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MAGNETIC FIELD OF UNIFORM DENSITY CURRENT CONFINED WITHIN AN ELLIPTICAL BOUNDARY

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Purpose

To determine the magnetic field of a uniform density current confined within an elliptical boundary. From the results of a single ellipse, to determine by superposition the field from two similar ellipses separated and carrying oppositely directed currents. The field within the bore of two ellipses displaced along the direction of the major axis is

$$H_{bore} = 4\pi J d \cdot \frac{2b}{a+b} \qquad \text{(emu)}$$

where J is the current density, d the displacement of each ellipse from the center, a and b the semiaxes of the ellipse.

Magnetostatics in Two Dimensions Using Complex Variable Notation

Let

$$H = H_x + iH_y \tag{1}$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \tag{2}$$

$$\frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{3}$$

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The equations of magnetostatics may then be written conveniently as

$$\frac{\partial H^*}{\partial z^*} = -2\pi i J \tag{4}$$

where J is the usual current density.

Elliptical Wire

The consequences of requiring that H* be linear in z and z* where

$$z = x + iy \qquad\qquad z^* = x - iy \tag{5}$$

will be explored. Thus let

$$H^* = Az + Bz^*. \tag{6}$$

Application of Eq. (4) gives

$$B = -2\pi i J \tag{7}$$

so that Eq. (6) represents a possible field within a constant current density region.

The boundary of an ellipse of semimajor axis a and semiminor axis b may be written in complex notation as

$$\frac{\left[\frac{1}{2}(z+z^*)\right]^2}{a^2} + \frac{\left[\frac{1}{2i}(z-z^*)\right]^2}{b^2} = 1 \tag{8}$$

or

$$z^* = \frac{(a^2+b^2)z - 2ab\sqrt{z^2-c^2}}{c^2}, \tag{9}$$

where

$$c^2 = a^2 - b^2. \tag{10}$$

The magnetic field expressed by Eq. (6) becomes on the boundary of the ellipse

$$H^*_{\text{ellip}} = Az - \frac{2\pi iJ}{c^2}\left[(a^2+b^2)z - 2ab\sqrt{z^2-c^2}\right]. \tag{11}$$

Note that this may be written as

$$H^*_{\text{ellip}} = \frac{\left[A - \frac{2\pi iJ}{c^2}(a^2+b^2)\right]^2 z^2 + \frac{16\pi^2J^2a^2b^2}{c^4}(z^2-c^2)}{\left[A - \frac{2\pi iJ}{c^2}(a^2+b^2)\right]z - \frac{4\pi iJab}{c^2}\sqrt{z^2-c^2}}. \tag{12}$$

In order that H* may be continued into the region outside the ellipse and become asymptotically the field due to line current choose

$$A - \frac{2\pi iJ}{c^2}(a^2+b^2) = -\frac{4\pi iJab}{c^2}$$

or

$$\boxed{A = 2\pi iJ\,\frac{a-b}{a+b}} \tag{13}$$

This removes the z-dependence from the numerator in Eq. (12) or

$$H^*_{\text{ellip}} = -\frac{4\pi iJab}{z+\sqrt{z^2+c^2}}\,. \tag{14}$$

Outside the current carrying region Eq. (4) requires that H^* be independent of z^* or analytic in z. Hence, Eq. (14) by analytic continuation is the expression of the field outside the ellipse.

$$\boxed{H^*_{out} = -\frac{4\pi iJab}{z+\sqrt{z^2+c^2}} .} \tag{15}$$

Having selected the constant A one has for the field inside the current carrying ellipse

$$\boxed{H^*_{in} = 2\pi iJ \left[\frac{a-b}{a+b} z - z^*\right]} \tag{16}$$

Displaced Ellipses

A magnetic dipole field may be constructed by superposing two solutions. Equations (15) and (16) with z replaced by $z - d$ are added to the same equations with z replaced by $z + d$ and J reversed in sign. Thus (for real d)

$$\boxed{H^*_{bore} = 4\pi iJd \cdot \frac{2b}{a+b} ,} \tag{17}$$

$$H^*_{in(R)} = 2\pi iJ\left[\frac{a-b}{a+b}(z-d) - (z^*-d)\right] + \frac{4\pi iJab}{z+d+\sqrt{(z+d)^2+c^2}}, \quad (18)$$

$$H^*_{in(L)} = -2\pi iJ\left[\frac{a-b}{a+b}(z+d) - (z^*+d)\right] - \frac{4\pi iJab}{z-d+\sqrt{(z-d)^2+c^2}}, \quad (19)$$

$$H^*_{out} = -4\pi iJab\left[\frac{1}{z-d+\sqrt{(z-d)^2-c^2}} - \frac{1}{z+d+\sqrt{(z+d)^2-c^2}}\right]. \quad (20)$$

As a curiosity one may observe that within the bore the field is uniform even for ellipses of different area so long as the eccentricity of each ellipse is the same. Furthermore, if d is made complex

$$H^*_{bore} = 4\pi iJ\,\frac{b(d+d^*) - a(d-d^*)}{a+b} \quad (21)$$

indicating that the field within the bore is uniform for any displacement of the ellipses.