national accelerator laboratory

TM-342
0402
0434.060

A FOOTNOTE ON GAS SCATTERING IN THE MAIN RING

J. A. MacLachlan

January 28, 1972

Calculations of the beam lifetime in the main ring as a function of pressure and aperture are in rough agreement with results of the accelerator experiment of 12/24.[1] It is not immediately obvious, however, what degree of precision should be expected from the one-dimensional diffusion model which was employed.[2] In the examination of the assumptions of the model a somewhat independent approach was obtained by calculating the scattering in a straightforward Monte Carlo. The MONACO[3] subroutine was modified to treat the multiple coulomb scattering (MCS) in the presence of the focusing accelerator lattice.

The use of a gaussian form

$$\frac{d\sigma}{d\Omega} \propto e^{-\theta^2/\langle\theta^2\rangle}$$

for the differential MCS cross-section is a great convenience in computation but is somewhat crude for thin scatterers. The amount of scattering required to remove a beam particle from the aperture is

$$\theta_m \approx w/\overline{\beta}$$

where $\overline{\beta}$ is the average value of the Courant-Snyder[4] β-function in the plane where the half aperture is w. For the main ring ($\overline{\beta}$ ~ 50) only about .2 mrad of scattering is required. Since

$$\langle\theta^2\rangle^{1/2} = \frac{.015}{\beta p}\sqrt{L/L_o}\ (1+\varepsilon)$$

one can see that for momentum p = 7.2 GeV/c and $\beta \approx 1$ a scatter of length L equal to one-hundredth of the radiation length L_o is sufficient to produce .2 mrad of scattering. Therefore, L ~ .4 gm/cm^2 and the gaussian form is still good.[5] The correction for thickness of the scatterer is approximately embodied in the choice

$$\varepsilon = .14 + .06 \log\ (L/L_o) = -.14.$$

This correction was not employed in the diffusion calculation and therefore was also omitted in the Monte Carlo calculation so that other errors, if any, might be discovered by comparison. It will be shown below how this error affects the result.

The effect of the focusing forces on the MCS should depend on average lattice parameters only because any significant scattering occurs over many oscillation and structure periods. The Courant-Snyder[4] invariant

$$W = \gamma\ x^2 + 2\alpha x x' + \beta x'^2,$$

where α, β and γ are the betatron functions and x is the transverse displacement of a particle from the equilibrium orbit, changes in first-order by the amount

$$\delta W = 2(\alpha_s x_s + \beta_s x_s')\ \delta x_s'$$

when the particle undergoes a scattering $\Delta x'_s$ at the point s. The amplitude A of the betatron oscillation is

$$A = \sqrt{W\beta}$$

so that

$$x = \sqrt{W\beta}\,\sin\left(\int \frac{dz}{\beta} + \phi\right).$$

Substituting this form for x into the expression for δW one has

$$\delta W = 2\sqrt{W\beta_s}\,\cos\left(\int_s \frac{dz}{\beta} + \phi\right)\delta x'_s\ .$$

The individual scatterings have no coherence with the betatron oscillation so the net of many scatterings is a gaussian distribution with mean W and standard deviation

$$\Delta W = \left[\sum_s (\delta w)^2\right]^{1/2} = \sqrt{2W\bar{\beta}}\left\langle \Delta x'^{\,2}_{MCS}\right\rangle^{1/2}$$

where $\Delta x'_{MCS}$ is the projected MCS scattering angle. Consequently,

$$A_{max} = \sqrt{(W+\Delta W)\beta_{max}} = \sqrt{W\beta_{max}} + \sqrt{\beta_{max}\bar{\beta}/2}\ \Delta x'_{MCS}$$

where $\Delta x'_{MCS}$ has been selected from the gaussian distribution for MCS. One sees that entire effect of the focusing on the amplitude is expressed in the numerical factor

$$\sqrt{\bar{\beta}\beta_{max}/2} = 55$$

applied to rms scattering angle. This expression is the one used at each step in the Monte Carlo calculation of the amplitude growth.

The treatment of the one-dimensional diffusion model[6] is specialized here to protons in a coasting beam. The notation parallels that of Ref. 6. Consider a distribution of particles $Y(y,\tau)$ in the variables

$$y = \frac{A^2}{w^2} = \frac{\beta_{max}W}{w^2}$$

$$\tau = t\,\frac{d}{dt}\langle y\rangle$$

where W_{max} is the value of the invarient W when the beam fills aperture of half width w and the coefficient of t is assumed to be a constant equal to the diffusion constant divided by W_{max}. The diffusion equation is then

$$\frac{\partial Y}{\partial \tau} = z\,\frac{\partial^2 Y}{\partial y^2} + \frac{\partial Y}{\partial y}.$$

