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MULTIPOLE FIELDS IN CIRCULAR CONCAVE POLES

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Introduction

A paper by G. Bosi[1] provides an exact calculation of a quadrupole field in a circular concave pole geometry. His method is readily generalized to the case of 2N-poles.

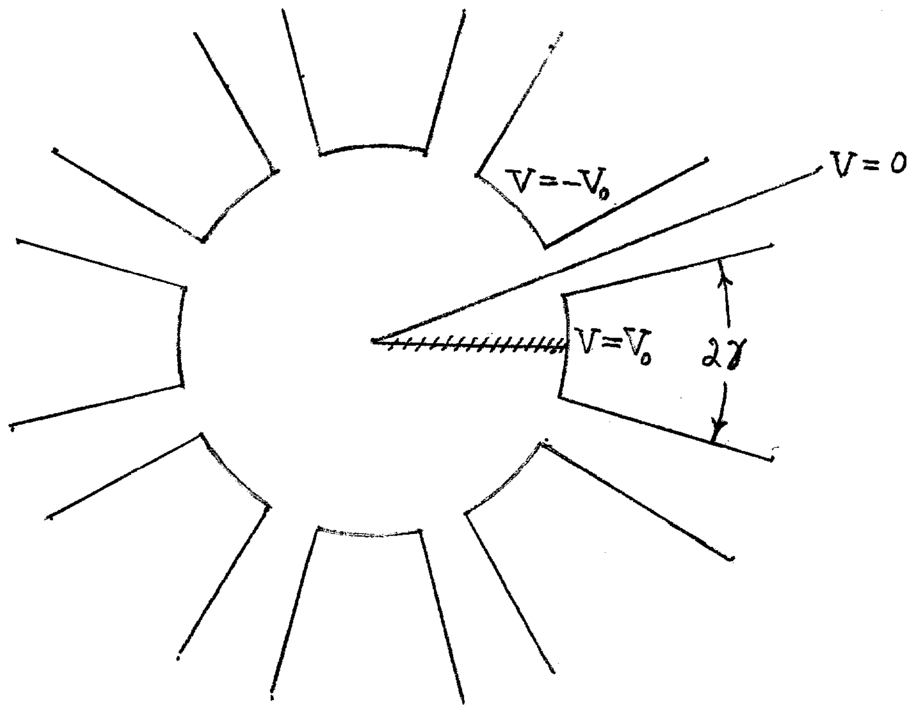

Fig. 1 z-Plane

The 2N-pole case is shown in Fig. 1 for the case of N = 4 (octupole). The complex z-plane is measured in units of the radius of the poles. Thus $|z| = 1$ is the circle on which the concave poles are drawn. The problem of finding the complex potential $W = U + iV$ where V is the potential function and U the stream function will be solved by complex variable transformations of the z-plane to the t-plane where the excitation is relatively clear.

Transformations

Transform the z-plane to the ω-plane using

$$\omega = \ell n z \ . \tag{1}$$

One sector of Fig. 1 will appear as

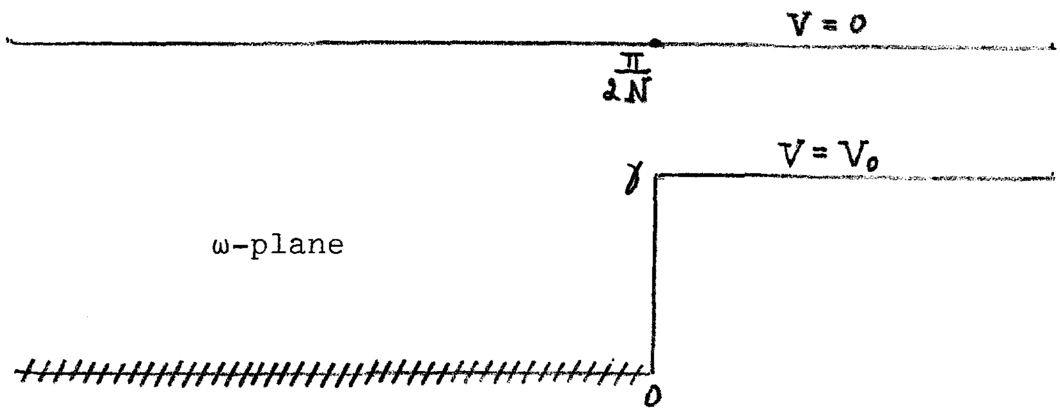

Fig. 2 Map of One Sector of Fig. 1 in ω-plane.

