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# Comments on ESME, a Tracking Code in the Longitudinal Phase Space

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# I. INTRODUCTION

The rf cavity, wall resistivity, space-charge impedance, and other discontinuities of the vacuum chamber are sources that generate bunch shape distortions as well as emittance growth in both the longitudinal and transverse phase spaces. In this paper, only the longitudinal phase space is examined. The understanding of these distortions and growths are extremely valuable in maintaining better performance of the accelerator or storage ring. Unfortunately, exact computations are, in general, impossible. The Vlasov equation even when linearized can provide perturbative information only when some other additional approximations are assumed. The situation of having bunch omissions in some buckets or an unequal number of particles per bunch makes the problem almost insolvable. As a result, a numerical tracking may be appealing. The tracking is carried out turn by turn. In order to shorten the computation time in the longitudinal space, one is inclined to calculate the potential seen by each particle in each turn using impedances in the frequency domain rather than using the wake potential in the time domain. In the frequency domain, however, the tracking in the longitudinal phase space is usually more difficult than the tracking in the transverse phase spaces. This is because the betatron frequencies are much bigger than the revolution frequency, whereas the synchrotron frequency is very much smaller.

ESME[1] is an example of such a code. It tracks bunches turn by turn in the longitudinal phase space by assuming an energy loss of

$$eV_i(\theta) = e\omega_0 \sum_{m=-\infty}^{\infty} \tilde{\rho}_i(m\omega_0) Z(m\omega_0) e^{in\theta} \ , \tag{1.1}$$

for the $i$th turn. In the above, $\tilde{\rho}_i$ is the *discrete* Fourier spectrum of the bunch in that particular turn, $Z(\omega)$ is the longitudinal coupling impedance, $\omega_0$ is the angular revolution frequency of the synchronized particle, and $\theta$ is the azimuthal angle along the storage ring. We would like to point out that Eq. (1.1), the dynamics of such tracking code, is nothing more than a tentative current-multiplied-by-impedance approach. As it stands, it is clear that it can at most represent some steady-state condition with all transient effects neglected. Therefore, such tracking in the frequency domain is definitely incorrect when the impedance $Z(\omega)$ has a time constant longer than the revolution period $T_0 = 2\pi/\omega_0$, or when the wake potential extends longer than one revolution of the storage ring.

This paper is not a criticism on ESME. This is because, in the first place, ESME was written to simulate the acceleration process and to understand the various rf maneuverings in the Fermilab Main Ring and is not intended to deal with sharp resonant driving forces. In the second place, the author[2] of ESME is now trying to extend the code to include driving impedance of sharp resonances. For these driving forces, he plans to track in the time domain instead.[1]

In this paper, we try to derive as rigorously as possible the energy loss per turn and indicate how it differs from Eq. (1.1). We also point out the condition under which Eq. (1.1) is correct. Some consequences of using Eq. (1.1) incorrectly are also discussed. Lastly, we point out how the tracking can be done correctly in the frequency domain when sharp resonant driving impedances are present.

## II. ENERGY LOSS PER TURN

### II.1 Canonical variable

A particle in a beam is characterized longitudinally by $\tau$, the time it arrives ahead of the synchronized particle at some pre-chosen reference point along the accelerator ring. We choose $\tau$ as the canonical variable. The choice of other canonical variables is also allowed.

### II.2 Wake potential

When a particle of unit charge traveling inside the vacuum chamber passes a structure, it leaves behind a wake. The *average* longitudinal potential seen by another particle lagging a time $\tau$ behind (Fig. 1) is called the wake potential $W(\tau)$. Note that the averaging over the structure is very crucial. It makes the wake potential a function of $\tau$ only and is independent of (1) the position of the leading particle relative to the structure, and (2) the lateral deviations of both particles from the axis of the vacuum chamber. Here, the particles travel with velocity $v$, which is less than $c$, the velocity of light. Thus, in general there is no causality requirement. However, it is the space-charge part of the wake that violates causality. Therefore, when space-charge contribution is excluded, $W(\tau) = 0$ when $\tau < 0$. Even with space charge, $W(\tau)$ drops to zero very rapidly as $\tau$ goes negative because the particle velocity is usually very close to $c$.

### II.3 Charge or current distribution

To measure the charge or current distribution in a beam, we choose a fixed reference point in the storage ring and put a detector there (Fig. 2). We record the amount of charge arriving when the time advance is between $\tau$ and $\tau + d\tau$. The result is $\rho(\tau)d\tau$, where $\rho(\tau)$ is a measure of charge distribution. The actual linear charge density is $\lambda(\tau) = \rho(\tau)/v$, where $v$ is the velocity of the synchronized particle. Note that this charge distribution is measured at a fixed point but at different times. Therefore, it is *not* a periodic function of $\tau$. This is important because, unlike Eq. (1.1), the spectrum of the charge distribution no longer just singles out the harmonics of the revolution frequency. On the other hand, there is something called the snapshot distribution $\rho(\tau)|_{\text{snapshot}}$, which is recorded by taking a snapshot camera picture above the accelerator ring. Here, we assume information at any point of the ring arrives at the camera at the same time. Thus, by definition, $\rho(\tau)|_{\text{snapshot}}$ is a periodic function of the ring. This distribution is useful in mathematical derivations, but is *not* what we measure with a detector. The two are identical only when the distribution does not change with time. Obviously, this condition can never be realized because there is always synchrotron motion. The properties of the two distributions are summarized in Table I.

| $\rho(\tau)$ | $\rho(\tau)\|_{\text{snapshot}}$ |
|---|---|
| Usual definition of charge density | mathematical definition only |
| measured at fixed location but at different time | measured at fixed time but at different location |
| not a periodic function of the ring | periodic function of the ring |

Table I: Comparison between charge density measured at a fixed location with charge density measured at a fixed time.

The charge density also depends on during which turn it is measured. Therefore, there is an independent variable, the turn number. However, it is more convenient to introduce a continuous variable instead, so that the charge distribution measured at any other location along the accelerator ring can be described. Also, it is obvious that a continuous variable can be dealt with much more easily in mathematical derivations. Time is not a good variable and should not be used because it is complicated by synchrotron motion and the acceleration process. We choose instead $s$, the distance along the closed orbit of the synchronized particle along the accelerator. Thus, the charge density is written as $\rho(\tau, s)$. Note that the canonical variable $\tau$ is also an implicit function of $s$.

Since we are dealing with a bunched beam, it is easy to distinguish for one turn the beginning and end of the charge distribution, which may contain many bunches. Thus, we can set

$$\rho(\tau, s) = 0 \qquad \text{except when } \tau_0 < \tau < \tau_0 + T_0 \ , \tag{2.1}$$

where $\tau_0$ is some arbitrarily chosen reference time advance and $T_0$ is the period of revolution of the synchronized particle for that particular turn.

### II.4 Potential seen by a particle due to a bunch

A particle at $s$ with time advance $\tau$ will experience a potential $V(\tau, s)$ due to a particle charge distribution. From the definition of the wake potential, this potential is given by[3]

$$V(\tau, s) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau' \rho(\tau', s - kC) W(kT_0 + \tau' - \tau) \ , \tag{2.2}$$

where $C$ is the length of the closed orbit of the synchronized particle, or $C = vT_0$. The summation over $k$ takes care of the contribution of the wake left by the charge distribution in previous turns. The lower limit of the summation can be extended to $k = -\infty$ and the lower limit of the integration can be extended to $\tau' = -\infty$, either when we consider the non-space-charge part of the wake where there is causality, or when we consider the space-charge contribution where there is no causality. Since the charge distribution is defined to carry values for one turn only in Eq. (2.1), the upper limit of integration can also be extended to $\tau' = +\infty$.