The initial and boundary conditions are

$$Y(y,0) = Y_o(y)$$

$$Y(1,\tau) = 0.$$

By separation of variables

$$Y(y,\tau) = \sum_i c_i J_o(\lambda_i\sqrt{y})\, e^{-\lambda_i^2\tau/4}$$

where

$$c_i = \frac{\int_o^1 Y_o(y) J_o(\lambda_i\sqrt{y})\,dy}{J_1^2(\lambda_i)},$$

the J's are ordinary Bessel functions, and λ_i are the zeros of J_o. The number of particles remaining in the beam at a given τ is

$$N(\tau) = \int_o^1 Y(y,\tau) = 2\sum_i \frac{e^{-\lambda_i^2\tau/4}}{\lambda_i J_1(\lambda_i)} \int_o^1 Y_o(y) J_o(\lambda_i\sqrt{y})\,dy.$$

The results for some interesting cases follow:[7]

1) $$Y_o(y) = \delta(y)$$

$$N(\tau) = 2 \sum_i e^{-\lambda_i^2 \tau/4} \Big/ \lambda_i J_i(\lambda_i)$$

2) $$Y_o(y) = 1 \qquad 0 \leq y < 1$$

$$N(\tau) = 4 \sum_i e^{-\lambda_i^2 \tau/4} \Big/ \lambda_i^2$$

3) $$Y_o(y) = \lambda_1 J_o(\lambda_1 \sqrt{y} \Big/ 2J_1(\lambda_1)$$

$$N(\tau) = e^{-\lambda_1^2 \tau/4}$$

All forms are seen to give an exponential decay with characteristic time given by λ_1 after sufficient time. The eigensolution in (3) above is a reasonably physical distribution which decays exponentially from $\tau = 0$.

The results of the Monte Carlo are in excellent agreement with the three forms above. The only point needing verification, therefore, is the numerical value for the diffusion constant

$$\frac{d\langle y\rangle}{dt} = \frac{1}{W_{max}} \frac{d}{dt} \langle W\rangle = \frac{D}{W_{max}} .$$

where D is the change in W caused by the scatterer encountered in one second. Using the previous expression for δW with the addition of the second-order term one has

$$D = \sum_s \delta W = \sum_s \left[2(\alpha_s\beta_s x_s + \beta_s x_s')\delta x_s' + \beta_s(\delta x_s')^2\right].$$

In the average over many scatterings (2,6) the second-order term accumulates to give

$$D = \overline{\beta}\left\langle \Delta x'_{MCS}{}^2 \right\rangle$$

while the first terms contain fluctuation effects. From this expression we have

$$\tau = t\,\frac{D}{W_{max}} = \frac{t\beta_{max}D}{w^2} = t\,\frac{\beta_{max}\overline{\beta}\left\langle \Delta x'_{MCS}\right\rangle^2}{w^2}.$$

Thus it can be seen that the failure to include the thin scattering correction $1 + \varepsilon$ in the expression for the rms scattering angle leads to a relative error of 2ε or nearly 30% underestimate for the decay time.

Complete numerical agreement has been obtained between the Monte Carlo and the diffusion calculation by adjusting the ΔW used in the Monte Carlo by about $\sqrt{2}$. The averaging performed to get D for the diffusion calculation and ΔW for the Monte Carlo are considerably different and apparently a discrepancy does exist. The arguments for ΔW are not very rigorous since one assumes in the sum over scatterings that "single" scatterings can be selected from the gaussian which represents the multiple scattering results. A derivation of ΔW in which one integrates the first power of the angle over the Rutherford scattering distribution shows that the numerical factor is not really $\sqrt{2}$ but depends in detail on things like the upper and lower cutoffs used. The discrepancy is therefore attributed to the rough argument used in the Monte Carlo.

One point worth mentioning which was quite obvious looking at the Monte Carlo distributions is that the one-dimensional results will overestimate the lifetime appreciably if the aperture in the other plane is within a factor of three of that in the plane considered. One can expect the decay in the two-dimensional case to go with half life

$$t_{1/2} \propto \frac{1}{1/w_x^2 + 1/w_y^2}$$

where w_x and w_y are the apertures in the perpendicular planes. Thus for the main ring the predicted beam life is reduced to nearly half that obtained from the one-dimensional formula.

References

1. Experimenters E. Malamud, A. Maschke, and S. Mori, "Accelerator Experiment--Pressure Dependence of the Decay of the Main-Ring Coasting Beam," L.C. Teng (1/11/72).
2. J.M. Greenberg and T.H. Berlin, Rev. Sci. Instr. 22, 293 (1951).
3. J. MacLachlan, T. Borak and M. Awschalom, TM-244.
4. E.D. Courant and H.S. Snyder, Ann. of Phys. 3, 1 (1958).
5. H. Øverås, CERN 60-18 Synchro-Cyclotron Div.
6. Kolmenky and Lebedev, Theory of Cyclic Accelerators, North Holland (1966).
7. L.C. Teng, private communication.