national accelerator laboratory

TM-179
2040

NONLINEAR TRANSFORMATIONS IN ACTION-ANGLE VARIABLES

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June 13, 1969

1. INTRODUCTION

The purpose of this note is to record the effects of a sequence of Moser transformations used in analysis of single-particle nonlinear resonances.[1] The calculations are carried out here in action-angle variables and the work is restricted to one dimension. (For resonance calculations in complex variables and in two dimensions, see Ref. 2.) Higher-order terms are calculated, both terms in which the $(n+1)^{st}$ order gives corrections to n^{th}-order resonances and terms in which a nonlinear term in the n^{th} order gives a new resonance in the $(n+1)^{st}$ order; this last can be referred to as a "higher-order resonance" and has apparently not been previously examined.

I began this work at LRL when P. F. Meads found higher-order resonances in digital computation for the Omnitron design work. I have also carried out some computational work and hope to be able to discuss the agreement between computational and analytical work in a later report. The cases of impulsive (δ-function) nonlinear forces treated here are aimed at this numerical work. It should also be remarked that the treatment of these δ-function terms is the main difference between this work and earlier work of Laslett in action-angle variables.[3]

The equation of motion treated in this report is

$$q'' + K(s)q + M(s)q^2 + N(s)\,q^3 = 0. \qquad (1.1)$$

Here s, the independent variable, is the arc length along the equilibrium orbit. Derivatives with respect to the independent variable are denoted by primes. The coefficients K, M, and N are periodic functions of s with period L. The dependent variable q is the transverse displacement from the equilibrium orbit. The equation of motion (1.1) is derivable from the Hamiltonian

$$H = H(p,q) = \frac{1}{2}p^2 + \frac{1}{2}Kq^2 + \frac{1}{3}Mq^3 + \frac{1}{4}Nq^4. \qquad (1.2)$$

The momentum p canonically conjugate to q is thus q', the tangent of the angle of the particle orbit with the equilibrium orbit.

The "order" of a term will be used throughout this report to mean the power of q or p in the Hamiltonian (1.2).

The Moser method is a sequence of canonical transformations, each of which removes the dependence on the independent variable s from one order to higher orders. The result is a Hamiltonian independent of s and therefore invariant. At any step in the sequence of transformations, a resonance term may be recognized because of its small denominator. These resonance terms are treated by a special canonical transformation to remove their dependence on s. In the one-dimensional case discussed in this report, the method will treat only one resonance and cannot give information on the combined effects of two or more different resonances.

2. TRANSFORMATION OF THE LINEARIZED MOTION

The dependence on s of the linear-motion terms (terms of second order in the Hamiltonian) can be removed by transformation to action-angle variables. In the linearized motion $[M(s) = N(s) = 0]$, the quadratic form

$$J = \frac{1}{2}(\gamma q^2 + 2\alpha q p + \beta p^2) = \frac{1}{2\beta}[q^2 + (\alpha q + \beta p)^2], \tag{2.1}$$

is an invariant; that is, $J' = 0$. Here α, β, and γ are the Courant-Snyder parameters of the linear-motion transformation matrix.[4] We take J as a new canonical momentum. The canonically conjugate coordinate is ψ, the polar angle in the (q,p) phase space. The transformation from (q,p) to (ψ_o, J_o) is

$$\begin{cases} q = -\sqrt{2\beta J_o}\,\cos\psi_o, \\ p = \sqrt{2J_o/\beta}\,(\sin\psi_o + \alpha\cos\psi_o). \end{cases} \tag{2.2}$$

The inverse transformation is

$$\begin{cases} J_o = \frac{1}{2\beta}[q^2 + (\alpha q + \beta p)^2], \\ \tan\psi_o = \alpha + \beta p/q. \end{cases} \tag{2.3}$$

