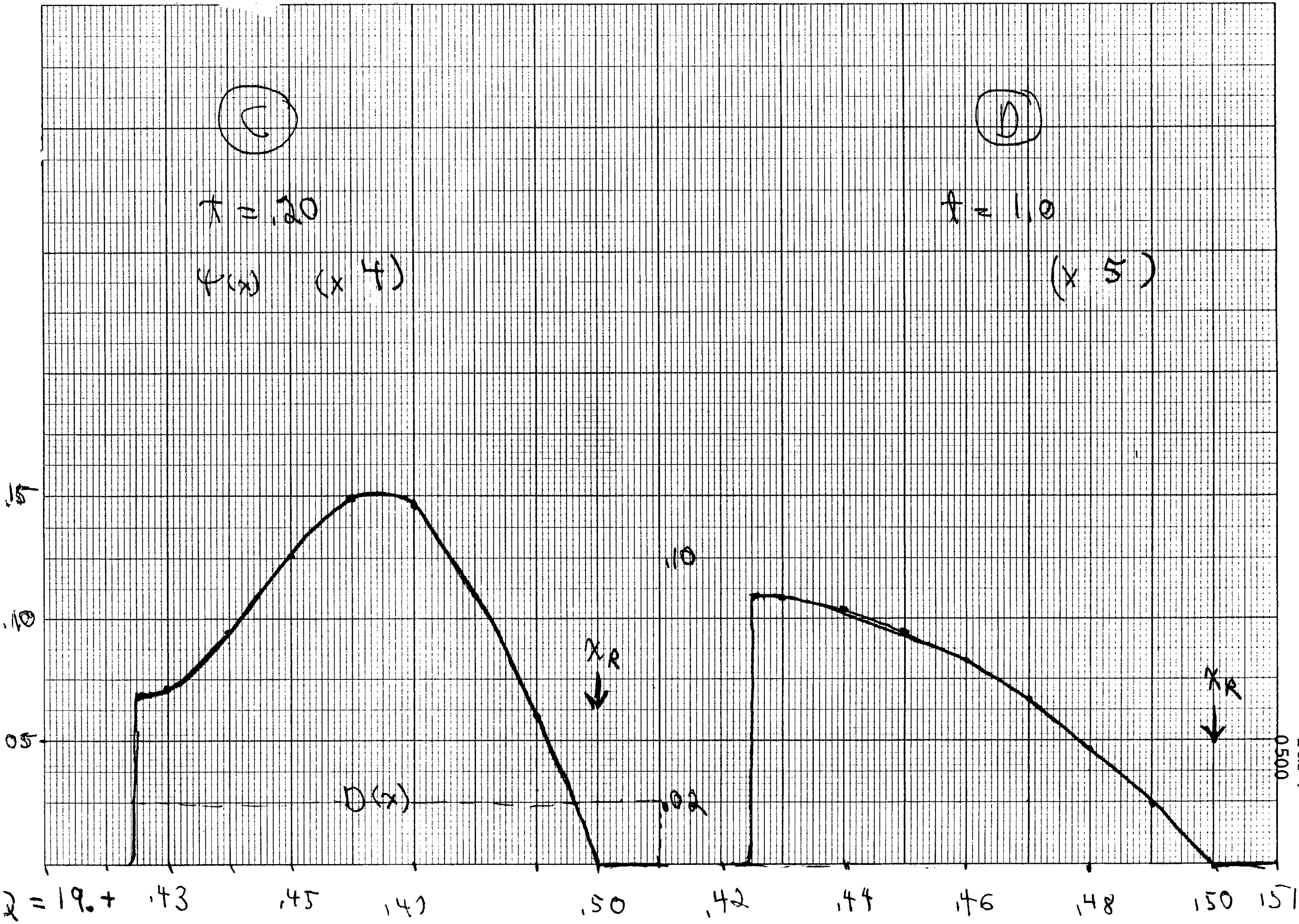
Figure 2 (Continued)