In Eq. (2.2), $T_0$ is assumed to be a constant. This is true when the particle beam is in storage. However, during an accelerating cycle, $v$ increases turn by turn and $T_0$ decreases turn by turn so that $C$ remains constant. In that case, we need to make the replacement

$$kT_0 \to \sum_{i=1}^{k} T_{0i} \qquad \text{when } k > 0 \ , \tag{2.3}$$

where $T_{0i}$ is the period for the previous $i$-th turn. It does not matter what the replacement is when $k < 0$, because the wake potential vanishes. Actually, the replacement in Eq. (2.3) is not quite necessary, because a wake will have a length of at the most a few turns, during which $T_0$ will not change very much.

When there are $M$ bunches in the accelerator, we can introduce $\rho_\ell(\tau, s)$ as the charge density of the $\ell$-th bunch, where $\tau$ is the distance advanced relative to the synchronized particle in the $\ell$-th bunch. Using Eq. (2.2), the potential experienced by a particle at $s$ in the $m$-th bunch with time advance $\tau$ can be written as

$$V_m(\tau, s) = \sum_{k=-\infty}^{\infty} \sum_{\ell=0}^{M-1} \int_{-\infty}^{\infty} d\tau' \rho_\ell(\tau', s - kC) W[kT_0 + (s_\ell - s_m)/v + \tau' - \tau] \ , \tag{2.4}$$

where $s_\ell$ and $s_m$ are the positions of the synchronized particles in the $\ell$-th and $m$-th bunch, and $(s_\ell - s_m)$ should not change from turn to turn. In the case of equally spaced bunches,

$$s_\ell - s_m = \frac{\ell - m}{M} C \ . \tag{2.5}$$

The longitudinal coupling impedance $Z(\omega)$ of the structure in the vacuum chamber is defined as the Fourier transform of the wake potential by

$$W(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega Z(\omega) e^{-i\omega\tau} \ . \tag{2.6}$$

The charge distribution can also be Fourier analysed by

$$\rho(\tau, s) = \int_{-\infty}^{\infty} d\omega \tilde{\rho}(\omega, s) e^{-i\omega\tau} \ . \tag{2.7}$$

Then, for one bunch, Eq. (2.2) can be rewritten in the frequency domain as

$$V(\tau, s) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \tilde{\rho}(\omega, s - kC) Z(\omega) e^{-i\omega(kT_0 - \tau)} \ . \tag{2.8}$$

Similarly, for $M$ bunches, Eq. (2.4) can be rewritten in the frequency domain as

$$V_m(\tau, s) = \sum_{k=-\infty}^{\infty} \sum_{\ell=0}^{M-1} \int_{-\infty}^{\infty} d\omega \tilde{\rho}_\ell(\omega, s - kC) Z(\omega) e^{-i\omega[kT_0 + (s_\ell - s_m)/v - \tau]} \ . \tag{2.9}$$

We note that, with the definition of Eq. (2.1), the Fourier transform of the charge density in the variable $\tau$ is for one turn only. Thus, the spectrum is continuous. The discrete nature comes about only when the charge distribution is measured for many turns (more correctly for infinite number of turns). The turn number is embedded in the variable $s$. As will be shown below in Eq. (3.9), we do have a discrete spectrum after summing up $k$ from $-\infty$ to $+\infty$.

## III. PERTURBATION OF CHARGE DISTRIBUTION

### III.1 Single bunch

Let us first study the simple situation of having only one bunch. If a bunch is well fitted to the bucket to start with, the charge distribution will not change at all. Therefore,

$$\rho(\tau, s - kC) \equiv \rho_0(\tau) \ ,$$

$$\tilde{\rho}(\omega, s - kC) \equiv \tilde{\rho}_0(\omega) \; . \tag{3.1}$$

The wake or coupling impedance can drive the bunch to oscillate in the bucket with frequencies $\Omega_n/2\pi$, $n = 1, 2, \ldots$. Since the rf provides a synchrotron oscillation frequency of $\omega_s/2\pi$, $\Omega_n \approx n\omega_s$ where $n$ is an integer. These are eigen-frequencies, whose values can only be obtained by solving a Vlasov-like eigen-equation. But only the perturbed part will oscillate. Thus, the charge distribution can be written as

$$\tilde{\rho}(\omega, s - kC) = \tilde{\rho}_0(\omega) + \sum_n \tilde{\rho}_n(\omega) e^{-i\Omega_n(s-kC)/v} \; , \tag{3.2}$$

where $\tilde{\rho}_n(\omega)$ are the perturbed distributions or eigenfunctions related to $\Omega_n$. Correspondingly, the voltage experienced can be written as

$$V(\tau, s) = V_0(\tau) + \sum_n V_n(\tau, s) \; . \tag{3.3}$$

This is illustrated schematically in Fig. 3.

Substituting Eq. (3.1) in Eq. (2.8), we get

$$V_0(\tau, s) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \tilde{\rho}_0(\omega) Z(\omega) e^{-i\omega(kT_0 - \tau)} \; . \tag{3.4}$$

With $\tilde{\rho}_0$ independent of $s$, the summation over $k$ can be performed. We obtain

$$V_0(\tau, s) = \omega_0 \sum_{p=-\infty}^{\infty} \tilde{\rho}_0(p\omega_0) Z(p\omega_0) e^{ip\omega_0\tau} \; , \tag{3.5}$$

which is independent of $s$. This implies that as $s$ increases, $V_0$ will not change.

Equation (3.5) implies that $V_0$ will not grow at all, but it will lead to a change of the equilibrium bunch shape and a modification of the synchrotron frequency. This can also be understood by assuming that the exponent $p\omega_0\tau/c$ is small. Thus,

$$e^{ip\omega_0\tau} \sim 1 + ip\omega_0\tau \; , \tag{3.6}$$

so that Eq. (3.5) can be written simply as

$$V_0(\tau, s) = A + B\tau \; , \tag{3.7}$$

where $A$ and $B$ are real constants. The voltage due to the unperturbed charge distribution is therefore a constant plus a force that is proportional to displacement when $\tau$ is sufficiently small. The former distorts the bunch shape by shifting the center of oscillation, while the latter adds to the rf force and modify the synchrotron frequency. The result is a distorted bunch with higher density towards the front.

The part of the distribution that oscillates with frequency $\Omega_n/2\pi$ contributes to the potential

$$V_n(\tau,s)=\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}d\omega\tilde{\rho}_n(\omega)Z(\omega)e^{-i\Omega_n(s-kC)/v-i\omega(kT_0-\tau)}\ . \tag{3.8}$$

Again the summation over $k$ can be carried out and we arrive at

$$V_n(\tau,s)=\omega_0e^{-i\Omega_ns/v}\sum_{p=-\infty}^{\infty}\tilde{\rho}_n(p\omega_0+\Omega_n)Z(p\omega_0+\Omega_n)e^{i(p\omega_0+\Omega_n)\tau}\ . \tag{3.9}$$

Equations (3.5) and (3.9) are the same as Eq. (2.2) but in the frequency space.