This canonical transformation is derivable from a generating function

$$G = G(q, \psi_o, s) = -\frac{1}{2}\frac{q^2}{\beta}(\tan\psi_o + \alpha) \tag{2.4}$$

by the rules

$$\begin{cases} P = \dfrac{\partial G}{\partial q} \\ J_o = -\dfrac{\partial G}{\partial \psi_o} \\ \hat{H} = H + \dfrac{\partial G}{\partial s}, \end{cases} \tag{2.5}$$

which reproduce the transformation equations. The new Hamiltonian is

$$\hat{H} = \frac{J_o}{\beta} - \frac{1}{3}(2\beta)^{3/2} M(s) J_o^{3/2} \cos^3\psi_o + \beta^2 N(s) J_o^2 \cos^4\psi_o. \tag{2.6}$$

In the linear approximation, Hamilton's equations are

$$\begin{cases} J_o' = \dfrac{\partial \hat{H}}{\partial \psi_o} = 0 \\ \psi_o' = \dfrac{\partial \hat{H}}{\partial J_o} = \dfrac{1}{\beta}. \end{cases} \tag{2.7}$$

The first equation reproduces the constancy of J_o in the linear approximation; the second shows a method for eliminating the s-dependence of the linear term J_o/β in $\hat{H}$. The phase change of the linear oscillation in one revolution of arc length C = NL is, from this equation,

$$\psi_o(C) - \psi_o(0) = \int_0^C \frac{ds}{\beta} = 2\pi\nu. \tag{2.8}$$

The independent variable

$$\phi = \int_0^s \frac{ds}{\nu\beta} , \tag{2.9}$$

advances by 2π in one revolution. The transformation from s to ϕ is canonical; the new Hamiltonian is

$$H_o = \hat{H}\frac{ds}{d\phi} = \nu J_o + A_3(\phi) J_o^{3/2} \cos^3 \psi_o + A_4(\phi) J_o^2 \cos^4 \psi_o , \tag{2.10}$$

where

$$\left\{ \begin{aligned} A_3(\phi) &= -\frac{2}{3}^{3/2} \nu \beta^{5/2} M(\phi), \\ A_4(\phi) &= \nu \beta^3 N(\phi). \end{aligned} \right. \tag{2.11}$$

The linear-motion part of H_o is νJ_o, independent of ϕ. In the linear approximation, trajectories in phase space are now circles. Curves of constant H_o are also curves of constant J_o. The total Hamiltonian is a periodic function of ϕ. The n^{th} order term, which came from the term proportional to q^n in the original Hamiltonian, now has the form

$$A_n(\phi) J_o^{n/2} \cos^n \psi_o . \tag{2.12}$$

It may be remarked that this tranformation differs from that of Snowdon, which introduces a coordinate

$$\xi = \psi_o - \frac{\nu s}{R} + \int \frac{ds}{\beta} ,$$

in place of ψ_o. Neither of these transformations is applicable to two-dimensional nonlinear cases, because either involves use of the amplitude-function β (in the linear transformation in our case, in the nonlinear parts in Snowdon's), which is different for the two dimensions.

3. NONLINEAR TRANSFORMATIONS

The Transformation

The purpose of the nonlinear transformations is to remove the ϕ-dependence from the n^{th} order and transform it to higher order. We take as the general form of the Hamiltonian

$$H_o = \nu J_o + A_n(\phi) J_o^{n/2} \cos^n \psi_o, \tag{3.1}$$

The transformation from (ψ_o, J_o) to the new variables is taken to be a function of ψ_o, J_1, and s. The transformation rules are

$$\left\{ \begin{aligned} \psi_1 &= \frac{\partial G_1}{\partial J_1}, \\ J_o &= \frac{\partial G_1}{\partial \psi_o}, \\ H_1 &= H_o + \frac{\partial G_1}{\partial s}. \end{aligned} \right. \tag{3.2}$$

We choose a generating function

$$G_1 = \psi_o J_1 + w(\psi_o, \phi) J_1^{n/2} \tag{3.3}$$

so that

$$\left\{\begin{aligned} J_o &= J_1 + \frac{\partial w}{\partial \psi_o} J_1^{\frac{n}{2}} \\ \psi_1 &= \psi_o + \frac{n}{2} w J_1^{\frac{n-2}{2}} \\ H_1 &= H_o + \frac{\partial w}{\partial \phi} J_1^{\frac{n}{2}} \end{aligned}\right.$$

We require that the transformation be a periodic function of ϕ, in order to keep the transformed Hamiltonian periodic.