The ω-plane can be generated by a Schwarz-Christoffel transformation from the t-plane according to

$$\omega = C_1 \int t^{-1}(1-t)^{\frac{1}{2}} \left(1-\frac{t}{a^2}\right)^{-\frac{1}{2}} dt + C_2 \tag{2}$$

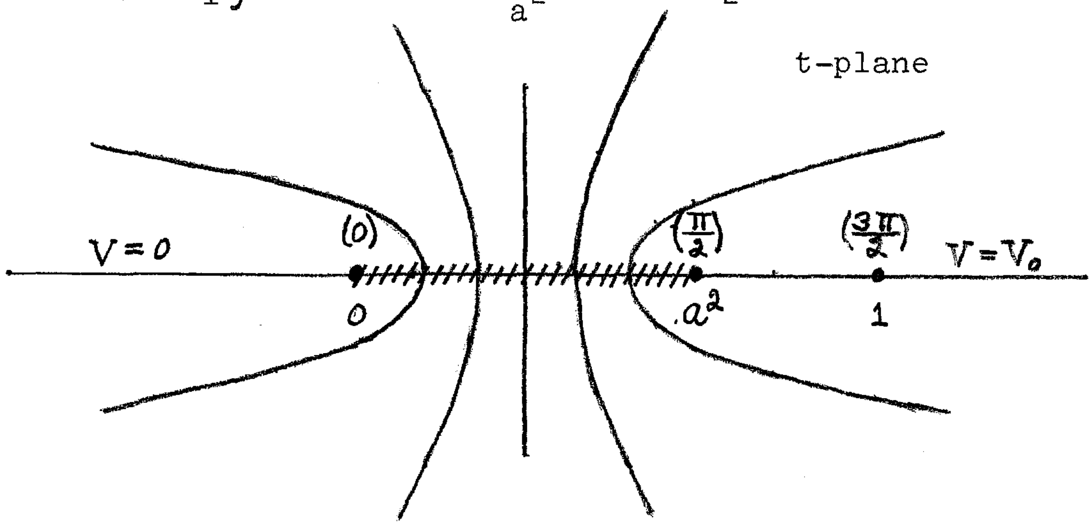

Fig. 3 Map of Fig. 2 Boundary to Real Axis of t-plane

Integration of Eq. (2) gives

$$\omega = C_1 \left\{ a\ \ell n\left(\frac{1+\xi}{1-\xi}\right) - \ell n\left(\frac{1+a\xi}{1-a\xi}\right) \right\} + C_2, \tag{3}$$

where

$$\xi = \frac{1}{a}(1-t)^{\frac{1}{2}}\left(1-\frac{t}{a^2}\right)^{-\frac{1}{2}} . \tag{4}$$

Putting $t = 1$ for which $\omega = i\gamma$ gives $C_2 = i\gamma$. Also putting t equal to a negative real number gives a real positive ξ such that $1 < \xi < \frac{1}{a}$. Equation (3) assuming that C_1 is real gives ω = Real Number + $ia\pi C_1 + i\gamma$. Hence from Fig. (1) $\frac{\pi}{2N} = a\pi C_1 + \gamma$. For $|t|$ near zero $\omega \to C_1\ \ell n\ t$ which gives $C_1 = \frac{1}{2N}$ since the imaginary part of ω for small negative t is $\pi/2N$. Thus

$$C_1 = \frac{1}{2N}, \qquad C_2 = i\gamma, \qquad a = 1 - \frac{2N}{\pi}\gamma . \tag{5}$$