### III.2 Multi-bunches

In order that the $M$ bunches have coupled motions, the bunches must oscillate in particular patterns. For the $\ell$-th bunch ($\ell=0,\ldots,M-1$), the bunch density can be written as

$$\rho_\ell(\tau,s)=\rho_0(\tau)+\sum_n\sum_\mu\rho_{\ell n\mu}(\tau)e^{-i\Omega_{n\mu}s/v}\ , \tag{3.10}$$

where the subscript $\mu=0$ to $M-1$ denotes the $M$ coupled modes and $n$ denotes other characteristic numbers. The eigen-frequency $\Omega_{n\mu}$ and the corresponding perturbed distributions or eigenfunctions $\rho_{\ell n\mu}$ are obtained by solving an eigen-equation like the Vlasov equation. The linearized Vlasov equation shows that the $\mu$-th coupled mode, defined as the oscillation of each bunch lagging the preceding one by a phase of $2\pi\mu/M$ in a snapshot sense, is characterized by equally-spaced bunches with

$$\rho_{\ell n\mu}[\tau,s+(s_\ell-s_0)]=\rho_{0n\mu}(\tau,s)e^{i2\pi\ell\mu/M}\ , \tag{3.11}$$

where the subscript 0 represents the 0-th bunch. Note that for the $\mu=0$ mode, all the bunches are oscillating in exactly the same way with the same phase at the same time but at different locations.

With only the coupled-mode frequency $\Omega_\mu/2\pi$, the extra potential seen by a particle in the $m$-th bunch is

$$\begin{aligned}V_{mn\mu}(\tau,s)\ &=\ \sum_{k=-\infty}^{\infty}\sum_{\ell=0}^{M-1}\int_{-\infty}^{\infty}d\omega\tilde{\rho}_{\ell n\mu}(\omega)e^{-i\Omega_{n\mu}[s-kC-(s_\ell-s_0)]/v}e^{i2\pi\mu\ell/M}\times\\&\quad\times Z(\omega)e^{-i\omega[kT_0+(s_\ell-s_0)/v-\tau]}\ ,\end{aligned} \tag{3.12}$$

where Eqs. (2.9) and (3.11) have been used. The summation over $k$ can be performed readily giving

$$V_{mn\mu}(\tau,s)\ =\ \omega_0\sum_{p=-\infty}^{\infty}\sum_{\ell=0}^{M-1}\tilde{\rho}_{0n\mu}(p\omega_0+\Omega_{n\mu})e^{-i\Omega_\mu[s-(s_\ell-s_0)]/v}e^{i2\pi\mu\ell/M}\times$$

$$\times Z(p\omega_0+\Omega_{n\mu})e^{i(p\omega_0+\Omega_{m\mu})[\tau-(s_\ell-s_0)/v]} \ . \tag{3.13}$$

In the case of equally spaced bunches, the summation over $\ell$ can be performed easily with the help of Eq. (2.5) to give

$$V_{mn\mu}(\tau,s) = M\omega_0 e^{-i\Omega_{n\mu}(s-mC/M)/v} e^{i2\pi\mu\ell/M} \sum_{q=-\infty}^{\infty} \tilde{\rho}_{0n\mu}(\omega')Z(\omega')e^{i\omega'\tau} \ , \tag{3.14}$$

where

$$\omega' = (Mq+\mu)\omega_0 + \Omega_{n\mu} \ . \tag{3.15}$$

## IV. TRACKING IN THE FREQUENCY DOMAIN

### IV.1 Broad-band impedance on single bunch or multi-bunches

Consider first the situation of having only one bunch. If the wake $W(\tau)$ has a length $\tau_c$ that is shorter than one revolution of the ring, or the impedance $Z(\omega)$ has a full width $\Delta\omega_r$ that is broad, *i.e.,*

$$T_0 > \tau_c = \frac{1}{\Delta\omega_r} \ , \tag{4.1}$$

the exponential term $\exp(i\omega kT_0)$ in Eq. (2.8) oscillates very fast when it is integrated over the broad band from $\omega_r - \Delta\omega_r/2$ to $\omega_r + \Delta\omega_r/2$, where $\omega_r$ is the center of the broad band. Thus, only $k=0$ contributes essentially, or

$$V(\tau,s) \approx \int d\omega\tilde{\rho}(\omega,s)Z(\omega)e^{-i\omega\tau} \ . \tag{4.2}$$

Since the impedance is of a broad band nature, the integral can be further reduced to a discrete sum, and we have

$$V(\tau,s) \approx \omega_0 \sum_{p=-\infty}^{\infty} \tilde{\rho}(p\omega_0,s)Z(p\omega_0)e^{-ip\omega_0\tau} \ . \tag{4.3}$$

This is exactly Eq. (1.1), or what ESME uses.

If there are $M$ bunches, and if the length of the wake is less than the distance between two bunches, *i.e.,*

$$\frac{T_0}{M} > \tau_c = \frac{1}{\Delta\omega_r} \ , \tag{4.4}$$

the factor $\exp[-i\omega(s_\ell - s_m)/v]$ in Eq. (2.9) will oscillate rapidly in the integration over $\omega$. Thus, only the same bunch $\ell = m$ will contribute, and Eq. (2.9) will be reduced to Eq. (4.3) also.

## IV.2 Narrow resonant impedance on single bunch

Sometimes, the coupling impedance $Z(\omega)$ is a narrow resonance with the width less than $\omega_0$. Then, we cannot neglect the summation over $k$ or the contribution from previous turns. This is, in fact, to be anticipated, because the wake extends to more than one complete revolution. Furthermore, the integral over $\omega$ cannot be reduced to a summation. Therefore, to perform tracking in the frequency domain, we have no choice but stick to Eq. (2.8), which is clearly different from Eq. (1.1) that ESME employs. The neglect of summation over past turns implies the failure to include transient effects. The reduction of the integral over frequency to a discrete sum over the harmonics explains why ESME cannot pick up the synchrotron satellites. As will be shown below, the satellites corresponding to positive frequencies contribute to growths while those corresponding to negative frequencies contribute to dampings. With only the harmonics of the revolution frequency, it is hard to visualize how the growths and dampings will evolve. As a result, it is natural that ESME, as it stands, will not be able to reproduce Robinson's instability.

Of course, in this case tracking can be done in the time domain using Eq. (2.2). However, we would like to investigate the way it can be performed in the frequency domain. We cannot use Eq. (3.9) because we do not know the eigen-frequencies $\Omega_n$ nor do we know distorted equilibrium density $\rho_0$ in Eq. (3.2). So we have to start with Eq. (2.8). But it is not as bad as it looks, because we only need to sum up in each case the past $\sim \omega_0/\Delta\omega_r$ turns, where $\Delta\omega_r$ is the width of the resonant driving impedance. Let us investigate how such a tracking is carried out and what sort of results will be produced.