The new Hamiltonian in mixed old and new variables is

$$H_1 = \nu J_1 + J_1^{\frac{n}{2}} \left[\frac{\partial w}{\partial \phi} + \nu \frac{\partial w}{\partial \psi_o} \right] + J_o^{\frac{n}{2}} A_n \cos^n \psi_o + \ldots ;$$

the old variables must be expressed in terms of the new in H_1. We write first

$$J_o^{\frac{n}{2}} = J_1^{\frac{n}{2}} + \frac{n}{2} \frac{\partial w}{\partial \psi_o} J_1^{n-1} + \ldots$$

and

$$H_1 = \nu J_1 + J_1^{\frac{n}{2}} \left[\frac{\partial w}{\partial \phi} + \nu \frac{\partial w}{\partial \psi_o} + A_n \cos^n \psi_o \right] + \frac{n}{2} \frac{\partial w}{\partial \psi_o} J_1^{n-1} A_n \cos^n \psi_o + \ldots$$

We then write ψ_o in terms of ψ_1 as

$$\psi_o = \psi_1 + \Delta\psi = \psi_1 - \frac{n}{2} w J_1^{\frac{n-2}{2}} ,$$

and write the Hamiltonian in new variables as

$$H_1 = \nu J_1 + J_1^{\frac{n}{2}} Q(\psi_1, \phi) + \frac{n}{2} J_1^{n-1} \left[\frac{\partial w}{\partial \psi_1} A_n \cos \psi_1 - w \frac{\partial Q}{\partial \psi_1} \right] + \ldots , \qquad (3.5)$$

where

$$Q(\psi, \phi) = \frac{\partial w}{\partial \phi} + \nu \frac{\partial w}{\partial \psi} + A_n \cos^n \psi. \qquad (3.6)$$

The last term of the Hamiltonian is the correction term, showing the effect of the transformation of the n^{th} order on higher orders. In the correction term, we have neglected the difference between ψ_1 and ψ_o, because we shall not carry the calculation to still higher orders.

Analysis of the n^{th} Order Term

Let us investigate the conditions under which the term $QJ_1^{\frac{n}{2}}$ can be transformed to zero. Then

$$Q(\psi, \phi) = \frac{\partial w}{\partial \phi} + \nu \frac{\partial w}{\partial \psi} + A_n \cos^n \psi = 0. \qquad (3.7)$$

The analysis can be carried out in complex exponentials, in real trigonometric functions, or by expanding $A_n \cos^n \psi$ in Fourier series. We shall use complex exponentials; we set

$$\cos^n \psi = \sum_{\ell} a_{n\ell} \, e^{i\ell\psi}, \qquad (3.8)$$

where ℓ takes the integral values $\pm n$, $\pm(n-2)$, etc., down to ± 1 for n odd and 0 for n even. In the third order,

$$\begin{cases} a_{33} = a_{-33} = \frac{1}{8} \\ a_{31} = a_{3-1} = \frac{3}{8}, \end{cases}$$

and in the fourth order,

$$\begin{cases} a_{44} = a_{4-4} = \dfrac{1}{16} \\[2ex] a_{42} = a_{4-2} = \dfrac{1}{4} \\[2ex] a_{40} = \dfrac{3}{8} \end{cases}$$

We then seek a solution of Eq. (3.7) of the form

$$w(\psi, \phi) = \sum_{\ell} w_{\ell}(\phi) e^{i\ell(\psi - \nu\phi)}. \tag{3.9}$$

Then (with primes now denoting derivatives with respect to ϕ)

$$w_{\ell}' = - a_{n\ell} A_n(\phi) e^{i\ell\nu\phi}$$

or

$$w_{\ell} = w_{\ell o} - a_{n\ell} \int_o^{\phi} A_n(\phi) e^{i\ell\nu\phi} d\phi,$$

where $w_{\ell o}$ is a constant of integration.

