Fermi National Accelerator Laboratory

FN-481

Fundamentals of Particle Tracking for the Longitudinal Projection of Beam Phasespace in Synchrotrons

James A. MacLachlan
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois 60510

April 15, 1988

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

Fundamentals of Particle Tracking for the Longitudinal Projection of Beam Phasespace in Synchrotrons

James A. MacLachlan

15 April 1988

Abstract

Turn-by-turn tracking of the longitudinal motion of the beam particles in a synchrotron is a useful technique for optimizing parameters for beam manipulation and studying the effect of beam-induced forces. This report gives a derivation of the single-particle, single-turn map and then shows how a realistic description of the time evolution of the longitudinal phasespace distribution can be obtained by adding to the single-particle map the effects of the beam upon itself. These collective effects include forces from spacecharge, image current interactions with cavity resonances and vacuum chamber, and beam feedback, *i.e.*, control of system parameters according to measured properties of the distribution to correct for errors in phase, radial position, and bunch shape. The types of problem for which useful results have been obtained are reviewed briefly.

1 Introduction

A beam of N particles is completely described by 6N functions of time, for example 3N coordinates and 3N components of the momentum. The beam at any particular time can be represented by N points in a six-dimensional space, the beam phasespace. Beam behavior in synchrotrons is often modeled by computer programs which iterate turn-by-turn or fractional-turn mappings derived from the single-particle equations of motion on a typical or otherwise interesting initial phasespace distribution. There is a strong dichotomy between the transverse motion where the forces just guide the beam and the longitudinal motion in which the momentum is boosted; the period of longitudinal (synchrotron) oscillations is commonly $\sim 10^2$ - 10^3 beam turns in a large proton synchrotron whereas the period for the transverse (betatron) oscillations is usually $\sim 10^{-1}$ - 10^{-2} turns so that the longitudinal motion is affected by the properties of the transverse motion averaged over many periods. Thus, the transverse and longitudinal dynamics are generally nearly uncoupled; the possibility of resonances between transverse and longitudinal modes does exist but is unimportant for typical systems. Transverse and longitudinal motion are usually treated in separate calculations, frequently with rather different techniques. Following the evolution of the distribution of transverse phasespace variables by iterative mapping is familiar to many; there are $\sim$ 10 well known transverse tracking codes providing a range of special features and maps to various orders of approximation to the equations of motion.

Turn-by-turn mapping of the longitudinal component of beam phasespace is likewise useful to determine the azimuthal distribution and momentum spread of the beam as it is acted upon by the rf systems. Longitudinal forces from collective effects like image currents, beam excitation of rf cavity resonances, and spacecharge can be included. Phase, radial position, and voltage amplitude feedback systems can be modeled in detail if desired. However, there is no collection of general, time-tested, programs for longitudinal phasespace tracking in any way comparable to those available for the transverse components. The major reason for this disparity is that the transverse behavior of beams is much richer in resonance phenomena *etc.* which need to be investigated. Until rather recently the job of the rf systems has usually been straightforward, and longitudinal phasespace properties have been unimportant for how well the beam serves its intended purpose. For colliding beam physics, however, detailed understanding of the longitudinal phasespace of the beam has proved important. Special rf procedures have been developed to tailor the beam at various stages[1,2], and the final luminosity depends on longitudinal as well as transverse phasespace density. This note is intended to cover the important concepts underlying a particular code, ESME[3], which has been written to provide as much generality and utility as practical for design calculations and the modeling of operating accelerators. An additional goal for ESME has been that it should be easy to use in simple applications. The details of program features and use are covered in program documentation.[4][1]

This note describes fundamentals of longitudinal phasespace tracking techniques. The emphasis is on "longitudinal motion"; less mention is made of "synchrotron oscillations". The words are significant because many approximations which are valid for the oscillatory motion of particles near the synchronous phase are not valid in processes like stacking and phase displacement acceleration. The fundamental mapping which is derived here can be used to derive the well-known differential equation for synchrotron oscillations by introducing appropriate approximations, as will be shown in a discussion of beam feedback.

The next section provides a detailed derivation of the discrete map describing the longitudinal single-particle motion. Section 3 is devoted to collective effects, *i.e.*, longitudinal forces on the particles arising from other beam particles. Beam feedback to system parameters is another kind of collective effect which is treated in Section 4. Some remarks regarding implementation and numerical methods are collected in Section 5. The final section includes some description and references for problems in which the techniques described have proved useful.

2 Single Particle Difference Equations

Difference equations are derived below for the turn-by-turn changes in particle energy and relative azimuth from an idealized model which retains the properties of a real synchrotron

[1] Ref. [4] is largely obsolete. A revised version incorporating major program revisions is being developed.(J. A. MacLachlan, S. Stahl, in preparation.)

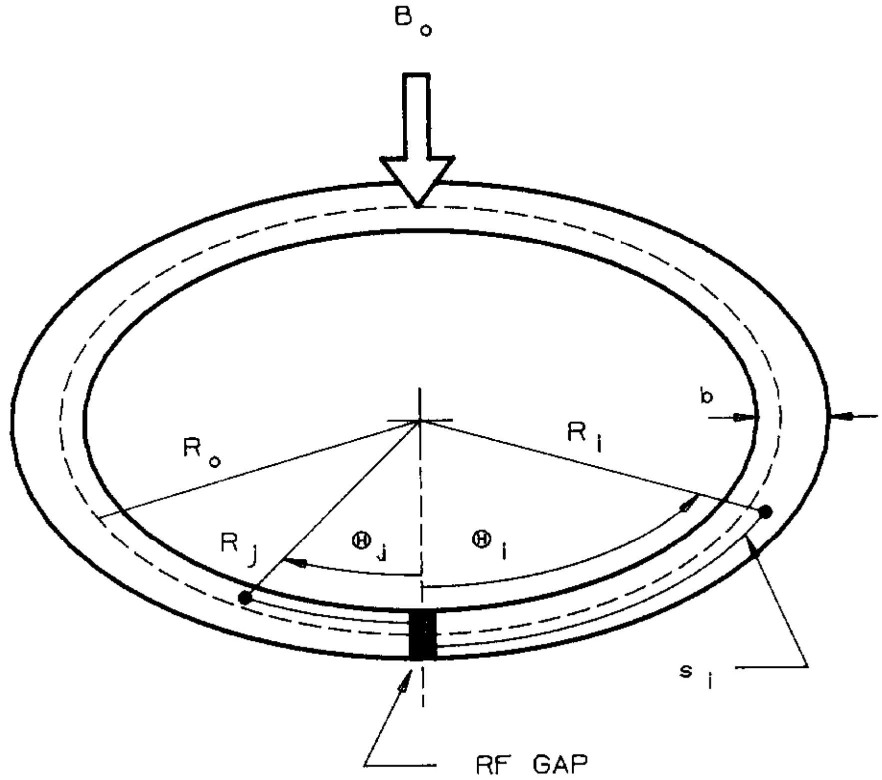

Figure 1: Conceptual model of a synchrotron showing properties and coordinates significant in longitudinal motion