We need to compute the energy gained by each particle in a turn, and real numbers are preferred. Let us start with a distribution

$$\rho(\tau,s) = \rho_0(\tau) + \rho_1(\tau)e^{\Omega_I s/v}\cos\frac{\Omega_R s}{v} \ . \tag{4.5}$$

where $\rho_0$ is a distribution that is well fitted to the bucket and the second term is some seed which oscillates with frequency $\Omega_R/2\pi$ which is taken here as approximately the synchrotron frequency $\omega_s/2\pi$ for the lowest mode and grows exponentially at a rate of $\Omega_I$. In real tracking, such seed is not actually necessary, because $\rho_0$ is *not* the distorted equilibrium distribution and therefore the rf voltage will generate some seed from $\rho_0$ automatically. Of course, Eq. (4.5) can be written simply as

$$\rho(\tau,s) = \rho_0(\tau) + \mathcal{R}e\left[\rho_1(\tau)e^{-i\Omega s/v}\right] \ , \tag{4.6}$$

where $\Omega = \Omega_R + i\Omega_I$.

Let us concentrate on the seed because this is the source of growth. The seed may be of the form

$$\rho_1(\tau) \sim \begin{cases} \sin \frac{2\pi\tau}{\sigma} & |\tau| < \sigma \ , \\ 0 & \text{otherwise} \ , \end{cases} \tag{4.7}$$

where $\sigma$ is a measure of the half length of the bunch. The spectrum of the seed is

$$\tilde{\rho}_1(\omega) \sim -\frac{2i}{\sigma} \frac{\sin \omega\sigma}{\omega^2 - (2\pi/\sigma)^2} \ . \tag{4.8}$$

We do not care so much about the detailed form of the spectrum except that it is imaginary and odd in $\omega$. Also it should have a width

$$\frac{2\pi}{\sigma} \gg \omega_r \ , \tag{4.9}$$

the frequency of the resonant impedance. If not, the bunch will not be affected by the resonance. Then, from Eq. (2.8), the voltage seen by a particle due to the seed is

$$V_1(\tau, s) = \mathcal{R}e \sum_{k\geq 0} \int d\omega \, \tilde{\rho}_1(\omega) Z(\omega) e^{-i(s-kC)\Omega/v} e^{-i\omega(kT_0-\tau)} \ . \tag{4.10}$$

To evaluate the perturbed potential, we have the choice of performing the integral first or summing over $k$ first. Let us consider the situation where we perform the integral first. We are going to state the results here. The derivations are given in the Appendix. The result is

$$V_1(\tau, s) = \mathcal{R}e \, \omega_0 \sum_{p=-\infty}^{\infty} \tilde{\rho}_1(p\omega_0 + \Omega) Z(p\omega_0 + \Omega) e^{i[(p\omega_0+\Omega)\tau - \Omega s/v]} \ , \tag{4.11}$$

which is the same as Eq. (3.9) as expected. If the growth rate is extremely small, we may consider $\Omega$ as real. For a short bunch, $(p\omega_0 + \Omega)\tau$ is small. Neglecting this, the result becomes

$$\begin{aligned} V_1(\tau, s) \approx & -\omega_0 \sin \frac{\Omega s}{v} \sum_{p>0} \{ i\tilde{\rho}_1(p\omega_0 + \Omega) Z_R(p\omega_0 + \Omega) - i\tilde{\rho}_1(p\omega_0 - \Omega) Z_R(p\omega_0 - \Omega) \} \\ & + \omega_0 \cos \frac{\Omega s}{v} \sum_{p>0} \{ i\tilde{\rho}_1(p\omega_0 + \Omega) Z_I(p\omega_0 + \Omega) + i\tilde{\rho}_1(p\omega_0 - \Omega) Z_I(p\omega_0 - \Omega) \} \ , \end{aligned} \tag{4.12}$$

where we have made use of the fact that $Z_R(\omega) = \mathcal{R}e \, Z(\omega)$ is even in $\omega$ whereas $Z_I(\omega) = \mathcal{I}m \, Z(\omega)$ and $\tilde{\rho}_1(\omega)$ are odd. The contribution of $p = 0$ is negligible and has been discarded. The second term in Eq. (4.12) begins with cosine which is in phase with the seed and therefore contributes to a change in synchrotron frequency. The first term begins with sine which is out of phase and therefore contributes to a growth for the $p\omega_0+\Omega$ part or damping for the $p\omega_0-\Omega$ part. Note that these two parts are preceded by different signs. As a result, we have a growth (damping) if the the resonance peak is nearer to (farther away from) $p\omega_0+\Omega$ than $p\omega_0-\Omega$. This is just Robinson's instability. Thus, the integration over $\omega$ has to be performed with binning much finer than the minimum of $\Omega$ and $\Delta\omega_r$. But this is not too tedious to do because $Z_R$ is very narrow. The term involving $Z_I$ comes with a plus sign. Since $Z_I$ is broad for a resonance, the second integral need not be done with very high accuracy with the exception of perhaps the small region around $\omega = \omega_r$.

The summation over $k$ is the summation over the interaction of the bunch with the long wake field left by the narrow resonance. As is shown in the Appendix, a summation of $\mathcal{N} \gtrsim \omega_0/(\Delta\omega_r+\Omega_I)$ previous turns is required. Actually in tracking, we compute Eq. (2.8), the total potential. The perturbed potential is imbedded inside. Therefore, a high degree of accuracy is required. The integral over $\omega$ is computed for each $k \geq 0$ and stop at $k = \mathcal{N}$ where the contribution becomes negligible. Knowing $V(\tau, s)$, the new current density can be computed. From it, we find the new spectral distribution. Then we go to the next turn. If N bins are used to store the spectral distribution, we need a storage size of $N\mathcal{N}$.

We can also do the summation over $k$ in Eq. (4.10) first. As is shown in the Appendix, exactly the same result of Eq. (4.11) is obtained. Here, the summation over $k$ is the summation over the perturbed density multiplied by an exponential,

$$\sum_{k\geq 0} \tilde{\rho}_1(\omega)e^{-ik(\omega-\Omega)T_0} \ . \tag{4.13}$$

Here, $\omega$ will be dominated by the poles of the impedance when the final integration over $\omega$ is performed. Thus, although the contribution of each term in Eq. (4.13) will not decrease as $k$ increases, the number of previous turns required in the sum is again $\mathcal{N} \gtrsim \omega_0/(\Delta\omega_r+\Omega_I)$. The same amount of storage space will be required. There is an advantage to this alternate process, because only one integral over $\omega$ will be required for each time advanced $\tau$. The price to pay is: the summation has to be done for each frequency of the distribution. Of course, in tracking we are not dealing with the perturbed density in Eq. (4.13). Instead, the sum is

$$\sum_{k\geq 0} \tilde{\rho}(\omega, s-kC)e^{-ik\omega T_0} \ , \tag{4.14}$$

where $\tilde{\rho}(\omega, s-kC)$ is the total charge density at the $k$-th previous turn. As a result, a high degree of accuracy is required so that the seed can be picked up.