We now impose the condition that w_{ℓ} be periodic in ϕ, that is, that

$$w_{\ell}\left(\psi, \phi + \frac{2\pi}{N}\right) = w_{\ell}(\psi, \phi).$$

From this condition it follows that

$$w_{\ell o}\left(1 - e^{\frac{2\pi i\ell\nu}{N}}\right) = a_{n\ell} \int_o^{\frac{2\pi}{N}} A_n(\phi) e^{i\ell\nu\phi} d\phi. \tag{3.10}$$

If the coefficient of $w_{\ell o}$ on the left is different from zero, we can solve for the $w_{\ell o}$ that gives a periodic solution. Then we have transformed to zero all n^{th} order terms and the new Hamiltonian is still periodic in ϕ.

If, on the other hand, the coefficient of $w_{\ell o}$ vanishes, we cannot find a non-trivial periodic solution unless the right-hand side also vanishes. The coefficient of $w_{\ell o}$ vanishes when $\ell\nu/N$ is an integer, or when

$$\ell\nu = mN, \tag{3.11}$$

with m an integer. Eq. (3.11) is a resonance relation and the terms for which it is satisfied are called "resonant" terms. What we have shown is that all terms in a given order except the resonant ones can be transformed to higher order, leaving the resonant terms for the special treatment discussed in the next section.

Even if the resonance relation (3.11) is not exactly satisfied, but if $(\ell\nu - mN)$ is small, the corresponding term in w_ℓ will be large, which will give large terms in higher orders. It is not even necessary that the resonant term be that for which $(\ell\nu - mN)$ is smallest, because the change of ν with amplitude may drive the motion away from this resonance. We therefore choose the resonant term and transform all other terms to higher order by the method above.

Resonant Terms

Only those terms for which $\ell\nu \simeq mN$ are left in the n^{th} order of the Hamiltonian. Furthermore, from Eq. (3.10), the integral

$$\int_0^{\frac{2\pi}{N}} A_n(\phi)e^{i\ell\nu\phi}d\phi,$$

must be different from zero. We shall call the particular values of ℓ and m for which resonance is possible ℓ_r and m_r. In the integral above, we can replace $\ell_r\nu$ by m_rN because they are approximately equal, thus expanding $A_n(\phi)$ in Fourier series:

$$\begin{cases} A_n(\phi) = \sum\limits_{m_r} A_{nm} e^{-imN\phi}, \\ A_{nm} = \dfrac{N}{2\pi} \displaystyle\int_0^{\frac{2\pi}{N}} A_n(\phi)e^{imN\phi}d\phi . \end{cases} \tag{3.12}$$

(Because $A_n(\phi)$ is real, $A^*_{nm} = A_{n-m}$.) Thus the term that drives the resonance $\ell_r\nu = m_rN$ is identified as the m_r^{th} harmonic of $A_n(\phi)$.

If the relation $\ell_r\nu \simeq m_rN$ is satisfied, so is the relation $-\ell_r\nu = -m_rN$, since ℓ and m come in positive and negative pairs. There are thus two resonant terms and the Hamiltonian is

$$H_1 = \nu J_1 + \Big[a_{n\ell_r} A_{nm_r} e^{i(\ell_r\psi_1 - m_rN\phi)}$$

$$+ a_{n-\ell_r} A_{n-m_r} e^{-i(\ell_r\psi_1 - m_rN\phi)} \Big] J_1^{\frac{n}{2}} + \dots ,$$

which can be written as

$$H_1 = \nu J_1 + 2a_{n\ell_r}\left|A_{nm_r}\right|\cos\left(\ell_r\psi_1 - m_rN\phi + \delta_{nm_r}\right) J_1^{\frac{n}{2}} + \dots , \tag{3.13}$$

where ℓ_r and m_r are now to be taken as non-negative numbers and δ_{nmr} is a phase angle. We have neglected here the higher-order terms, to which we shall return in Sec. 4.

There are two kinds of resonant terms, those with $\ell_r = 0$ and $\ell_r \neq 0$, that require separate treatment.

