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HANDY FORMULAS FOR DIPOLE COILS WITH CIRCULAR IRON SHIELDS

L. C. Teng

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To obtain easy-to-evaluate analytical solutions we shall approximate the "cos θ-coil" by an annular current (Fig. 1) with the cos θ-distribution, i.e. the current density (current per unit area) within the annular region between a_1 and a_2 is

$$I = I_0 \cos\theta \tag{1}$$

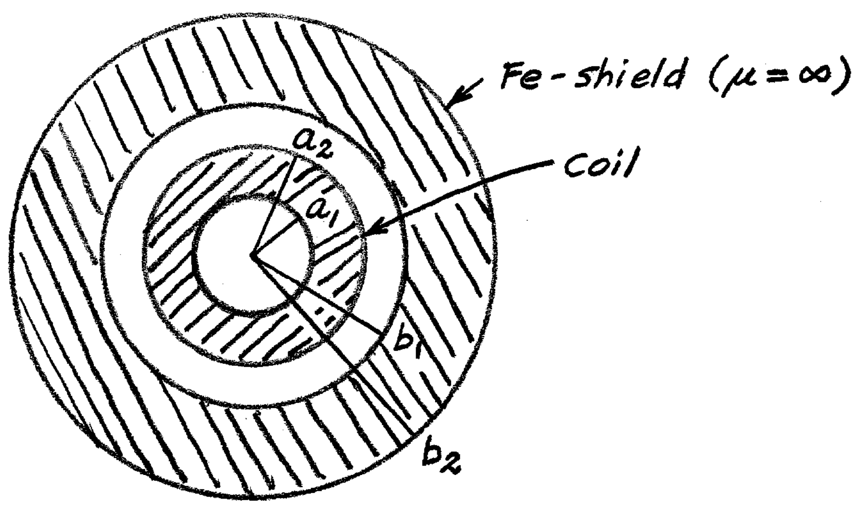

Figure 1

The concentric circular Fe-shield is assumed to have infinite permeability. The field in such a magnet was given by

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J. P. Blewett ("Iron Shielding for Air Core Magnets," pp. 1042-1051, Proc. of the 1968 Summer Study on Superconducting Devices and Accelerators, Brookhaven Nat. Lab.).

For $r<a_1$

$$\left\{\begin{aligned} B_r &= 2\pi I_o\left[(a_2-a_1) + \frac{1}{3b_1^2}\left(a_2^3-a_1^3\right)\right]\sin\theta \\ B_\theta &= 2\pi I_o\left[(a_2-a_1) + \frac{1}{3b_1^2}\left(a_2^3-a_1^3\right)\right]\cos\theta \end{aligned}\right. \tag{1}$$

For $a_1<r<a_2$

$$\left\{\begin{aligned} B_r &= 2\pi I_o\left[a_2 + \frac{1}{3b_1^2}\left(a_2^3-a_1^3\right) - \frac{a_1^3}{3r^2} - \frac{2}{3}r\right]\sin\theta \\ B_\theta &= 2\pi I_o\left[a_2 + \frac{1}{3b_1^2}\left(a_2^3-a_1^3\right) + \frac{a_1^3}{3r^2} - \frac{4}{3}r\right]\cos\theta \end{aligned}\right. \tag{2}$$

$$A_z = -2\pi I_o\left[a_2 + \frac{1}{3b_1^2}\left(a_2^3-a_1^3\right) - \frac{a_1^3}{3r^2} - \frac{2}{3}r\right] r\cos\theta \tag{3}$$

For $a_2<r$

$$\left\{\begin{aligned} B_r &= \frac{2\pi}{3} I_o\left(a_2^3-a_1^3\right)\left(\frac{1}{b_1^2} + \frac{1}{r^2}\right)\sin\theta \\ B_\theta &= \frac{2\pi}{3} I_o\left(a_2^3-a_1^3\right)\left(\frac{1}{b_1^2} - \frac{1}{r^2}\right)\cos\theta \end{aligned}\right. \tag{4}$$

where all quantities are in emu (length in cm, current in abAmp = 10A, field in Gauss, energy in erg, and force in dyne). This approximation should give parameters accurate to 5% and useful for first-order comparisons of gross design features.