Potential

Smythe[2] gives the solution of the potentials of a charged strip. For the origin shown in Fig. 3 and choosing the imaginary part of the complex potential W as the potential function one has from Sm 4.22 (3)

$$t - \frac{a^2}{2} = -\frac{a^2}{2} \sin\left[i\frac{\pi}{V_o}\left(W - i\frac{V_o}{2}\right)\right] \tag{6}$$

or

$$W = -\frac{iV_o}{\pi}\sin^{-1}\left(1-\frac{2t}{a^2}\right) + i\frac{V_o}{2} = i\frac{V_o}{\pi}\cos^{-1}\left(1-\frac{2t}{a^2}\right). \tag{7}$$

Using Pierce[3] (644-645) this may be written as

$$W = i\frac{2V_o}{\pi}\tan^{-1}\sqrt{\frac{\frac{t}{a^2}}{1-\frac{t}{a^2}}} = i\frac{2V_o}{\pi}\sin^{-1}\left(\frac{\sqrt{t}}{a}\right) . \tag{8}$$

Multipole Expansion

One desires to obtain a series of expansions of Eq. (8) in powers of z. First obtain an expansion of Eq. (8) in powers of t. From Gradsteyn and Ryzhik[4] (1.641) and (8.335) one has

$$W = i\frac{2V_o}{\pi} \cdot \frac{1}{\sqrt{\pi}} \sum_{n=o}^{\infty} \frac{a^{-(2n+1)}}{2n+1} \cdot \frac{\Gamma(n+\frac{1}{2})}{n!} t^{n+\frac{1}{2}} . \tag{9}$$

Next one needs t expanded in a power series in ω. First obtain ω as a power series in t. To this end expand dω/dt from Eq. (2). Abramowitz and Stegun[5] (3.6.8) give

$$(1-t)^{\frac{1}{2}} = \sum_{n=\infty}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} t^n \tag{10}$$

and

$$(1-\frac{t}{a^2})^{-\frac{1}{2}} = \sum_{m=o}^{\infty} (-1)^m a^{-2m} \binom{-\frac{1}{2}}{m} t^m . \tag{11}$$

Hence from Eq. (2) and Eq. (5) after juggling indices

$$\frac{d\omega}{dt} = \frac{1}{2N} \sum_{n=o}^{\infty} \sum_{m=o}^{n} (-1)^n a^{-2m} \binom{\frac{1}{2}}{n-m} \binom{-\frac{1}{2}}{m} t^{n-1} . \tag{12}$$

Abramowitz and Stegun[5] (6.1.21) and (6.1.17) also give

$$\binom{\frac{1}{2}}{n-m} = \frac{\Gamma(\frac{3}{2})}{\Gamma(n-m+1)\Gamma(\frac{3}{2}-n+m)} = \frac{-\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(n-m-\frac{1}{2})(-1)^{n-m}}{(n-m)!\pi} \tag{13}$$

and

$$\binom{-\frac{1}{2}}{m} = \frac{\Gamma(\frac{1}{2})}{\Gamma(m+1)\Gamma(\frac{1}{2}-m)} = \frac{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})(-1)^m}{m!\pi} \tag{14}$$

or

$$\frac{d\omega}{dt} = -\frac{1}{4\pi N} \sum_{n=o}^{\infty} \sum_{m=o}^{n} a^{-2m} \frac{\Gamma(m+\frac{1}{2})\Gamma(n-m-\frac{1}{2})}{m!(n-m)!} t^{n-1} . \tag{15}$$

Integration yields

$$\omega = \omega_1 + \frac{1}{2N} \ell n t - \frac{1}{4\pi N} \sum_{n=1}^{\infty} \sum_{m=o}^{n} a^{-2m} \frac{\Gamma(m+\frac{1}{2})\Gamma(n-m-\frac{1}{2})}{m!(n-m)!} \cdot \frac{t^n}{n} , \tag{16}$$

where the constant of integration is found by comparing Eq. (16) with Eq. (3) for small $|t|$. Thus

$$\omega_1 = \frac{1}{2N} \left\{ a \; \ell n\left(\frac{1+a}{1-a}\right) - \ell n\left(\frac{4a^2}{1-a^2}\right) \right\}. \tag{17}$$