(i) Terms with $\ell_r = 0$. These terms occur only in even order because ℓ has only odd values in odd orders. Furthermore, $m_r = 0$ when $\ell_r = 0$ and the term is therefore independent of both ψ and ϕ and has the form

$$2 a_{no} |A_{no}| J_1^{\frac{n}{2}}. \qquad \text{(n even)}$$

These terms change ν as a function of J, or equivalently, with amplitude. If there were never any resonant terms with $\ell_r \neq 0$, the Hamiltonian could be transformed to

$$H = \nu J + \frac{3}{4} |A_{40}| J_1^2 + \frac{5}{8} |A_{60}| J_1^3 + \ldots ,$$

and the equations of motion would be

$$\begin{cases} \dfrac{dJ}{d\phi} = - \dfrac{\partial H}{\partial \psi} = 0 . \\ \\ \dfrac{d\psi}{d\phi} = \dfrac{\partial H}{\partial J} = \nu + \dfrac{3}{2} |A_{40}| J_1 + \dfrac{15}{8} |A_{60}| J_1^2 + \ldots , \end{cases}$$

so that J would be a constant of the motion and the effective ν value would depend on J or, equivalently, on amplitude.

(ii) Terms with $\ell_r \neq 0$. These are the terms that can give rise to unstable motion. In them, the dependence on ϕ is contained in the argument of the cosine in the form $\ell_r\psi_1 - m_r N\phi$. A transformation from ψ_1 to $\psi_2 = \psi_1 - m_r N\phi/\ell_r$, that is, to a rotating coordinate system in phase space, will remove this dependence. This transformation can be derived from the generating function

$$G_2 = G_2(\psi_1 J_2, \phi) = J_2\left(\psi_1 - \frac{m_r N\phi}{\ell_r}\right). \qquad (3.14)$$

Then

$$\begin{cases} J_1 = \dfrac{\partial G_2}{\partial \psi_1} = J_2, \\[2ex] \psi_2 = \dfrac{\partial G_2}{\partial J_2} = \psi_1 - \dfrac{m_r N\phi}{\ell_r}, \\[2ex] H_2 = H_1 + \dfrac{\partial G_2}{\partial \phi} = H_1 - J_2 \dfrac{m_r N}{\ell_r}, \end{cases}$$

and the new Hamiltonian is

$$H_2 = \left(\nu - \frac{m_r N}{\ell_r}\right) J_2 + 2a_{n\ell_r} \left|A_{nm_r}\right| J_2^{\frac{n}{2}} \cos\left(\ell_r \psi + \delta_{nm_r}\right) + \cdots \qquad (3.15)$$

If terms of higher order are neglected, this Hamiltonian is independent of ϕ and is therefore a constant of the motion. The stability of the motion can be predicted from this invariant.

4. HIGHER-ORDER TERMS

There are two reasons for interest in higher-order terms. First, when there is a resonance in the n^{th} order, the stability limits will be affected by those terms that change ν with amplitude (terms independent of ψ and ϕ). Even when there is no resonance in n^{th} order, but one exists in $(n+1)^{st}$ order, its stability limits will be affected by such change-of-ν terms transformed from the n^{th} to the $(n+1)^{st}$ order.

Second, the higher-order terms can give new resonances. The cases of the preceding section, in which $|\ell_r|$ has the form $|n - 2t|$ (t an integer), are not the only possible resonances that can arise from a given term in the original Hamiltonian of Eq. (1.2).

The next higher-order term is given in Eq. (3.5) as

$$\left\{\begin{aligned} &\frac{n}{2} J_1^{n-1} \left[\frac{\partial w}{\partial \psi_1} A_n(\phi)\cos^n \psi_1 - w \frac{\partial Q}{\partial \psi_1} \right] \\ &Q = \frac{\partial w}{\partial \psi_1} + \nu \frac{\partial w}{\partial \phi} + A_n \cos^n \psi_1 . \end{aligned}\right. \tag{4.1}$$

Let us consider the two terms separately. We must also distinguish between the cases when there is or is not a resonance in the n^{th} order. We can use Eqs. (3.8) and (3.9) to write the first term as

$$R(\psi_1, \phi) = \frac{\partial w}{\partial \psi_1} A_n(\phi)\cos^n \psi_1 = iA_n(\phi) \sum_{\ell_1, \ell_2} \ell_1 w_{\ell_1}(\phi) a_{n\ell_2} e^{i[(\ell_1+\ell_2)\psi - \ell_1 \nu\phi]} .$$