significant for the purely longitudinal dynamics. An adequate characterization of the entire magnet ring becomes very simple if all of the rf is treated as though located at a single narrow gap. The necessary parameters are then just the length C_0 of a reference equilibrium orbit approximately centered in the vacuum chamber, the mean value of the normal magnetic field on that orbit,

$$B_0 = \frac{1}{C_0} \oint B_n \, ds \ ,$$

and α_p which relates change in orbit radius to change in particle momentum:

$$\alpha_p = \frac{\Delta R/R}{\Delta p/p} \ .$$

The model, sketched in Fig. 1, reduces to a toroidal metal vacuum chamber of minor radius b, major radius $R_0 - b$, where $R_0 = C_0/2\pi$, situated in an azimuthally symmetric magnetic field B_0 normal to the midplane of the torus. The metal chamber is interrupted at one azimuth by an insulating gap across which rf voltage may appear. For calculating collective effects the vacuum chamber may be endowed with an arbitrary, frequency-dependent, complex longitudinal coupling impedance, and the rf may have a spectrum of frequencies.

2.1 The Reference Orbit and Quantities Defined Thereon

The reference orbit is a circle of radius R_0 lying in the midplane of the toroidal vacuum chamber; the magnetic field on the orbit is a constant B_0 normal to the midplane. The reference momentum p_0 is the momentum of a proton which follows the orbit:

$$p_0 = eB_0R_0 .$$

Other expressions for p_0 are

$$p_0 = m_0c\beta_0\gamma_0 = m_0\gamma_0v_0 = \frac{\beta_0E_0}{c} ,$$

where E_0 is the total energy, v_0 the speed, β_0 & γ_0 the usual relativistic kinematic parameters, and m_0 the proton rest mass. The angular velocity of particles on the reference orbit is

$$\Omega_0 = \frac{v_0}{R_0} .$$

The definition of the reference orbit does not depend on the rf voltage; it may not be a possible equilibrium orbit when rf is present. Because R_0 is taken to be rigidly fixed by the accelerator geometry, p_0 changes in direct proportion to B_0; it does not depend on any other parameters of the system.

2.2 Synchronous Trajectory and Quantities Defined On It

When the rf acts, one looks for a trajectory of length C_s not necessarily equal to C_0 on which a particle of momentum p_s can circulate. It is called a trajectory rather than an orbit because the particle must be at the right place at the right time. In particular, it must be at the rf gap at the time the voltage is appropriate for the other parameter values. R_0 is fixed, but frequency, voltage amplitude, rf phase, and magnetic field may all be changing slowly, *i.e.* so that $\dot{p}_s/p_s \ll \Omega_s$ and $\dot{C}_s/C_s \ll \Omega_s$, where the dot means total derivative with respect to time and Ω_s is the angular velocity of the particle circulation. The particle is called the synchronous particle because its arrivals at the rf gap are synchronized with the rf period, and the trajectory it follows is called the synchronous trajectory. One need not require that the synchronous trajectory be physically realizable; it may, for example, lie outside of the vacuum chamber while the beam one is interested in circulates quite happily at a different momentum inside the chamber. The property of synchronism implies that the rf voltage waveform is nearly periodic. A general periodic waveform with recurrence frequency $f = \omega/2\pi$ can be written as a real fourier series

$$V(\omega t) = V_0 \sum_{k=1} a_k \sin(k\omega t + \psi_k) . \tag{1}$$

The assumption of slow variation of parameters will be taken to mean that V is precisely this periodic function of t with constant parameters during a circulation period of the synchronous particle but that the parameters can change in small steps turn-by-turn. The basic expression of the synchronous condition is

$$\omega = h\Omega_s ,$$

where h is an integer called the harmonic number of the rf. Frequently each of the non-zero terms in the sum for V will represent an independent rf system. In that case the numbers $h_k = kh$ relating the system frequencies to the synchronous circulation frequency are the harmonic numbers for those systems.

Denote by t_n the time that the synchronous particle crosses the gap at the end of its n-th passage around the ring. The phase at that time is $\varphi_{s,n}$, called the synchronous phase for turn n. It, and consequently t_n, are implicitly defined by

$$\dot{p}_{s,n} = eV(\varphi_{s,n})/C_{s,n} \tag{2}$$

and an additional condition on the slope of V to establish phase stability. The basic requirement for phase stability is that the particle having momentum p_s but arriving at the gap with a small time error should find an rf voltage that changes its momentum in the sense to reduce the time error on the next turn. This phase focusing may be expressed by the condition

$$\begin{array}{lll} V'(\varphi_s) & > 0 & (p_s < p_T) \\ V'(\varphi_s) & < 0 & (p_s > p_T)\,, \end{array}$$

where p_T is the transition momentum, the solution of

$$\left(\frac{\partial\Omega(p)}{\partial p}\right)_{B=const} = 0\,.$$

It is determined by the properties of the guide field. More commonly quoted than the transition momentum is the transition energy $E_T = cp_T/\beta_T$ or $\gamma_T = E_T/m_0c^2$ which is given directly by the magnet lattice parameter α_p:

$$\gamma_T = \alpha_p^{-\frac{1}{2}}\,.$$

For a general periodic waveform there may be more than one phase for which the conditions defining φ_s are satisfied. Such a waveform is atypical, but does occur in practice. In such a case an arbitrary choice of one such point for φ_s is permitted, *e.g.*, the one surrounded by the largest region of bounded motion.