### IV.3 Narrow resonances on multi-bunches

Here, we have to start from Eq. (2.9) for the tracking. Either summing over $k$ first or integrating over $\omega$ first, we get something very similar to Eq. (4.12). Folding negative frequency onto positive frequency, the perturbed voltage involving $Z_R$ is

$$V_{1\mu}(\tau,s)|_{Z_R} \propto \sin\frac{\Omega s}{v}\sum_{q>0}\{i\tilde{\rho}_{01}[(qM+\mu)\omega_0+\Omega]Z_R[(qM+\mu)\omega_0+\Omega]$$
$$-i\tilde{\rho}_{01}[(qM-\mu)\omega_0-\Omega]Z_R[(qM-\mu)\omega_0-\Omega]\}\ , \qquad (4.15)$$

where $\tilde{\rho}_{01}$ is the density of the seed of the 0-th bunch. We see that if the driving resonance is near $(qM+\mu)\omega_0+\Omega$, it is also near $[(q-1)M+(M-\mu)]\omega_0-\Omega$. This implies that when coupled-bunch mode $\mu$ is activated to grow, mode $M-\mu$ is damped. This type of damping has never been reproduced by ESME.[4] For this reason, we doubt whether the negative frequency-contributions have been handled correctly in ESME.[5]

To have the bunches coupled, the length of the wake can be as short as the distance between two adjacent bunches, or the width of the resonance $\Delta\omega_r$ can be as wide as $\sim M/T_0$. For the Fermilab Booster, there is a rf parasitic resonance at $f_r\sim 85.8$ MHz with $Q\sim 3300$. The width is $\Delta\omega_r/2\pi\sim f_r/Q=26$ kHz or a wake of characteristic length $cQ/f_r=11.5$ km or about 24 turns. On the other hand, the synchrotron frequency is $\omega_s/2\pi\sim 2.2$ kHz. Thus, the $\pm\Omega$ in Eq. (4.15) can be neglected and only sampling of the revolution harmonics will be accurate enough. For the same reason, in performing the integration over $\omega$, we only need binning $\lesssim\Delta\omega_r$. However, there are two modes that are different. They are mode $\mu=0$ and $\mu=M/2$ if $M$ is even. Like the Robinson's instability criterion, these modes will grow or damp depending on whether the center of the driving resonance is nearer to the $+\Omega$ or $-\Omega$ satellites. Therefore, for these two modes binning $\lesssim\omega_s$ is required. There has been some studies of coupled-bunch instability induced by rf parasitic resonances in the Fermilab Booster using ESME.[6] Because of the neglect of contributions due to previous turns, the neglect of synchrotron satellites, and the possible mistreatment of negative frequencies, these studies may not have been correct.

## V. CONCLUSION

We have shown that ESME, as it stands, is correct when the driving impedance is of broad band nature or when the wake does not extend to more than a revolution around the accelerator. For resonance that has a wake of length more than a revolution, Eq. (2.8) or (2.9) should be used, if we want the tracking to be performed in the frequency domain. The integral over $\omega$ that involves $\mathcal{R}e\,Z$ has to be done with binning $\lesssim\omega_s$, the synchrotron frequency for the situation of a single bunch or for $M$ bunches of coupled mode $\mu = 0$ and $M/2$ (if $M$ is even). For the other coupled modes, the binning needs to be less than the width of the resonance only. The integral that involves $\mathcal{I}m\, Z$ can be done with much bigger binning. Of course, all the computations should have an accuracy that the seed can be picked up.

To evaluate Eq. (2.8) or (2.9), we have the option of either doing the summation over $k$ first or performing the integration over $\omega$ first. In each case, a storage of $N\mathcal{N}$ is required, where $N$ is the number points to store the charge spectral density and $\mathcal{N} \gtrsim \omega_0/(\Delta\omega_r + \Omega_I)$ is the number of previous turns needs to be summed.

## APPENDIX

The voltage $V(\tau,s)$ induced by the perturbed density in the frequency domain has been derived in Section III by summing over $k$ from $-\infty$ to $\infty$. In this appendix, we try to derive the result by summing over *previous* turns only. This is what we do when tracking in the frequency domain. We do it in two ways: performing the integration first or performing the summation first. We point out the assumptions made in each method.

For simplicity, we shall use the perturbed density $\tilde{\rho}_1(\omega)$ of Eq. (4.8). Note that the "poles" at $\omega = \pm 2\pi/\sigma$ are spurious. In fact, with proper definition, $\tilde{\rho}_1(\omega)$ is entire. Thus, we have the choice of positioning these two "poles" slightly below the real $\omega$-axis. The only assumption about the impedance $Z(\omega)$ is that it has poles in the lower half $\omega$-plane only, which is a result of causality. Thus, in the integral of Eq. (4.10), there are poles in the lower half $\omega$-plane only.

Let us denote by

$$\bar{\rho}_1(\omega) = \frac{1}{\omega^2 - (2\pi/\sigma)^2} \tag{A.1}$$

the portion of $\tilde{\rho}_1(\omega)$ after the exponentially $\omega$-dependent parts have been removed. Then, Eq. (4.10) can be rewritten as

$$V_1(\tau,s) = -\mathcal{R}e \sum_{k\geq 0} \frac{e^{-i\Omega(s-kC)/v}}{\sigma} \int d\omega \bar{\rho}_1(\omega) Z(\omega) \left[ e^{-i\omega(kT_0-\tau-\sigma)} - e^{-i\omega(kT_0-\tau+\sigma)} \right] \; . \tag{A.2}$$

Noting that $|\tau| < \sigma$, the integration over $\omega$ gives

$$V_1(\tau,s) = \mathcal{R}e \frac{2\pi i}{\sigma} \sum_{\ell} \mathrm{Res} \left\{ \bar{\rho}_1(\omega) Z(\omega) e^{-i\Omega s/v} \times \right.$$

$$\left. \times \left[ \sum_{k>0} e^{-i(\omega-\Omega)kT_0} e^{i\omega(\tau+\sigma)} - \sum_{k\geq 0} e^{-i(\omega-\Omega)kT_0} e^{-i\omega(\sigma-\tau)} \right] \right\}_{\omega=\omega_\ell} , \tag{A.3}$$

where $\omega_\ell$ denotes the poles of $\bar{\rho}_1(\omega)Z(\omega)$ which may be of any order, and Res denotes the corresponding residues. The summation over $k$ can be performed yielding

$$V_1(\tau,s) = \mathcal{R}e \frac{2\pi i}{\sigma} \sum_{\ell} \mathrm{Res} \left\{ \bar{\rho}_1(\omega) Z(\omega) e^{-i\Omega s/v} \left[ \frac{e^{-i(\omega-\Omega)T_0} e^{i\omega(\sigma+\tau)}}{1 - e^{-i(\omega-\Omega)T_0}} - \frac{e^{-i\omega(\sigma-\tau)}}{1 - e^{-i(\omega-\Omega)T_0}} \right] \right\}_{\omega=\omega_\ell} . \tag{A.4}$$

The summation is justified when $\mathcal{I}m\,\omega_\ell - \Omega_I < 0$, with $\omega_\ell$ being the poles of $Z(\omega)$. This is mostly true since $\mathcal{I}m\,\omega_\ell = -\omega_r/2Q = -\Delta\omega_r$, and $\Omega_I > 0$ for a growing seed to our interest so that $|\exp[-i(\omega_\ell - \Omega)T_0]| < 1$, where $Q$ and $\Delta\omega_r$ are the figure of merit Q-factor and full width of the resonance. Therefore, in actual tracking the number of previous turns required is

$$\mathcal{N} \gtrsim \frac{\pi}{T_0(\Delta\omega_r + \Omega_I)} = \frac{\omega_0}{\Delta\omega_r + \Omega_I} \; . \tag{A.5}$$

There may be cases where $\Omega_I < 0$ for a finite period of time before it becomes positive. The above summation can be generalized to include such cases without fundamental changes in the treatment, although the number $\mathcal{N}$ in Eq. (A.5) will have to be enlarged.