This term is periodic in both ψ and ϕ and can therefore be expanded in a

Fourier series

$$R(\psi,\phi) = \sum_{\ell,m} R_{\ell m} e^{i(\ell\psi - mN\phi)}, \tag{4.2a}$$

and we find that

$$R_{\ell m} = \frac{iN}{2\pi} \sum_{\ell_1} \ell_1 a_{n(\ell-\ell_1)} \int_0^{\frac{2\pi}{N}} A_n(\phi) w_{\ell_1}(\phi) e^{i(mN - \ell_1\nu)\phi} d\phi. \tag{4.2b}$$

If there is a resonance $\ell_r \nu = m_r N$ in the n^{th} order, then the terms containing w_{ℓ_r} and $w_{-\ell_r}$ are missing from $R_{\ell m}$.

Consider now the second term of Eq. (4.1). If there is no resonance in n^{th} order, then Q is identically zero and the second term makes no contribution to higher orders. If there is a resonance in n^{th} order, then

$$Q = 2a_{n\ell_r} |A_{nm}| \cos\left(\ell_r \psi_1 - m_r N\phi + \delta_{nm_r}\right),$$

and

$$w = \sum_{\ell}{}' w_\ell(\phi) e^{i\ell(\psi_1 - \nu\phi)},$$

where the prime on the summation is to remind us that the terms with $\ell = \pm\ell_r$ are not included. Then

$$S(\psi_1,\phi) = w \frac{\partial Q}{\partial \psi_1} = -\ell_r 2 a_{n\ell_r} |A_{nm}| \times$$

$$\times \sum_{\ell}{}' w_\ell(\phi) e^{i\ell(\psi_1 - \nu\phi)} \sin\left(\ell_r \psi_1 - m_r N\phi + \delta_{nm_r}\right),$$

can make a contribution of interest only in certain cases. A contribution of interest would be to a) the change-of-ν-with-amplitude term (the term independent of ψ and ϕ), b) a different higher-order resonance, or c) another term of the same ($\ell_r \nu = m_r N$) resonance.

In order to make a contribution in case a), there must be an ℓ such that $\ell \pm \ell_r = 0$. But these terms of w are specifically excluded from the sum. The second term therefore makes no contribution to case a).

Case b) is beyond the reach of the theory. If we have made a transformation to rotating coordinates to remove the ϕ-dependence from the resonant term with $\ell_r \nu = m_r N$, we cannot treat a different resonance without reintroducing ϕ into the original resonant term. We therefore cannot treat case b).

Only the terms of w with $\ell = 0$ or $|\ell| = 2\ell_r$ can make a contribution in case c), since we must have $\ell \pm \ell_r = \ell_r$. Terms with $\ell = 0$ occur only in even order. Thus, if there is a resonant term proportional to J^2, there will be a higher-order term of the same resonance proportional to J^3. This kind of correction is the only contribution that the second term of Eq. (4.1) can make to the theory.

In order to derive more specific results for higher-order terms, we need to specify $A_n(\phi)$. In the next section, we shall treat two such particular cases.

5. APPLICATIONS

$A_n(\phi)$ a Fourier Series

We write

$$A_n(\phi) = \sum_{m\geq 0} C_{nm} \cos mN\phi + D_{nm} \sin mN\phi \ , \tag{5.1a}$$

in real representation. We can also write

$$A_n(\phi) = \sum_m A_{nm} e^{-imN\phi} , \tag{5.1b}$$

and

$$\begin{cases} A_{nm} = \frac{1}{2}\left(C_{nm} + i D_{nm}\right) & (m > 0), \\ A_{no} = C_{no}, \\ A_{nm} = \frac{1}{2}\left(C_{n|m|} - iD_{n|m|}\right) & (m < 0). \end{cases} \tag{5.1c}$$