In addition to C_s, p_s, Ω_s, and φ_s there are further quantities defined with reference to the synchronous trajectory which will be used in deriving the single-particle difference equations. The synchronous radius is $R_s = C_s/2\pi$ and r_s is defined as $r_s = R_s - R_0$. Generally $r_s < b \ll R_0$, where b is the beampipe radius. The mean guide field is denoted B_s so one can write

$$p_s = eB_sR_s = m_0c\beta_s\gamma_s = m_0\gamma_sv_s = \frac{\beta_sE_s}{c}$$

as done for the same parameters on the reference orbit. The quantity α_p can be used to relate B_s to B_0. Define $\Delta_p = (p_s - p_0)/p_0$; then

$$eB_sR_s = eB_s(1+\alpha_p\Delta_p)R_0 = (1+\Delta_p)eB_0R_0$$

so that finally

$$B_s = \frac{1+\Delta_p}{1+\alpha_p\Delta_p}B_0 \approx (1+\Delta_p-\alpha_p\Delta_p)B_0\ . \tag{3}$$

α_p is itself a function of momentum and is given as a Taylor expansion relative to its value on the reference orbit:

$$\alpha_p(p_s) = \alpha_0 + \Delta_p\alpha_1 + \Delta_p^2\alpha_2 + \Delta_p^3\alpha_3 + \cdots\ .$$

The coefficients α_i are obtained from lattice codes like SYNCH[5] *et al.*

The synchronous phase can change at constant R_s because of coordinated changes in B_s and the frequency f, but it may of course change at constant B_s because of changes in f with resulting radial displacement:

$$\dot{p}_s = \frac{d}{dt}\,eB_sR_s = e(\dot{r}_sB_s + R_s\dot{B}_s)\ .$$

Note that this equation does not give exact $\dot{p}_s$ explicitly because of the implicit dependence of the righthand side on $\dot{p}_s$ through the expansion of α_p. In a tracking calculation where the small change in p_s is evaluated every beam turn, the evaluation of the righthand side is not a problem.

2.3 Difference Equations in the Lab Coordinate System

In fig. 1 a particle in the ring is shown located by its azimuthal coordinate

$$\Theta_i = \frac{s_i}{R_i}\ ,$$

where s_i is the arc length of its trajectory measured positive and negative with respect to the gap. Thus, Θ_i is a cyclic variable

$$-\pi \le \Theta_i \le \pi\ .$$

Define $t = 0$ to be a time that the synchronous particle departs the gap and t_n to be the time at which it completes the n-th turn and receives its n-th energy kick at the gap. Each passage will take a time interval $\tau_{s,n} = 2\pi h/\omega_n$, where the rf angular frequency is treated as constant at its average value for the duration of a turn. Thus,

$$t_n = \sum_{j=1}^{n}\tau_{s,j}\ .$$

At the end of the n-th turn Θ_i will have the value

$$\begin{aligned}\Theta_{i,n} &= [\Theta_{i,n-1} + \Omega_{i,n}\tau_{s,n} + \pi]_{mod(2\pi)} - \pi \\ &= \left[\Theta_{i,n-1} + 2\pi\frac{\Omega_{i,n}}{\Omega_{s,n}} + \pi\right]_{mod(2\pi)} - \pi\ ,\end{aligned} \tag{4}$$

where $\Omega_{i,n}$ is the angular velocity of the i-th particle. Its total energy will be

$$E_{i,n} = E_{i,n-1} + eV\left(\varphi_{s,n} - h\frac{\Omega_{s,n}}{\Omega_{i,n}}\Theta_{i,n}\right) . \tag{5}$$

In derivations of differential equation(s) for synchrotron oscillation one usually writes the argument of V in eq. 5 as $\varphi_{s,n} - h\Theta_{i,n}$, but the implicit assumption that angular velocities of the synchronous particle and the particle of interest are equal is not justifiable in general.

In both difference equations the ratio of the angular velocities of particle i and the synchronous particle appear. To use the equations one needs to relate these angular velocities to each other through a function of system parameters. In what immediately follows the fractional difference of a quantity between its value on the trajectory of particle i and its value on the synchronous trajectory will appear in the notation

$$\delta_q = \frac{q_{i,n} - q_{s,n}}{q_{s,n}} ,$$

where q is any of the quantities R, p, β, and E. Then

$$\Omega_{i,n} = \frac{v_{i,n}}{R_{i,n}} = \frac{\beta_{s,n}c(1+\delta_\beta)}{R_{s,n}(1+\delta_R)} = \Omega_{s,n}\frac{1+\delta_\beta}{1+\delta_R} = \Omega_{s,n}\frac{\beta_{i,n}R_{s,n}}{\beta_{s,n}R_{i,n}} . \tag{6}$$

The difference equations, eqs. 4 and 5, become

$$\Theta_{i,n} = \left[\Theta_{i,n-1} + 2\pi\frac{1+\delta_\beta}{1+\delta_R} + \pi\right]_{mod(2\pi)} - \pi \tag{7}$$

and

$$E_{i,n} = E_{i,n-1} + eV(\varphi_{s,n} - h\frac{1+\delta_R}{1+\delta_\beta}\Theta_{i,n}) . \tag{8}$$

This form of the mapping could be as convenient as any for calculation, but both δ_R and δ_β can be related to δ_p to give forms suggestive of familiar results. To first order in δ_p

$$\delta_\beta = \gamma_{i,n}^{-2}\delta_p \tag{9}$$

and by definition

$$\delta_R = \tilde{\alpha}_p\delta_p , \tag{10}$$

where $\tilde{\alpha}_p$ is the value of α_p local to the synchronous trajectory and expanded about it rather than the reference orbit:

$$\tilde{\alpha}_p = \tilde{\alpha}_0 + \delta_p\tilde{\alpha}_1 + \delta_p^2\tilde{\alpha}_2 + \delta_p^3\tilde{\alpha}_3 + \cdots .$$

The expansion coefficients $\tilde{\alpha}_i$ are obtained from α_i:

$$\begin{array}{ccccccccccc}
\tilde{\alpha}_0 & = & \alpha_0 & + & \Delta_p\alpha_1 & + & \Delta_p^2\alpha_2 & + & \Delta_p^3\alpha_3 & + & \cdots \\
\tilde{\alpha}_1 & = & \alpha_1 & + & 2\Delta_p\alpha_2 & + & 3\Delta_p^2\alpha_3 & + & \cdots & & \\
\tilde{\alpha}_2 & = & \alpha_2 & + & 3\Delta_p\alpha_3 & + & \cdots & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \cdots .
\end{array}$$

The Θ-equation may be written

$$\Theta_{i,n} = [\Theta_{i,n-1} + 2\pi(1+\delta_\beta)(1-\delta_R+\delta_R^2-\delta_R^3 \pm \cdots) + \pi]_{mod(2\pi)} - \pi$$

$$= \left[\Theta_{i,n-1} + 2\pi[(\delta_\beta - \delta_R - (\delta_\beta-\delta_R)\delta_R + (\delta_\beta-\delta_R)\delta_R^2 \pm \cdots] + \pi\right]_{mod(2\pi)} - \pi\,.$$

Using the expressions for δ_β and δ_R in terms of δ_p only in the common factor

$$\Theta_{i,n} = \left[\Theta_{i,n-1} + 2\pi\delta_p \frac{\gamma_{i,n}^{-2} - \tilde{\alpha}_p}{1+\delta_R} + \pi\right]_{mod(2\pi)} - \pi\,.$$

Defining $\tilde{\eta}_i$ by analogy to $\eta = \alpha_0 - \gamma_s^{-2}$

$$\tilde{\eta}_i = \tilde{\alpha}_p - \gamma_{n,i}^{-2}$$

one has

$$\Theta_{i,n} = \left[\Theta_{i,n-1} - \frac{2\pi\tilde{\eta}_i}{1+\delta_R}\delta_p + \pi\right]_{mod(2\pi)} - \pi\,. \tag{11}$$

It will be shown below that approximations of the character $\delta_R = 0$, $\tilde{\eta}_i = \eta$, *etc.* can lead to well-known results. Such approximations are often very good, but the object here is to write the difference equations with the widest possible domain of validity. There is not much more one can do with the Θ-equation in general.