Since one expects z^N to be the first term in the multipole expansion of the potential W, first expand z^N in a power series in t. For $|t|$ small from Eq. (1) and (17) one has

$$z^N = e^{N\omega_1} \sqrt{t} . \tag{18}$$

Therefore one expects

$$e^{N(\omega-\omega_1)} = \sum_{n=o}^{\infty} \alpha_n \, t^{n+\frac{1}{2}} . \tag{19}$$

To find α_n first define for convenience

$$Q_n = -\frac{1}{4\pi} \cdot \frac{1}{n} \sum_{m=o}^{n} a^{-2m} \frac{\Gamma(m+\frac{1}{2})\Gamma(n-m-\frac{1}{2})}{m!(n-m)!} . \tag{20}$$

From Eqs. (16) and (20)

$$e^{N(\omega-\omega_1)} = t^{\frac{1}{2}} e^{\sum_{n=1}^{\infty} Q_n t^n} . \tag{21}$$

Comparing this with Eq. (19) one has

$$\sum_{n=o}^{\infty} \alpha_n t^n = e^{\sum_{k=1}^{\infty} Q_k t^k} . \tag{22}$$

But the coefficients α_n are given by Cauchy's integral formula as

$$\alpha_n = \frac{1}{2\pi i} \oint_C \frac{e^{\sum_{k=1}^{\infty} Q_k t^k}}{t^{n+1}} dt, \tag{23}$$

where C is any simple contour surrounding the origin. For $t = o$ one sees from Eq. (22) that

$$\alpha_o = 1. \tag{24}$$

To find the other α_n consider $\sum_{j=1}^{n} j\alpha_{n-j} Q_j$. From Eq. (23) one has

$$\frac{1}{2\pi i} \oint_C \frac{1}{t_n} \sum_{j=1}^{n} jQ_j t^{j-1} e^{\sum_{k=1}^{\infty} Q_k t^k} dt = \sum_{j=I}^{n} j\alpha_{n-j} Q_j . \tag{25}$$

Again, for convenience let

$$P(t) = \sum_{j=1}^{\infty} Q_j t^j . \tag{26}$$

Then

$$P'(t) = \sum_{j=1}^{\infty} jQ_j t^{j-1} = \sum_{j=1}^{n} jQ_j t^{j-1} + R(t), \tag{27}$$

where the expansion of R(t) in powers of t to t^n as the lowest order terms. Hence Eq. (25) becomes

$$\sum_{j=1}^{n} j\alpha_{n-j}\, Q_j = \frac{1}{2\pi i} \oint_c \frac{1}{t^n}[P'(t)-R(t)]e^{P(t)}\, dt. \tag{28}$$

The term in R(t) does not yield a simple pole and the term in $P'(t)$ yields after integration by parts.

$$\sum_{j=1}^{n} j\alpha_{n-j}\, Q_j = \frac{1}{2\pi i} \left\{ \frac{e^{P(t)}}{t^n} \Bigg| + n \oint_c \frac{e^P}{t^{n+1}}\, dt \right\}. \tag{29}$$

The first term on the RHS is zero since the contour is closed. After using Eq. (23), one finally has

$$\alpha_n = \frac{1}{n} \sum_{j=1}^{n} j\, \alpha_{n-j}\, Q_j\ , \tag{30}$$

which is the recursion relation from which all the α_n may be found.