The constant of integration of the transformation function w can be calculated. For the case in which $\ell\nu/N$ is not integral,

$$w_{\ell o} = \frac{a_{n\ell}}{1 - e^{2\pi i\ell\nu/N}} \int_o^{\frac{2\pi}{N}} A_n(\phi)\, e^{i\ell\nu\phi} d\phi = i a_{n\ell} \sum_m \frac{A_{nm}}{\ell\nu - mN} .$$

The transformation function is then

$$w(\psi, \phi) = i \sum_{\ell} \sum_m \frac{a_{n\ell} A_{nm}}{\ell\nu - mN} e^{i(\ell\psi + mN\phi)} . \tag{5.2}$$

When there is a resonance in n^{th} order, say $\ell_r \nu = m_r N$, the term with $\ell = \ell_r$, $m = m_r$ and the term with $\ell = -\ell_r$, $m = -m_r$ are to be dropped from the double sum of Eq. (5.2).

Substitution into Eq. (4.2b) gives

$$R_{\ell m} = -\sum_{\ell_1} \sum_{m_1} \ell_1 a_{n(\ell-\ell_1)} a_{n\ell_1} \frac{A_{nm_1} A_{n(m-m_1)}}{\ell_1 \nu - m_1 N} ,$$

where the same resonant terms are to be dropped when there is an n^{th}-order resonance.

(i) Resonance in n^{th} order – Change-of-ν-with-amplitude term.

We combine terms of $\pm\ell$ and $\pm m$ to find

$$R_{oo} = \frac{2C_{no}^2}{\nu} \sum_{\ell>0} a_{n\ell}^2 - \nu \sum_{\substack{\ell>0 \\ m>0}} \ell^2 a_{n\ell}^2 \frac{C_{nm}^2 + D_{nm}^2}{m^2 N^2 - \ell^2 \nu^2} + \frac{1}{2} \ell_r a_{n\ell_r}^2 \frac{C_{nm_r}^2 + D_{nm}^2}{\ell_r \nu + m_r N} . \tag{5.3}$$

For example, for the resonance $3\nu = m_r N$ in the third order, the correction term in the Hamiltonian is

$$\frac{3}{32} J^2 \left\{ \left[\frac{5}{\nu} C_{30}^2 + \frac{3}{8} \left(C_{3m_r}^2 + D_{3m_r}^2 \right) \left[\frac{1}{m_r N + 3\nu} - \frac{6\nu}{(m_r N)^2 - \nu^2} \right] \right. \right.$$
$$\left. - \frac{9}{4} \nu \sum_{\substack{m \neq 0 \\ m \neq m_r}} \left(C_{3m}^2 + D_{3m}^2 \right) \left[\frac{1}{m^2 N^2 - 9\nu^2} + \frac{1}{m^2 N^2 - \nu^2} \right] \right\} \tag{5.4}$$

Laslett has calculated[3] this term in the special case with Hamiltonian

$$H = \frac{1}{2} p^2 + \frac{1}{2} \left(\frac{\nu}{R} \right)^2 x^2 + \frac{1}{3} x^3 \left[b_o + b_1 \cos \frac{Ns}{R} \right] .$$

Both his direct calculation and our method give for the invariant near $3\nu = N$

$$H = \frac{\nu J}{R} - \frac{1}{24}\left(\frac{2R}{\nu}\right)^{3/2} J^{3/2} b_1 \cos 3\psi +$$

$$+ \frac{1}{96}\frac{R^4}{\nu^3} J^2 \left\{ 3b_1^2 \left[\frac{6\nu}{N^2-\nu^2} - \frac{1}{N+3\nu} \right] - \frac{40 b_o^2}{\nu} \right\}$$

(ii) <u>No resonance in n^{th} order — Change-of-ν with amplitude term.</u>

This is the same as the preceding problem, except that is simplified because there are no resonant terms requiring special treatment. Then

$$R_{oo} = \frac{2C_{no}^2}{\nu} \sum_{\ell > 0} a_{n\ell}^2 - \nu \sum_{\substack{\ell > 0 \\ m > 0}} \frac{\ell^2 a_{n\ell}^2 \left(C_{nm}^2 + D_{nm}^2\right)}{m^2 N^2 - \ell^2 \nu^2} . \tag{5.5}$$