The phase of the E-equation can be treated in the same fashion:

$$\begin{aligned} E_{i,n} &= E_{i,n-1} + eV(\varphi_{i,n}) \\ \varphi_{i,n} &= \varphi_{s,n} - h\frac{\Omega_{s,n}}{\Omega_{i,n}}\Theta_{i,n} = \varphi_{s,n} - h\left(1 + \frac{\tilde{\eta}_i\delta_p}{1+\delta_\beta}\right)\Theta_{i,n}\,. \end{aligned} \tag{12}$$

These equations in the forms eqs. 4 & 5 or eqs. 7 & 8 constitute a computable one-turn single-particle map for the longitudinal motion essentially without approximation. Eqs. 11 & 12 involve the first order approximation eq. 9 for δ_β which is excellent at high energy.

2.4 Difference Equations in the Synchronous Frame

It is conventional to describe synchrotron oscillations with rf phase and a variable expressing energy or momentum difference between the particle of interest and the synchronous particle. The phase $\varphi = \omega t$ increases monotonically, unlike the cyclic variable Θ which was introduced in section 2.3. One can pass from the preceding description to one based on phase or time quite simply by changing the definition of the azimuthal variable. Define

$$\theta_i = \frac{s_i}{R_i} \quad (0 \le s_i = v_i t)\,.$$

The azimuthal difference equation becomes

$$\theta_{i,n} = \theta_{i,n-1} + \Omega_{i,n}\tau_{s,n} \quad (\theta_{s,n} = \theta_{s,n-1} - 2\pi)\,.$$

The uninteresting secular increase in θ_i of 2π per turn can be removed by transforming to what one may well call a rotating frame by defining a new variable

$$\vartheta_{i,n} = \theta_{i,n} - 2n\pi$$

so that

$$\vartheta_{i,n} = \vartheta_{i,n-1} + 2\pi\left(\frac{\Omega_{i,n}}{\Omega_{s,n}} - 1\right) . \tag{13}$$

This looks simple and clean. Why should one bother with the awkward looking $mod(2\pi)$ appearing in the lab frame Θ-equation? The difficulty lies with the E-equation. As before one must relate the rf phase to the azimuthal variable:

$$\varphi_{i,n} = \varphi_{s,n} - h\frac{\Omega_{s,n}}{\Omega_{i,n}}\theta_{i,n} = \varphi_{s,n} - h\frac{\Omega_{s,n}}{\Omega_{i,n}}(\vartheta_{i,n} + 2n\pi) .$$

For cases in which the beam is captured in a phase stable region centered on $\varphi_{s,n}$ the approximation $\Omega_{i,n} = \Omega_{s,n}$ is excellent.[2] For these cases

$$\varphi_{i,n} = \varphi_{s,n} - h\vartheta_{i,n} , \tag{14}$$

where integral multiples of 2π have been dropped. Thus, the E-equation in this frame is approximately

$$E_{i,n} = E_{i,n-1} + eV(\varphi_{s,n} - h\vartheta_{i,n}) .$$

The synchronous frame is one in which the phase origin is φ_s and the energy $\varepsilon_{i,n}$ is relative to the synchronous energy $E_{s,n}$:

$$\phi_{i,n} = \varphi_{i,n} - \varphi_{s,n}$$

and

$$\varepsilon_{i,n} = \varepsilon_{i,n-1} + eV(-h\vartheta_{i,n}) - eV(0) .$$

Writing these results entirely in (ε, ϕ) coordinates one has

$$\phi_{i,n} = \phi_{i,n-1} - 2\pi h\left(\frac{\Omega_{i,n}}{\Omega_{s,n}} - 1\right) \tag{15}$$

and

$$\varepsilon_{i,n} = \varepsilon_{i,n-1} + eV(\phi_{i,n}) - eV(0) . \tag{16}$$

This is a convenient form of the map to use when the particle of interest is in an rf bucket. It will be helpful later in discussing beam feedback.

[2] It is not necessary that a derivation of a differential equation for synchrotron oscillations depend on the approximation $\Omega_{i,n} = \Omega_{s,n}$; to avoid the approximation avoid the azimuthal variables $\theta_{i,n}$ and $\vartheta_{i,n}$ by retaining time explicitly as the independent variable.

3 Beam Induced Longitudinal Forces

The detailed treatment of the interaction of a beam particle with others through a direct particle-particle force and through longrange and shortrange wakefields excited in the vacuum chamber, rf cavities, *etc.* is beyond the scope of this note. What will be described is perhaps best characterized as an engineering approximation which makes a good phenomenological model.

The beam of particles circulating in the accelerator constitutes a current. The action of the beam as a whole on an individual particle is manifest as an additional accelerating or decelerating voltage acting on the particle each turn. The quantity relating this voltage to the beam current is an impedance of some kind — the net of longitudinal coupling impedance to the vacuum chamber, the shunt impedance of cavity resonances, the self impedance (longitudinal space charge force), *etc.* The constituent impedances are all frequency dependent, and the beam current is not a pure harmonic ac current. Therefore, one fourier analyzes the beam and calculates the voltage generated by each component.

3.1 Fourier Series for Beam Induced Voltage

At the end of the n-th turn the beam has a distribution of particles in azimuth and total energy which can be represented as $Ne\Psi_n(E,\Theta)$ where N is the total number of particles in the beam and Ψ_n is normalized to one. If the spread in particle velocity is small, the distribution will circulate some few turns without change of form. Then the current distribution is just

$$I_{b,n}(\Theta) = \frac{\bar{\Omega}_n Ne}{2\pi}\int \Psi_n(E,\Theta)\,dE = \frac{\bar{\Omega}_n Ne}{2\pi}\Lambda_n(\Theta)\,,$$

where $\bar{\Omega}_n$ is the angular velocity of circulation for a particle having the mean energy

$$\bar{E}_n = \int_{-\pi}^{\pi}\int E\Psi_n(E,\Theta)\,dE\,d\Theta\,.$$

The distribution $\Lambda_n(\Theta)$ circulates with negligible change of shape if

$$\frac{\sigma_\tau}{\tau} = \frac{\eta\sigma_E}{\beta^2\bar{E}} \ll 1\,,$$

where the σ's are the rms spreads resulting from the energy variance

$$\sigma_E^2 = \iint (E-\bar{E})^2\Psi(E,\Theta)\,dE\,d\Theta\,.$$

Generally the width of the energy distribution will be severely limited by available aperture. The mapping approach additionally requires what will be called a quasistatic current distribution $I_{b,n+1}(\Theta) \approx I_{b,n}(\Theta)$ implying $\bar{\Omega}_n \approx \Omega_{s,n}$. If one wants to write the current at the gap as a function of time there is the usual change of sign between phase and azimuth:

$$I_b(\Omega_s t) = \frac{Ne\Omega_s}{2\pi}\sum_m \Lambda_m e^{i(-m\Omega_s t+\psi_m)}\,. \tag{17}$$

The fourier series for the current has been written with real amplitudes multiplied by complex phase factors, that is, as a sum of phasors.