The next step requires the expansion of t in powers of z. From Eq. (18) one sees that the first term is

$$t^{\frac{1}{2}} = (ze^{-\omega_1})^N = e^{N(\omega-\omega_1)}. \tag{31}$$

However a 2N-pole expansion has terms in the sequence (Z^N, Z^{3N}, Z^{5N}, etc.). Hence, one expects

$$t^{\frac{1}{2}} = e^{N(\omega-\omega_1)} \sum_{k=o}^{\infty} A_k\, e^{2kN(\omega-\omega_1)}\ . \tag{32}$$

Since $t^{n+\frac{1}{2}}$ is desired raise Eq. (32) to the $n^{\underline{th}}$ power. Thus

$$t^{\frac{n}{2}} = e^{nN(\omega-\omega_1)} \sum_{k=o}^{\infty} A_k\, e^{2kN(\omega-\omega_1)^n}$$

$$= e^{nN(\omega-\omega_1)} \sum_{k_1=o}^{\infty} \sum_{k_2=o}^{\infty} \cdot\cdot \sum_{k_n=o}^{\infty} A_{k_1} A_{k_2} \cdot\cdot A_{k_n} e^{2N(\omega-\omega_1)(k_1+k_2+\cdot\cdot k_n)} . \tag{33}$$

Juggling the indices such that $k_1 + k_2 + \cdot\cdot k_n = m$, one may write Eq. (33) as

$$t^{\frac{n}{2}} = e^{nN(\omega-\omega_1)} \sum_{m=1}^{\infty} \beta_{m,n} e^{2mN(\omega-\omega_1)} . \tag{34}$$

Since Eq. (31) obtains for small $|t|$ it follows that

$$\beta_{o,1} = 1 . \tag{35}$$

Next, notice that the use of Eq. (34) in the identity, $t^{\frac{1}{2}(n+1)} = t^{\frac{n}{2}} t^{\frac{1}{2}}$, gives

$$\sum_{m=o}^{\infty} \beta_{m,n+1} e^{2mN(\omega-\omega_1)} = \sum_{k=o}^{\infty} \sum_{\ell=o}^{\infty} \beta_{k,n} \beta_{\ell,1} e^{2(k+\ell)N(\omega-\omega_1)} . \tag{36}$$

Let $k+\ell = m$ and equate coefficients of like powers

$$\beta_{m,n+1} = \sum_{k=o}^{\infty} \beta_{k,n} \beta_{m-k,1} . \tag{37}$$

Finally Eq. (34) gives for $t^{r+\frac{1}{2}}$

$$t^{r+\frac{1}{2}} = e^{(2r+1)N(\omega-\omega_1)} \sum_{m=o}^{\infty} \beta_{m,2r+1} e^{2mN(\omega-\omega_1)} . \tag{38}$$

Substituting this into Eq. (19) yields

$$e^{N(\omega-\omega_1)} = \sum_{r=o}^{\infty} \alpha_r \sum_{m=o}^{\infty} \beta_{m,2r+1} e^{[2(m+r)+1]N(\omega-\omega_1)} . \tag{39}$$

Rewriting the sum using m + r = n gives

$$e^{N(\omega-\omega_1)} = \sum_{n=o}^{\infty} \sum_{r=o}^{n} \alpha_r \, \beta_{n-r,\ 2r+1} \, e^{(2n+1)N(\omega-\omega_1)} . \tag{40}$$

Equating coefficients of like powers on each side gives

$$\alpha_o \, \beta_{o,1} = 1 \tag{41}$$

$$\sum_{r=o}^{n} \alpha_r \, \beta_{n-r,\ 2r+1} = 0 \quad (n = 1,2,3, \text{ etc.}). \tag{42}$$

Since $\alpha_o = 1$, Eq. (41) simply confirms Eq. (35). Equation (42) may be written as

$$\alpha_o \, \beta_{n,1} + \sum_{r=1}^{n} \alpha_r \, \beta_{n-r,\ 2r+1} = 0 \tag{43}$$

or

$$\beta_{n,1} = - \sum_{r=1}^{n} \alpha_r \, \beta_{n-r,\ 2r+1} \, . \tag{44}$$

Using Eq. (38) one may substitute into Eq. (9) to give

$$W = i\frac{2V_o}{\pi} \frac{1}{\sqrt{\pi}} \sum_{n=o}^{\infty} \frac{a^{-(2n+1)}}{2n+1} \cdot \frac{\Gamma(n+\frac{1}{2})}{n!} \cdot$$

$$\sum_{m=o}^{\infty} \beta_{m,2n+1} \, e^{[2(m+n)+1]N(\omega-\omega_1)} \tag{45}$$