When $\omega_\ell = \pm 2\pi/\sigma$ are the poles of $\tilde{\rho}_1$, there is in fact no summation, because only $k = 0$ contributes. When $k \neq 0$, the two terms in Eq. (A.3) actually cancel each other due to the fact that there are no poles in $\tilde{\rho}(\omega)$ originally. It can be easily verified that for $\omega_\ell = \pm 2\pi/\sigma$, the expression in Eq. (A.4) is in fact identical to the $k = 0$ contribution.

Take the first term inside the square brackets of Eq. (A.4). It can be rewritten as

$$\frac{e^{-i(\omega-\Omega)T_0} e^{i\omega(\sigma+\tau)}}{1 - e^{-i(\omega-\Omega)T_0}} = \frac{e^{-i(\omega-\Omega)T_0/2} e^{i(\omega-\Omega)(\sigma+\tau)} e^{i\Omega(\sigma+\tau)}}{2i \sin(\omega-\Omega)T_0/2} \; , \tag{A.6}$$

and further as

$$\frac{e^{i\frac{(\omega-\Omega)T_0}{2\pi}\left[\frac{2\pi(\sigma+\tau)}{T_0} - \pi\right]} e^{i\Omega(\sigma+\tau)}}{2i \sin(\omega-\Omega)T_0/2} \; . \tag{A.7}$$

Now the summation formula

$$\sum_{p=-\infty}^{\infty} \frac{e^{ip\theta}}{p-\nu} = \begin{cases} -\dfrac{\pi}{\sin \pi\nu} e^{i\nu(\theta-\pi)} & 0 < \theta < \pi \\ \\ -\pi \cot \pi\nu & \theta = 0 \end{cases} \tag{A.8}$$

is used to obtain

$$\frac{e^{-i(\omega-\Omega)T_0} e^{i\omega(\sigma+\tau)}}{1 - e^{-i(\omega-\Omega)T_0}} = -\frac{1}{2\pi i} \sum_{p=-\infty}^{\infty} \frac{e^{i2\pi p(\sigma+\tau)/T_0} e^{i\Omega(\sigma+\tau)}}{p - (\omega-\Omega)T_0/2\pi} \; . \tag{A.9}$$

Similarly, the second term in the square brackets of Eq. (A.4) can be reduced to a summation over $p$,

$$\frac{e^{-i\omega(\sigma-\tau)}}{1 - e^{-i(\omega-\Omega)T_0}} = -\frac{1}{2\pi i} \sum_{p=-\infty}^{\infty} \frac{e^{-i2\pi p(\sigma-\tau)/T_0} e^{-i\Omega(\sigma-\tau)}}{p - (\omega-\Omega)T_0/2\pi} \; . \tag{A.10}$$

The net result is

$$V_1(\tau,s) = \mathcal{R}e\, -\frac{2i\omega_0}{\sigma}\sum_p\sum_\ell \mathrm{Res}\left\{\frac{\tilde{\rho}_1(\omega)Z(\omega)}{\omega'-\omega}\right\}_{\omega=\omega_\ell} \sin\omega'\sigma e^{i(\omega'\tau-\Omega s/v)} \ , \tag{A.11}$$

where $\omega_0 = 2\pi/T_0$ and we have used the abbreviation $\omega' = p\omega_0 + \Omega$. We next make the assumption (to be proved below) that

$$\tilde{\rho}_1(\omega')Z(\omega') = \sum_\ell \mathrm{Res}\left\{\frac{\tilde{\rho}_1(\omega)Z(\omega)}{\omega'-\omega}\right\}_{\omega=\omega_\ell} \ , \tag{A.12}$$

and obtain

$$V_1(\tau,s) = \mathcal{R}e\, \omega_0 \sum_p \tilde{\rho}_1(\omega')Z(\omega')e^{i(\omega'\tau-\Omega s/v)} \ , \tag{A.13}$$

which is exactly Eq. (4.11).

We next start over again from Eq. (A.2) by performing the summation over $k$ first. It is clear that for the first term in the square brackets, $k=0$ does not contribute and is therefore excluded. Eventually the integration will be performed by closing the contour in the lower $\omega$-plane. We try to distort the contour from $(-\infty,\infty)$ plus the lower half circle to $C + C'$. As shown in Fig. 4, $C'$ circles the poles of $\bar{\rho}(\omega)$ in the clockwise direction while $C$ runs just above all the poles of $Z(\omega)$ and closes by the lower half circle. Therefore, when $\Delta\omega_r + \Omega_I > 0$, the contour $C$ can be safely taken such that $\mathcal{I}m\,\omega - \Omega_I < 0$ along $C$ (see Fig. 4). Then the summations over $k$ can be carried out giving

$$V_1(\tau,s) = \mathcal{R}e\,\frac{-1}{\sigma}\int_{C+C'} d\omega\bar{\rho}_1(\omega)Z(\omega)e^{-i\Omega s/v}\left[\frac{e^{i\omega(\sigma+\tau)}e^{-i(\omega-\Omega)T_0}}{1-e^{-i(\omega-\Omega)T_0}} - \frac{e^{-i\omega(\sigma-\tau)}}{1-e^{-i(\omega-\Omega)T_0}}\right] . \tag{A.14}$$

As is discussed above, we need not worry about the poles of $\bar{\rho}(\omega)$ because the contribution comes from only the term with $k=0$ so that there is in fact no summation. If we terminate the series by summing up only the previous $\mathcal{N}$ turns in tracking, $\mathcal{N}$ is again given by Eq. (A.5). Equation (A.14) can now be written as

$$V_1(\tau,s) = \mathcal{R}e\,\frac{-1}{\sigma}\int_{C+C'} d\omega\bar{\rho}_1(\omega)Z(\omega)e^{-i\Omega s/v}\times$$

$$\times\left[\frac{e^{-i(\omega-\Omega)T_0/2}e^{i(\omega-\Omega)(\sigma+\tau)}e^{i\Omega(\sigma+\tau)}}{2i\sin(\omega-\Omega)T_0/2} - \frac{e^{i(\omega-\Omega)T_0/2}e^{-i(\omega-\Omega)(\sigma-\tau)}e^{-i\Omega(\sigma-\tau)}}{2i\sin(\omega-\Omega)T_0/2}\right] , \tag{A.15}$$

and using again the summation formula in Eq. (A.8),

$$V_1(\tau,s)=\mathcal{R}e\,\frac{1}{2\pi i\sigma}\sum_{p=-\infty}^{\infty}\int_{C+C'}d\omega\bar{\rho}_1(\omega)Z(\omega)e^{-i\Omega s/v}\times$$

$$\times\left[\frac{e^{i(\Omega+2\pi p/T_0)(\sigma+\tau)}}{p-(\omega-\Omega)T_0/2\pi}-\frac{e^{-i(\Omega+2\pi p/T_0)(\sigma-\tau)}}{p-(\omega-\Omega)T_0/2\pi}\right]\ , \tag{A.16}$$