The current produces a beam induced voltage through the total longitudinal coupling impedance $Z_{||}(\omega)$; this quantity evaluated at $m\Omega_s$ will be denoted as the phasor $Z_m e^{i\chi_m}$. The beam-induced voltage is applied to each particle at time t_n when the synchronous particle is at the gap; that voltage depends on the relative phase between the particle and the current. The synchronous particle has phase 0. Thus, the energy increment for the i-th particle on the n-th turn resulting from the voltage induced by the beam current is

$$eV^b_{i,n} = \frac{Ne^2\Omega_s}{2\pi}\sum_m \Lambda_{m,n} Z_m e^{i(m\Theta_{i,n}+\psi_{m,n}+\chi_m)} . \tag{18}$$

3.2 Impedance Sources

The various contributions to the longitudinal coupling are calculable or measurable. Some can be evaluated from averaged properties of the vacuum chamber and others from calculation or measurement of the fields induced in particular structures like rf cavities, beam pickups, bellows, *etc.* Measurements may include beam dynamics experiments and a variety of bench tests of components. With enough such information one can construct a frequency-impedance table which represents the electromagnetic environment of the beam in a particular accelerator with considerable realism. A general model depending only on the average vacuum chamber radius b, the wall resistivity, and average beam radius a is often quite adequate.

3.2.1 Self-impedance

The self-impedance arising from the direct interparticle forces is usually derived from an electrostatic calculation in the beam rest frame which is then transformed to the lab frame.[3] The force arises from the gradient of the azimuthal charge distribution which can be calculated from the fourier series for $\Lambda_n(\Theta)$. The impedance representing the direct particle-particle force is obtained as $Z^{sc}_m e^{i\chi^{sc}_m}$ by relating the fourier coefficients of the spacecharge energy increment with the fourier components of the beam current. The result is

$$\frac{Z^{sc}_m}{m} = -i\frac{Z_0 g}{2\beta\gamma^2} , \tag{19}$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0} = 377\Omega$ and g is a factor containing the dependence on beam and vacuumchamber transverse dimensions. For a uniform cylindrical beam of radius a centered in a smooth beampipe of radius b

$$g = 1 + 2\log(\frac{b}{a}) .$$

[3]done in a notation consistent with that used in this note in Ref. [6]

The impedance is pure imaginary because the bunch as a whole can not gain or lose energy through internal forces. The reactance is capacitive. The sign is consistent with the origin of other contributions to the coupling impedance in the flow of image current in vacuum-chamber wall and various structures. Thus, the self-impedance term can be added directly to other contributions. The spacecharge impedance is usually neglected at high energies but is frequently dominant at injection. It may have sizeable effects on bunch shape matching across transition because it is defocusing below transition and focusing above.

3.2.2 Resistive Wall Impedance

At least for frequencies below the microwave cutoff of the beampipe, coupling impedance to the vacuum chamber can generally be analyzed as the impedance to the flow of the image current in the beampipe or in a special structure. In particular, the impedance arising from a smooth walled metal vacuum chamber is just the surface resistance to the image current[7]

$$\frac{Z_m^w}{m} = \frac{1}{m}\frac{2\pi R_0}{2\pi b}R_{surf} = (1+i)\sqrt{\frac{\bar{\Omega}\rho\mu_0}{2m}}\frac{R_0}{b},$$

where μ_0 is the free-space permittivity and ρ is the volume resistivity. The reactive component is inductive. Because of the $\omega^{-\frac{1}{2}}$ behavior, this term is important only at low frequencies where the image current treatment is appropriate.

3.2.3 Resonant Structures

Not only the rf cavities but such objects as kickers and bellows may have relatively sharp resonances overlapping the spectrum of the beam current. To a first approximation one can usually ignore the fundamental resonance of the rf system because it is so strong that compensatory measures are almost always used. However, for intense beams the effects of beam loading can be a dominant factor in beam behavior. Because the model being described relates induced voltage on the gap to beam current via an impedance it is clearly taking into account only the steady state solution for the response of the resonator. Thus it seems that beam loading with missing bunches, for example, will be very poorly represented if the cavity filling time is comparable to the circulation period. The model, however, does better than one might suppose because in practical cases the distribution of charge changes very little between turns so that the driving term for the transient solution is basically the distribution representing the difference between the charge distributions for successive turns rather than the full charge corresponding to the missing bunches. For the fundamental this remark is pertinent even when the circulation frequency is changing rapidly because the fundamental is of course tuned along with the change in beam energy. For parasitic resonances of the cavities and other devices, however, the use of the steady state solution should tend to overestimate the voltage induced in the resonator by a beam harmonic sweeping across the resonant frequency because there is not sufficient time to develop the steady state fully. A more precise treatment of the transient case is complicated;

note that the usual dispersion theory treatment of longitudinal instabilities uses the same approximation of an impedance to characterize any resonator.

Represent the resonator as an equivalent parallel RLC circuit as shown in fig. 2. The form in fig. 2a is the conventional choice; the parameter R_{sh} has a direct interpretation with respect to the coupling between resonator and beam. However, the circuit of fig. 2b looks more as one might idealize a real cavity, *viz.*, a pure gap capacitance resonating with a lossy inductive reactance. It is shown in the appendix, where the details of the circuit problem are discussed, that the two circuits are equivalent if the resistance parameters are related by $R_{sh} = Q^2 R_{series}$. The impedance can be found by summing the admittance of the parallel branches:

$$Y = \frac{1}{R_{sh}} + \frac{1}{i\omega L} + i\omega C \, .$$

Defining $\omega_0^{-2} = LC$ and $Q = \omega_0 R_{sh} C$ one finds

$$Y = \frac{1}{R_{sh}} + \omega_0 C\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \frac{1}{R_{sh}} + i\frac{Q}{R_{sh}}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

so that

$$Z^{res} = \frac{R_{sh}}{1 + iQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \, . \tag{20}$$

A convenient form for narrow resonances can be obtained by defining $\Delta\omega = \omega - \omega_0$ so that to first order in $\Delta\omega/\omega$

$$Z^{res} \approx \frac{R_{sh}}{1 + 2iQ\Delta\omega/\omega} \, .$$

Writing the denominator in polar form gives $1 + 2iQ\Delta\omega/\omega = e^{i\xi}/\cos\xi$ where ξ, called the tuning angle, is

$$\xi = \arctan(2Q\Delta\omega/\omega) \, . \tag{21}$$

The resonator impedance can then be written

$$Z^{res} \approx R_{sh}\cos\xi e^{-i\xi} \, . \tag{22}$$

The phase of the impedance is inductive below resonance and capacitive above.