Then, setting m + n = p and juggling the indices one has

$$W = i\frac{2V_o}{\pi} \frac{1}{\sqrt{\pi}} \sum_{p=o}^{\infty} \sum_{n=o}^{p} \frac{a^{-(2n+1)}}{2n+1} \cdot$$

$$\frac{\Gamma(n+\frac{1}{2})}{n!} \, \beta_{p-n,2n+1} \, e^{(2p+1)N(\omega-\omega_1)} \tag{46}$$

or

$$W = i\frac{2V_o}{\pi}\frac{1}{\sqrt{\pi}}\Bigg\{a^{-1}\,\Gamma(\tfrac{1}{2})\,\beta_{o,1}\,e^{N(\omega-\omega_1)}$$

$$+\left[a^{-1}\,\Gamma(\tfrac{1}{2})\,\beta_{1,1}+\frac{a^{-3}}{3}\,\Gamma(\tfrac{3}{2})\,\beta_{o,3}\right]e^{3N(\omega-\omega_1)}$$

$$+\left[a^{-1}\,\Gamma(\tfrac{1}{2})\,\beta_{2,1}+\frac{a^{-3}}{3}\,\Gamma(\tfrac{3}{2})\,\beta_{1,3}+\frac{a^{-5}}{5}\,\frac{\Gamma(\frac{5}{2})}{2!}\,\beta_{0,5}\right]e^{5N(\omega-\omega_1)}$$

$$+\,\cdots\cdots\Bigg\}. \qquad (47)$$

From Eq. (37)

$$\beta_{o2}=\beta_{o1}\beta_{o1} \qquad \beta_{o3}=\beta_{o2}\beta_{o1}$$

$$\beta_{o4}=\beta_{o3}\beta_{o1} \qquad \beta_{o5}=\beta_{o4}\beta_{o1}\ . \qquad (48)$$

Hence, since from Eq. (35) $\beta_{o1} = 1$, one has

$$\beta_{o1}=\beta_{o2}=\beta_{o3}=\beta_{o4}=\beta_{o5}=1\ . \qquad (49)$$

From Eq. (44)

$$\beta_{1,1}=-\alpha_1\beta_{o,3}=-\alpha_1 \qquad (50)$$

and

$$\beta_{2,1}=-(\alpha_1\beta_{1,3}+\alpha_2\beta_{o,5})\ . \qquad (51)$$

But, from Eq. (37)

$$\beta_{12}=\beta_{o1}\beta_{11}+\beta_{11}\beta_{o1}$$

$$\beta_{1,3}=\beta_{o,2}\beta_{1,1}+\beta_{1,2}\beta_{o1}\ . \qquad (52)$$

Thus

$$\beta_{1,3} = -3\alpha_1 \tag{53}$$

and

$$\beta_{2,1} = 3\alpha_1{}^2 - \alpha_2 \ . \tag{54}$$

From Eqs. (30) and (20)

$$\alpha_1 = Q_1 = \frac{1}{4} \cdot \frac{1-a^2}{a^2} \tag{55}$$

$$\alpha_2 = Q_2 + \frac{1}{2} Q_1{}^2 = \frac{1}{32a^4} (4-4a^2) \quad . \tag{56}$$

Inserting the various β-coefficients into Eq. (47) gives for the multipole expansion

$$W = i\frac{2V_o}{\pi} \frac{1}{\sqrt{\pi}} \left\{ \frac{1}{a}\left(ze^{-\omega_1}\right)^N + \frac{1}{12a^3} (3a^2-1)\left(ze^{-\omega_1}\right)^{3N} \right.$$

$$\left. + \frac{1}{80a^5} (1-10a^2 + 15a^4)\left(ze^{-\omega_1}\right)^{5N} + \cdots \right\} . \tag{57}$$

Note that for all multipoles it is possible to eliminate the next higher term in the series by choosing $a = 1/\sqrt{3}$ or from Eq. (5)

$$N\gamma = 38.04^\circ \ . \tag{58}$$

Acknowledgement

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