or

$$V_1(\tau,s)=\mathcal{R}e-\frac{2}{\sigma T_0}\sum_{p=-\infty}^{\infty}\int_{C+C'}d\omega\frac{\bar{\rho}_1(\omega)Z(\omega)}{\omega-(p\omega_0+\Omega)}\sin\omega'\sigma e^{-i\Omega s/v+i\omega'\tau}\ , \tag{A.17}$$

where $\omega'=p\omega_0+\Omega$. The new poles generated, $\omega=p\omega_0+\Omega$, are outside the contour $C+C'$ regardless of the sign of $\Omega_I$ since $\Omega_I-\mathcal{I}m\,\omega_\ell>0$. Doing the integration over $\omega$, we get

$$V_1(\tau,s)=\mathcal{R}e\,\frac{4\pi i}{\sigma T_0}\sum_{p=-\infty}^{\infty}\sum_{\ell}\text{Res}\left\{\frac{\bar{\rho}_1(\omega)Z(\omega)}{\omega-(p\omega_0+\Omega)}\right\}_{\omega=\omega_\ell}\sin\omega'\sigma e^{i(\omega'\tau-\Omega s/v)}\ , \tag{A.18}$$

which reduces to

$$V_1(\tau,s)=\mathcal{R}e\,\omega_0\sum_p\bar{\rho}_1(\omega')Z(\omega')e^{i(\omega'\tau-\Omega s/v)}\ , \tag{A.19}$$

which is the same as Eq. (A.13) when Eq. (A.12) is applied.

Now we are ready to examine the validity of Eq. (A.12). In general, we can write

$$\bar{\rho}_1(\omega')Z(\omega')=\sum_\ell\text{Res}\left\{\frac{\bar{\rho}_1(\omega)Z(\omega)}{\omega'-\omega}\right\}_{\omega=\omega_\ell}+g(\omega')\ , \tag{A.20}$$

where $\omega'$ is any point in the complex plane and $g(\omega')$ is analytic at all the poles $\omega_\ell$. For example, if $\omega_\ell$ is a pole of order $n$,

$$\text{Res}\left\{\frac{\bar{\rho}_1(\omega)Z(\omega)}{\omega'-\omega}\right\}_{\omega=\omega_\ell}=\frac{1}{(n-1)!}\frac{d^{n-1}}{d\omega^{n-1}}\left\{\frac{(\omega-\omega_\ell)^n\bar{\rho}_1(\omega)Z(\omega)}{\omega'-\omega}\right\}_{\omega=\omega_\ell}\ , \tag{A.21}$$

In the special case that all the poles are of first order, Eq. (A.20) reduces simply to the more familiar form

$$\bar{\rho}_1(\omega')Z(\omega')=\sum_\ell\frac{\text{Res}\{\bar{\rho}_1(\omega_\ell)Z(\omega_\ell)\}}{\omega'-\omega_\ell}+g(\omega')\ , \tag{A.22}$$

But with all the exponentially $\omega'$-dependent factors removed, $\bar{\rho}_1(\omega')Z(\omega')$ is analytic at $\omega'=\infty$. Therefore, $g(\omega')$ can only be a constant. However, a dc resistance does not interact with the perturbed density. Thus, $g(\omega')$ must vanish.

## REFERENCES


1. J.A. MacLachlan, Fermilab Report FN-481.

2. J.A. MacLachlan, private communication.

3. Some of the treatment can be found in, for example, A.W. Chao, *Physics of High Energy Particle Accelerators,* SLAC Summer School, 1982, Ed. M. Month, p.353.

4. S.A. Bogacz, private communication.

5. J.A. MacLachlan claims that the negative frequency contributions have been handled correctly (private communication).

6. S. Stahl and S.A. Bogacz, Phys. Rev. **D37** 1300 (1988).

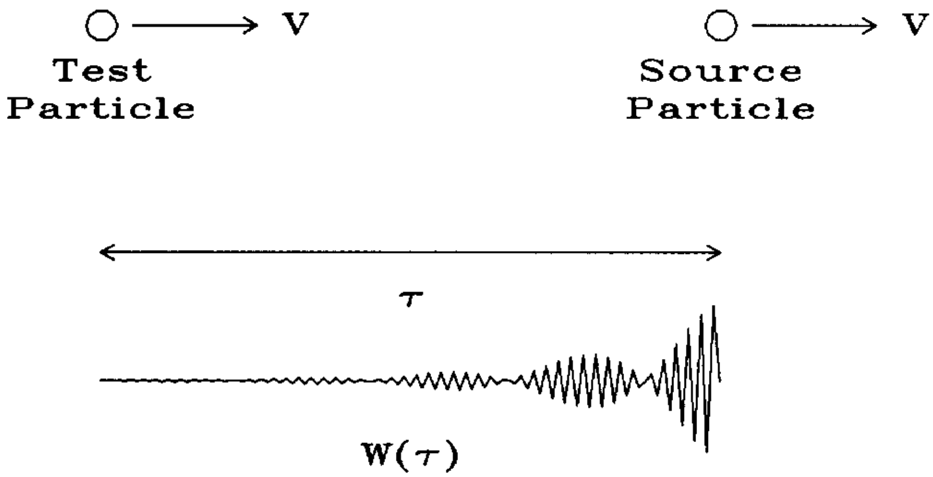


Fig. 1. Wake potential $W(\tau)$ is the electric field experienced by a test particle due to a source particle at a time $\tau$ ahead.

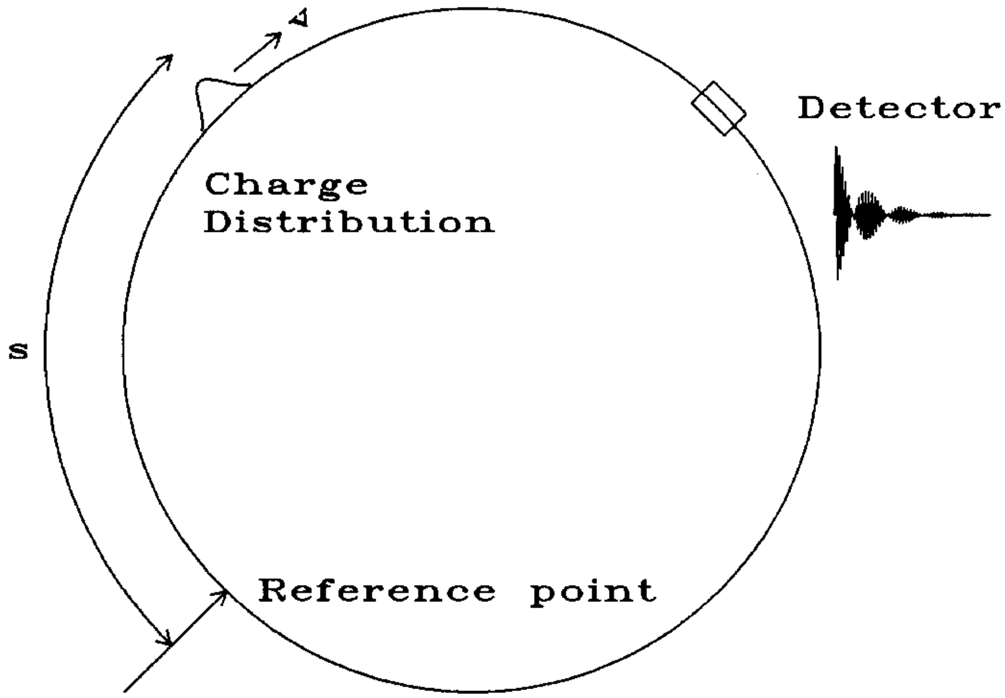


Fig. 2. The detector measures the charge or current distribution in a beam. Note that the measurement is made at a fixed location but at different times.