For n = 3, the correction term in the Hamiltonian is

$$-\frac{3}{32} J^2 \left\{ \frac{5}{\nu} C_{30}^2 - \frac{9\nu}{4} \sum_{m \neq 0} \left(C_{3m}^2 + D_{3m}^2\right) \left(\frac{1}{m^2 N^2 - 9\nu^2} + \frac{1}{m^2 N^2 - \nu^2} \right) \right\} \tag{5.6}$$

(iii) <u>No resonance in n^{th} order — New resonance in higher order.</u>

As an example, let us calculate the 4^{th} order term with $\ell = 4$ arising from n = 3. The result is

$$\frac{3}{64} J^2 \sum_{m_1} \left(\frac{3}{3\nu - m_1 N} + \frac{1}{\nu - m_1 N} \right) \left[A_{3m_1} A_{3(m-m_1)} e^{i(4\psi - mN\phi)} + A_{3(-m_1)} A_{3(-m+m_1)} e^{-i(4\psi - mN\phi)} \right] \tag{5.7}$$

Thus a Hamiltonian with only sextupole (n = 3) terms will give rise to fourth-order resonances through this correction process. In physical terms, one can say that the sextupole term distorts the waveform of the oscillation so that the first and second harmonics combine to make an equivalent third harmonic.

<u>$A_n(\phi)$ a δ- function</u>

We choose $\phi = 0$ to be at the location of the δ-function. Thus

$$A_n(\phi) = A\delta(\phi). \tag{5.8}$$

Then

$$w_{\ell o} = \frac{a_{n\ell}A}{1 - e^{\frac{2\pi i\ell\nu}{N}}},$$

and

$$w(\psi,\phi) = \sum_{\ell} \frac{a_{n\ell}A}{\left(e^{\frac{-2\pi i\ell\nu}{N}} - 1\right)} e^{i\ell(\psi - \nu\phi)}. \tag{5.9}$$

As in the Fourier-series case, the equations for w must take into account the effects of resonances in the n^{th} order. We shall omit those terms for which $\ell\nu/N$ is an integer, denoting this by a prime on the sum. Then

$$R_{\ell m} = \frac{iN}{2\pi} \sum_{\ell_1}{}' \frac{\ell_1 a_{n\ell_1} a_{n(\ell - \ell_1)} A^2}{e^{\frac{-2\pi i\ell_1\nu}{N}} - 1}. \tag{5.10}$$

Note that $R_{\ell m}$ is independent of m, because the δ-function contains all harmonics equally.

(i) Resonance in n^{th} order—Change-of-ν-with-amplitude term.

$$R_{oo} = \frac{iNA^2}{2\pi} \sum_{\ell_1 > 0}{}' \ell_1 a^2_{n\ell_1} \cot \frac{\ell_1 \pi\nu}{N} . \tag{5.11}$$

Thus for the third-order resonance $3\nu = mN$, the correction term in the Hamiltonian is

$$- \frac{27}{256} J^2 \frac{NA^2}{\pi} \cot \frac{\pi\nu}{N} . \tag{5.12}$$

(ii) No resonance in n^{th} order—Change-of-ν-with amplitude term.

The result is just Eq. (5.11) without the prime. For $n = 3$, the correction term is

$$- \frac{9}{256} \frac{NA^2}{\pi} J^2 \left[3 \cot \frac{\pi\nu}{N} + \cot \frac{3\pi\nu}{N} \right] . \tag{5.13}$$

(iii) No resonance in n^{th} order—New resonance in higher order. As we did in the Fourier-series example, we shall calculate the $\ell = 4$ terms in fourth order arising from $n = 3$. The result is

$$- \frac{9}{256} \frac{NA^2 J^2}{\pi} \left\{ \left(3 \cot \frac{3\pi\nu}{N} + \cot \frac{\pi\nu}{N} \right) \cos (4\psi - mN\phi) - \sin(4\psi - mN\phi) \right\} \tag{5.14}$$

We have explored numerically the effects of this term and shall report the results in a later report.

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