A. Coil and Shield Parameters

The maximum field $\hat{B}$ on the Fe-shield is at $r = b_1$, $\theta = \frac{\pi}{2}$, and is given by Eq. (4) to be

$$\hat{B} = 4\pi I_o \frac{a_2^3 - a_1^3}{3b_1^2} . \tag{5}$$

In order that the Fe-shield does not saturate thereby spoiling the uniformity of the field inside a_1, we must have $\hat{B} \leq 20$ kG. The uniform field B_o inside the bore $(r<a_1)$ is given by Eq. (1) to be

$$B_o = 2\pi I_o \left[(a_2 - a_1) + \frac{1}{3b_1^2} \left(a_2^3 - a_1^3 \right) \right]$$

$$= 2\pi I_o (a_2 - a_1) + \frac{\hat{B}}{2}$$

This gives

$$\boxed{2\pi I_o (a_2 - a_1) = B_o - \frac{\hat{B}}{2}} \tag{6}$$

which shows that the contribution of the Fe-shield to the bore-field is $\frac{\hat{B}}{2}$ ($\leq$10 kG) and gives the thickness $a_2 - a_1$ of the annular coil for a given assumed current density I_o.

The inner radius b_1 of the Fe-shield is, then, given directly by Eq. (5) and (6) to be

$$\boxed{b_1^2 = \frac{4\pi}{3\hat{B}} I_o \left(a_2^3 - a_1^3\right) = \frac{1}{3} \frac{2B_o - \hat{B}}{\hat{B}} \left(a_1^2 + a_1 a_2 + a_2^2\right)} \quad (7)$$

The maximum flux $\hat{\Phi}$ to be returned by the Fe-shield is given by Eq. (4) to be

$$\hat{\Phi} = \int_o^{\pi/2} b_1 B_r (r=b_1)\, d\theta = b_1 \hat{B}.$$

If we demand that the maximum flux density in the Fe-shield shall not exceed, say, 0.8 $\hat{B}$ we get for the thickness $b_2 - b_1$ of the shield

$$0.8\,\hat{B}\,(b_2 - b_1) = \hat{B} b_1$$

or

$$\boxed{b_2 = 2.25\, b_1} \quad (8)$$

The total current I_t in the coil is

$$\boxed{I_t = 2I_o \int_{a_1}^{a_2} r\, dr \int_o^{\pi/2} \cos\theta\, d\theta = I_o \left(a_2^2 - a_1^2\right)} \quad (9)$$

B. Stored Energy

The stored energy U is given by

$$U = \frac{1}{2} \int_{a_1}^{a_2} r\, dr \int_o^{2\pi} d\theta\, I_z A_z$$

with $I_z = -I = -I_o \cos\theta$ and A_z given by Eq. (3). This gives

$$\boxed{U = \pi^2 I_o^2 \left\{\frac{1}{3}\left(a_2^3-a_1^3\right)\left[a_2 + \frac{1}{3b_1^2}\left(a_2^3-a_1^3\right)\right] - \frac{1}{3}\, a_1^3(a_2-a_1) - \frac{1}{6}\left(a_2^4-a_1^4\right)\right\}} \tag{10}$$

C. Stress in Coil

The stress (force per unit volume) in the coil is given by

$$S_r = I\, B_\theta \qquad\qquad S_\theta = -I\, B_r .$$

It is interesting to integrate over r and leave the θ dependence explicit. This gives

$$\left\{\begin{aligned} f_r &= \int d\theta \int_{a_1}^{a_2} r\,dr\, IB_\theta = \int g_r(1+\cos 2\theta)\, d\theta \\ f_\theta &= \int d\theta \int_{a_1}^{a_2} r\,dr(-IB_r) = \int g_\theta \sin 2\theta\, d\theta \end{aligned}\right. \tag{11}$$