3.2.4 Broadband Model

In any accelerator there is coupling impedance unaccounted for by spacecharge, resistive wall, and known sharp resonances. The observed character of this residuum is that generally it is a smooth function of frequency that is inductive in phase and has growing real part up to some frequency about one half the lowest frequency f_c for wave propagation in the beampipe.[8] Above this frequency the real part decreases and the reactive part is capacitive. It has been shown that the spectrum of resonances produced by discontinuities in the vacuum chamber like bellows and size steps becomes very dense at high frequencies and furthermore that the very strong coupling of these "resonators" to the beampipe

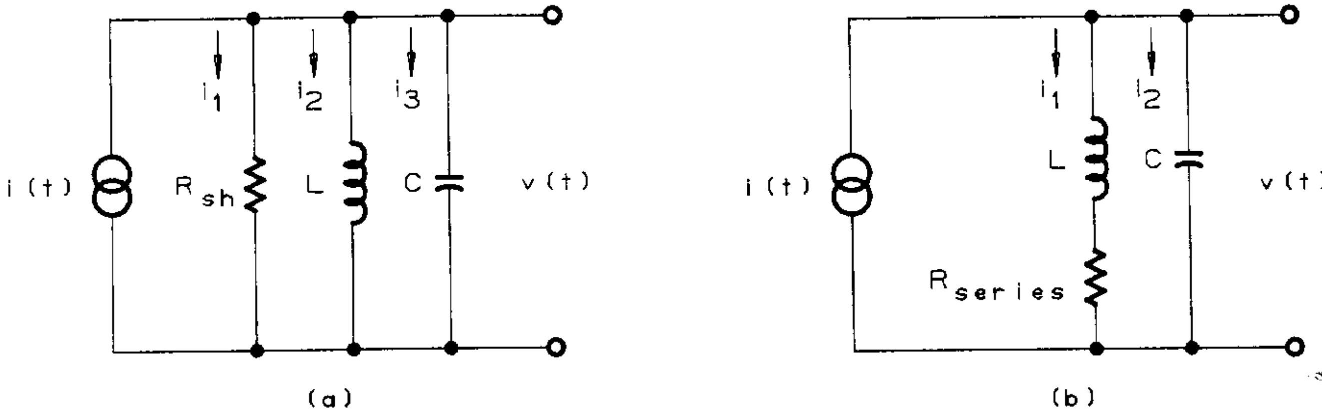

Figure 2: Equivalent RLC circuits for an rf cavity or other high-Q resonator. (a) Conventional representation (b) Physical analog representation

at frequencies $O(f_c)$ dominates the character of the spectrum regardless of the particular discontinuities.[9] This situation is well approximated by the impedance of a single resonator of $Q = 1$, $\omega_0 = O(f_c)$, and $R/n = 1 - 50\ \Omega$ where $n = [\omega_0/\Omega_s]_{integer}$. The magnitude is contained in the resistance parameter which is a measure of the overall smoothness of the vacuum chamber. The value R/n for the Fermilab Booster is approximately 40Ω; the main ring and the CERN SPS come in at about 7Ω; the design for the SSC calls for $R/n < 1\Omega$.

For an existing machine broadband parameters can be estimated from coasting beam energy loss or the threshold for microwave instability. For design simulations an $\omega_0 \approx b/c$ and a choice of R/n based on comparing vacuum chamber design to existing examples should give reasonable guidance to the effects to be expected from broadband coupling impedance. From the very low Q it follows that this impedance represents the effect of extremely shortrange wakefields. Therefore, it is significant primarily for microwave instability, *i.e.*, single-bunch instability driven by wakes shorter than the bunch length. The threshold for microwave instability is known, so tracking calculations are relevant primarily to a few special processes where the threshold will be crossed quickly enough that the dynamic effects are of interest or where the outcome including effects of the instability are of interest. An example of a process of the latter type is the "bunch coalescing" operation in the Fermilab main ring wherein a high intensity bunch is prepared from several less intense bunches for use in the Tevatron collider.[1,10] For this case one wants to know precisely how the final bunch brightness depends on the margin by which the threshold is exceeded.

3.3 Transient Response of a Resonator

Although it has been argued above that representing a resonator by the impedance of an equivalent circuit is generally adequate, there are conditions for which the assumption is dubious. If a fourier component of the beam current is sweeping rapidly across a resonance and the object of the calculation is a careful evaluation of the effect of that resonance, then one can calculate in detail the time dependance of the induced voltage.[4] Calculating from the impedance one has voltage only with the time dependance of the driving current. However, if the driving current is not essentially fixed during a few filling times $2\pi Q/\omega_0$, there are significant transients having time dependance given by the resonant frequency. Unless the Q is extremely high these transients will last very few beam turns, but of course a new one arises each turn. Short decay time makes it practical to calculate the total resonator voltage each turn from the then-current particle distribution and the distributions for a few prior turns.

In an article on the excitation of an RLC resonator by a linearly swept frequency Hok[11][5] treats the problem in which the frequency of the excitation is the only time-dependent parameter. He obtains a result, meaningful in the present context as well, for how rapidly the frequency can change without invalidating the impedance model. For the voltage obtained from the impedance to be correct to better than five percent for all t the criterion

$$\dot{\Omega}_s < \frac{m}{8}\left(\frac{\Omega_s}{Q}\right)^2$$

must be satisfied, where m is the harmonic number of the resonance. To appreciate the domain of relevance of the discussion here, take parameters similar to those for the Fermilab Booster, viz., $R_0 = 75\text{m}$, $\Omega_s \approx 3 \times 10^6$, and acceleration from 200 MeV to 8 GeV in 1/30 second. Assume there is a spurious cavity resonance of $Q \sim 10^4$ at $m = 400$. The rate of change for Ω_s is approximately

$$\frac{\Delta\Omega_s}{\Delta t} = \frac{c}{R_0}\frac{\Delta\beta}{\Delta t} \sim \frac{3 \times 10^8}{75}\frac{.94 - .57}{.03} = 5 \times 10^7 \,.$$

The rhs of the inequality is $\sim 4 \times 10^6$, so the impedance model is not correct. However, if the spurious resonance is swamped to a modest $Q \sim 10^3$ level, the quasi-steadystate solution is fine. Note that $Q \sim 10^4$ is rather extreme; at mode $m = 400$ the filling time for the resonator is about 25 turns. Therefore, this example requires carrying about that many different tables of voltage *vs.* azimuth in the Green's function approach discussed in the next section.