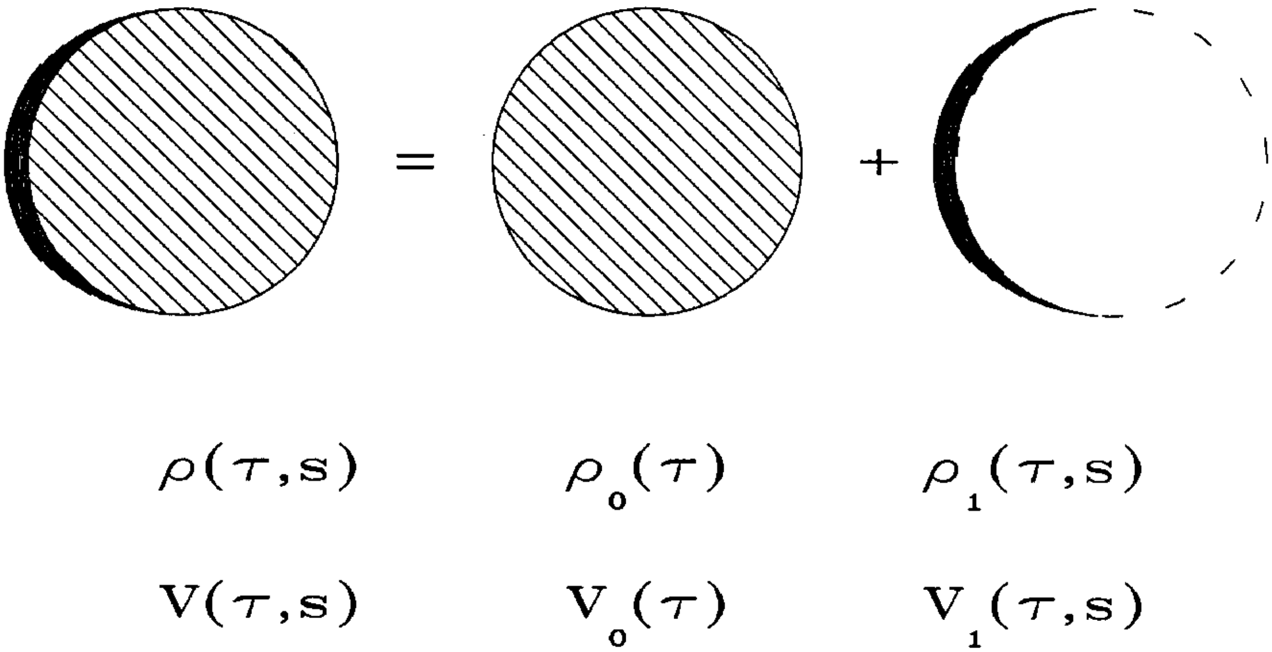


Fig. 3. A bunch distribution in the longitudinal phase space consists of a unperturbed part $\rho_0(\tau')$ which is fitted to the bucket and is therefore independent of $s$, plus a perturbed part $\sum_n \rho_n(\tau', s)$ which oscillates in the bucket. The voltage experienced by a test particle with time advanced $\tau$ can be divided into a corresponding unperturbed part $V_0(\tau)$ plus a perturbed part $\sum_n V_n(\tau, s)$.

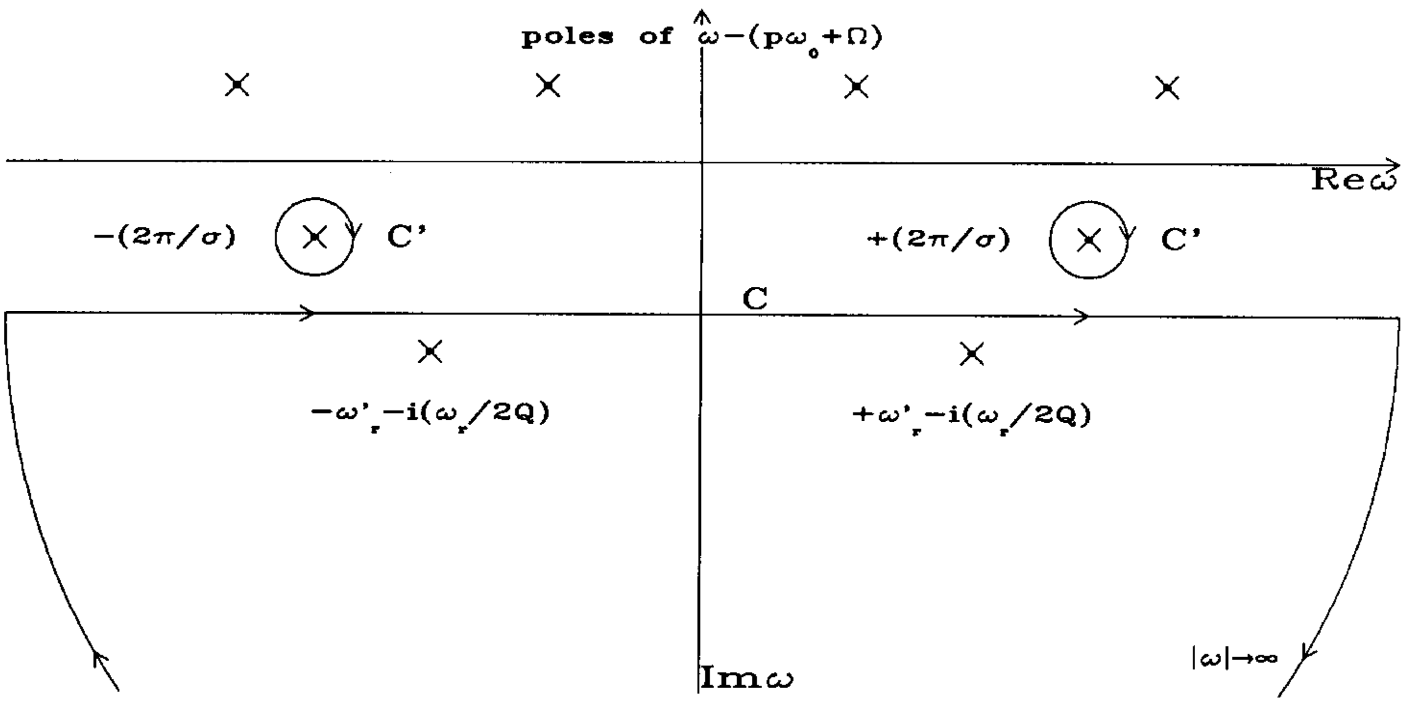


Fig. 4 The contour of integration from $-\infty$ to $+\infty$ and closing by a semicircle of infinite radius in the lower $\omega$-plane is distorted into $C'$ which circles the poles at $\pm 2\pi/\sigma$ and $C$ which runs just above all the poles of $Z(\omega)$ and closes by the lower half circle at infinity.