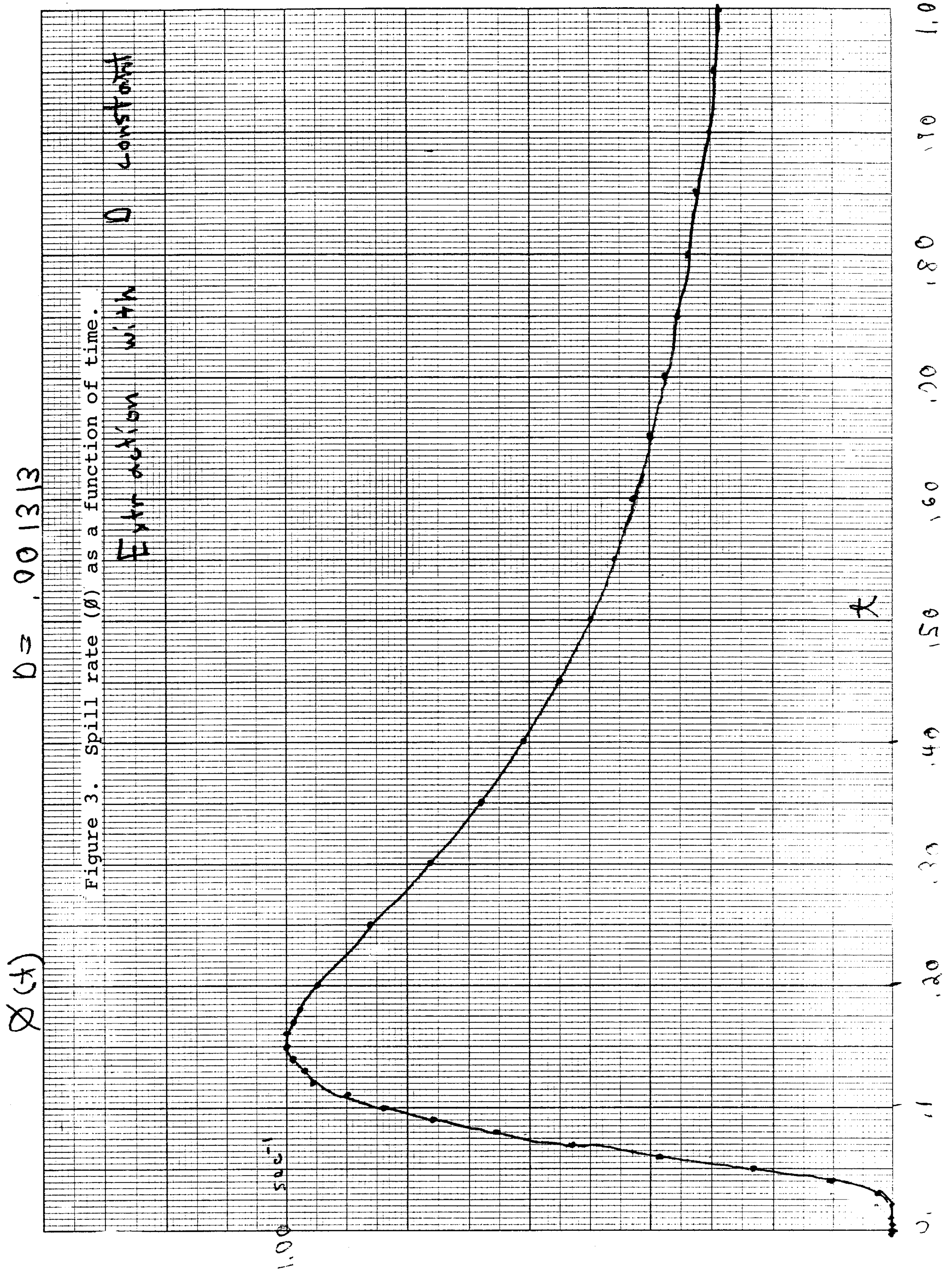


Figure 3. Spill rate (Ø) as a function of time.

Figure 4. Spill rate Ø and noise voltage V_n with the voltage increasing in order to maintain a constant extraction rate.