The two algorithms described are proposals for dealing with fast traversal of resonances. The first, based on a Green's function approach, is most natural for low-Q resonances where many harmonics of the rotation frequency are included in the bandwidth but transients

[4] In the program ESME only the steadystate response of a resonator is taken into account. There has been no test of the ideas developed in this section.

[5] Thanks to J. E. Griffin for this reference.

persist for very few turns. The second, based on the excitation of the resonance by fourier components of the beam current, is suited to high-Q resonances where a few harmonics are in the resonance bandwidth. Although neither of these algorithms have been used in tracking calculations, the computational demands look reasonable for at least some interesting cases. In principle there is no fundamental difficulty in tracking simulation of systems for which transient response of resonances plays a major role.

3.3.1 Green's Function Solution

It is shown in the appendix that the voltage produced in a resonant circuit of the type shown in fig. 2a by a single passage of a discrete circulating charge q is[6]

$$\tilde{v}(t) = -\frac{q}{C} e^{-\alpha t} \left(\cos\beta t - \frac{\alpha}{\beta} \sin\beta t \right) , \tag{23}$$

where $\alpha = \omega_0/(2Q) = 1/(2R_{sh}C)$, $\beta^2 = \omega_0^2 - \alpha^2$, and the minus sign on q appears because the excitation is produced by the image charge of the beam. To calculate the beam induced voltage one forms the the longitudinal charge distribution by histogramming number of particles *vs.* azimuth and folds that distribution expressed as a function of time with the Green's function $\tilde{v}(t)$. The folding integral should be evaluated for all times corresponding to the bins defining $\Lambda_n(\Theta)$ so that the voltage acting on a given particle can be found by determining which bin the particle populates. The solutions $\tilde{v}$ for prior turns can be updated to the current turn by relabelling bins to account for the phase change $\beta\tau_s$ and by reducing all values by $e^{-\alpha\tau_s}$. The voltage acting on a given particle is evaluated by summing the voltage corresponding to its azimuthal bin over however many voltage-azimuth tables are retained. The algorithm is easier to give in words or computer code than in conventional mathematical notation. The idea is straightforward, but the number of subscripts gives an illusion of great intricacy.

A calculation of the kind suggested may seem to require a formidable amount of computing, but the binning need not be very fine to represent the interaction of the beam with resonators of frequency of the order of the rf fundamental. Nonetheless, if the Q is high the calculation requires the retention of many prior-turn transients. However, when the Q is high presumably only a few beam harmonics excite the resonance. It is then reasonable to try calculating excitation of transients by sinusoidal currents.

3.3.2 Solution for Beam Current Harmonics

One can fourier analyze the charge distribution each turn and calculate the response produced by harmonics near the resonant frequency exciting the resonator during one turn with initial conditions being the response calculated for the previous turn. It is shown in the appendix that the voltage on a high-Q resonator excited by the m-th fourier component

[6] I am grateful for the guidance from unpublished notes of J. E. Griffin on transient solutions for the RLC resonator, but I have departed sufficiently from the urtext that I claim authorship at least for any errors.

of the image current $i_m \cos(m\Omega_s t + \chi_m)$ is

$$v_m(t) = i_m R\left[\cos(m\Omega_s t + \chi_m - \xi) - e^{-\alpha t}\cos(\beta t + \chi_m - \xi)\right]\cos\xi$$
$$+ \quad v_0 e^{-\alpha t}\cos\beta t + \left[\dot{v}_0 - \frac{i_m \cos\chi_m}{C}\right] e^{-\alpha t}\frac{\sin\beta t}{\beta} \,. \tag{24}$$

where α and β are the same as in eq. 23, ξ is the tuning angle

$$\xi = \arctan(2Q\frac{\omega - \beta}{\omega_0}) \,,$$

and v_0 and $\dot{v}_0$ are the initial conditions for the resonator voltage and its rate of change. The definition of tuning angle is slightly different from the circuit theory expression eq. 21 but numerically the same for high Q. The voltages v_m can be evaluated each turn for every harmonic within the resonance band by retaining just two quantities per harmonic from the previous turn. Thus, unless the the number of harmonics involved is very large, this method will be preferred to the Green's function method. Regardless of computational convenience the use of eq. 24 is limited by the absence of several terms of $O(Q^{-1})$.

4 Beam Feedback

It is possible in principle to specify *a priori* the magnet ramp, frequency program, *etc.* to accelerate beam to a desired energy without significant loss in beam quality, but in practice feedback loops operate to modify system parameters according to measured properties of the beam. The most important of these controls the phase of the rf drive by comparing the bunch arrival times to rf phase to hold the bunch centroids at the synchronous phase. Because the beam current is translated into a change in rf voltage, the transfer functions of the loops are dynamic impedances which act in general to reduce net coupling impedance. Common global loops, *i.e.* loops affecting all bunches more or less the same, are for bunch phase and radial position. Longitudinal dampers for individual bunch motion are also phase feedback devices of very great bandwidth. Although they are not treated in detail below they can be represented in the difference equations as azimuth-dependent feedback. RF amplitude may be controlled to eliminate bunchwidth oscillation deriving from mismatch between bucket and bunch; amplitude feedback generally makes sense only as applied to each bunch separately, when beams are stored for long periods. There are in addition likely to be local loops which lock individual power amplifier output phases to the drive; these tuning loops do not enter directly into the beam dynamics except that they should of course be set to hold the correct sign of the tuning angle to avoid Robinson instability.[12] In a beam loading study the dynamic tuning curve of the rf fundamental could be relevant.

4.1 Phase Feedback

The fundamental property of synchronous acceleration is that the beam is bunched and that the bunches arrive at the rf gap(s) at the correct phase to receive the desired energy increment. Difference in phase from the synchronous value causes particles to oscillate in phase and energy about the synchronous values; phase error of the bunch centroid causes coherent oscillation of the bunch about a center (E_s, φ_s). The negative effects of this synchrotron oscillation are many including obvious ones like use of additional momentum aperture and growth of effective emittance because of nonlinear smearing of the initially coherent motion. By correcting rf phase according to bunch arrival times many times per synchrotron oscillation period, the errors can be kept small and dilution practically eliminated.