with

$$\left\{\begin{aligned} g_r &= \pi I_o^2 \left\{\frac{1}{2}\left(a_2^2-a_1^2\right)\left[a_2 + \frac{1}{3b_1^2}\left(a_2^3-a_1^3\right)\right] + \frac{1}{3}\, a_1^3\, \ln\frac{a_2}{a_1} - \frac{4}{9}\left(a_2^3-a_1^3\right)\right\} \\ g_\theta &= -\pi I_o^2 \left\{\frac{1}{2}\left(a_2^2-a_1^2\right)\left[a_2 + \frac{1}{3b_1^2}\left(a_2^3-a_1^3\right)\right] - \frac{1}{3}\, a_1^3\, \ln\frac{a_2}{a_1} - \frac{2}{9}\left(a_2^3-a_1^3\right)\right\}. \end{aligned}\right. \tag{12}$$

D. Force on Displaced Coil

When the coil is exactly centered it is in unstable equilibrium. When the coil is displaced from the center the magnetic force tends to increase the displacement.

When the coil is displaced in the x-direction by ξ (first-order infinitesimal) currents are generated at $r = a_1$ and $r = a_2$ with θ-linear densities

$$\begin{cases} \dfrac{di}{a_1 d\theta} = -I_o \xi \cos^2\theta = -\dfrac{I_o \xi}{2} - \dfrac{I_o \xi}{2} \cos 2\theta \\[2ex] \dfrac{di}{a_2 d\theta} = I_o \xi \cos^2\theta = \dfrac{I_o \xi}{2} + \dfrac{I_o \xi}{2} \cos 2\theta \end{cases}$$

each consisting of a monopole and a quadrupole current shell. The field is, then, a superposition of the original dipole field and the field produced by these current shells, which is also given in analytical form by J. P. Blewett (Ibid). The total force on the coil is the sum of the force by the dipole field on the shell-currents and that by the shell-current field on the dipole current. (The force by the field on its own generating current vanishes because of symmetry.) The calculation is lengthy, but the result is simple. The x and y forces are

$$\boxed{\begin{aligned} F_x/\xi &= \frac{1}{16} \hat{B}\ (4B_o + \hat{B}) \\ F_y/\xi &= 0 \end{aligned}} \qquad (13)$$

Similarly, when the coil is displaced in the y-direction by η the current shells are

$$\begin{cases} \dfrac{di}{a_1 d\theta} = -I_o\eta \sin\theta \cos\theta = -\dfrac{I_o\eta}{2} \sin 2\theta \\ \\ \dfrac{di}{a_2 d\theta} = I_o\eta \sin\theta \cos\theta = \dfrac{I_o\eta}{2} \sin 2\theta . \end{cases}$$

The forces on the coil are

$$\boxed{\begin{cases} F_x/\xi = 0 \\ \\ F_y/\xi = \dfrac{3}{16} \hat{B}^2 \end{cases}} \qquad (14)$$

E. Examples

Given below are the parameters for three magnets with bore diameters 2-1/2 inch (a_1 = 3.175 cm), 2-inch (a_1 = 2.54 cm), and 1-1/2 inch (a_1 = 1.905 cm), each having a coil with current density of either 20 kA/cm^2 (I_o = 2000 abAmp/cm^2) or 30 kA/cm^2 (I_o = 3000 abAmp/cm^2). The central dipole field is taken to be 45 kG ($=B_o$) and the maximum field on the Fe-shield is taken to be 20 kG ($=\hat{B}$).

	I_o = 20 kA/cm^2			I_o = 30 kA/cm^2		
a_1(cm)	3.175	2.540	1.905	3.175	2.540	1.905
a_2(cm)	5.960	5.325	4.690	5.032	4.397	3.762
b_1(cm)	8.676	7.509	6.350	7.742	6.566	5.395
b_2(cm)	19.52	16.90	14.29	17.42	14.77	12.14
I_t(kA)	509	438	367	457	386	316
U(J/cm)	717	522	360	597	419	274
g_r(kg/cm rad)	164	144	124	141	122	103
$-g_\theta$(kg/cm rad)	504	424	345	469	389	309
F_x/ξ(kg/cm^2)	255	255	255	255	255	255
F_y/η(kg/cm^2)	76.5	76.5	76.5	76.5	76.5	76.5