By linearizing the difference equations for the phasespace motion of individual particles about the synchronous point (E_s, φ_s) one gets equations which are valid for sufficiently small distances from it and apply equally well to the motion of the centroid of a distribution of particles all sufficiently close. One can pass from difference equations to differential equations without the benefit of mathematical formalities by rewriting eqs. 15 & 16 in a suggestive form

$$\begin{aligned}\frac{\Delta\phi}{\Delta n} &= 2\pi h\left(1-\frac{1+\delta_\beta}{1+\delta_R}\right)\\ \frac{\Delta\varepsilon}{\Delta n} &= eV(\phi)-eV(0)\end{aligned}$$

and adopting the suggestion that ϕ and ε be reinterpreted as smooth functions of a continuous variable n. Then in successive steps we can discard the precise calculation of the phase slip per turn:

$$\begin{aligned}\frac{d\phi}{dn} &\approx 2\pi h(\delta_r-\delta_\beta)\\ &= \frac{2\pi h}{\beta^2}(\tilde{\alpha}_p-\gamma^{-2})\delta_E\\ &\approx \frac{2\pi h\eta}{\beta^2 E_s}\varepsilon\,. \end{aligned} \tag{25}$$

The last step involves replacing α_p at the bunch centroid by its value on the reference orbit and taking it to be independent of momentum. Linearize the rf waveform around φ_s:

$$V(\phi)\approx V(\varphi_s)-\phi V'(\varphi_s)\,.$$

Taking the usual $V(\varphi)=\mathbf{V}\sin(\varphi)$

$$\frac{d\varepsilon}{dn}=e\mathbf{V}\phi\cos\varphi_s\,. \tag{26}$$

Differentiating the ϕ equation eq. 25 and substituting for $\frac{d\varepsilon}{dn}$ from eq. 26 gives a linear second order equation:

$$\frac{d^2\phi}{dn^2}-\frac{2\pi h\eta e\mathbf{V}\cos\varphi_s}{\beta^2 E_s}\phi=0\,.$$

Below transition $\eta < 0$ and $V'(\varphi_s) > 0$ so that the differential equation

$$\phi'' + \nu_s^2 \phi = 0$$

describes bounded harmonic oscillation with frequency (synchrotron tune)

$$\nu_s = \sqrt{\frac{2\pi h|\eta|e\mathbf{V}\cos\varphi_s}{\beta^2 E_s}} .$$

A term in the differential equation proportional to $-\phi'$ damps the oscillations; therefore add to ϕ each turn $g\frac{d\phi}{dn}$ getting

$$\phi'' + \nu_s^2(g\phi' + \phi) = \phi'' - 2\alpha\phi' + \nu_s^2\phi = 0 ,$$

where the $\frac{d\phi}{dn}$ is determined in practice by differencing the centroid position each turn and g is the loop gain. For initial conditions $\phi = \phi_0$ and $\phi' = 0$ the solution is

$$\phi(n) = \phi_0 e^{-\alpha n} \cos\beta n ,$$

where $\beta^2 = \nu_s^2 - \alpha^2$. The most rapid decrease of $\phi(n)$ is obtained in the critically damped condition $\beta = 0$. Then because $\alpha_c^2 = \nu_s^2$ one obtains

$$-2\alpha_c = \nu_s^2 g_c = -2\nu_s$$

so that

$$g_c = -\frac{2}{\nu_s} .$$

In both real accelerator and simulation the nonlinearity of the oscillation will be washing out the error signal as the feedback is trying to correct it. The realistic situation may also be dynamic because of the change of ν_s during the oscillation. Another way in which a realistic feedback system differs from the idealized model above is that precise knowledge of ϕ' implies loop response at least to the beam circulation frequency whereas the typical phase loop has an upper cutoff f_u a few times the synchrotron frequency. The first approximation to limited frequency response is to sample the phase at a rate $O(f_u)$. However, it is possible to represent the loop transfer function in detail by by differencing weighted averages of the phases saved over a number of turns corresponding to f_l, the lower cutoff. The loop should roll off below the synchrotron frequency to avoid biasing the rf frequency by noise or fighting with intentional radial (frequency) offset.

4.2 Radial Position Feedback

Note that in the phase feedback it is not the phase itself that is fed back from the beam but ϕ', which is in quadrature. Another way to get a quadrature signal from the phase is to integrate it. An integrating loop with response past the synchrotron frequency will clearly track the synchrotron oscillation and try to correct it. However, such a loop that cuts off below the synchrotron frequency will provide a signal that is the average value of the phase integrated over the synchrotron period. If bunch arrival time has a non-zero average the circulation frequency must be slipping with respect to the clocking frequency. That is, the rf frequency must be offset from the nominal defining the target synchronous orbit; there is a radial position offset. Within the context described above radial position feedback could be implemented via such an integrating phase feedback loop; however, it is more likely that the phase will be locked to the beam frequency and the radial position corrected by a separate, non-interlaced loop working from a position detector or detectors. In this case the clocking frequency must be $\Omega_s = \omega/h$ and the average of the synchronous phase must be zero. The radial feedback then serves to correct an external frequency program to the magnetic field program. It acts very weakly on the phase loop as an open loop control on the parameter ν_s via E_s. The question of appropriate gain on radial feedback of this kind is not related to beam dynamics but rather questions of noise, precision and delay in the radial position measurement. A deliberate offset from the the reference orbit may be introduced, but, by definition, there is no average offset from the the synchronous orbit.

5 Implementation

This section goes little beyond listing in general terms necessary and desirable functions for a program for tracking simulation. Program design is very important in achieving adaptability and ease of use, but there is little explicit discussion of it here. However, even a very general list suggests natural high-level modularity, which in fact provides the foundation for the outline of a specific program.

The program will contain a kernel routine for mapping a distribution of particles. This is the part of the code where most of the computer time is used and the first place, therefore, to try for computational efficiency. However, mapping code *per se* is likely to be less than five percent of a general purpose program. In the usual order of activation the following functions will be required:

- Description of accelerator and rf systems
- Reading of explicit external time programs or parameters for generic time functions for magnetic field, rf voltage, *etc.*
- Description for and generation of initial phasespace distribution
- Specification of required output
- Tracking
- Calculation on each turn of quantities derived from the distribution for time histories and for collective effects on the mapping equation parameters
- Periodic numerical output

- Periodic graphical output
- Plots of quantities recorded each turn *vs.* time or one another

In both input and internal structure most subsystems can be treated semi-independently because there is a modest number of parameters which directly affect the beam and therefore have global importance, whereas the parameters which define how a given subsystem generates or reacts to global parameters are usually significant only to a small group of closely related functions.

The time dependences are specified most generally by explicit tables of parameters *vs.* time, but this mode is too tedious for general use. However, there are relatively few time functions that appear in common practice, so it will generally be preferable to select from a collection of generic time functions and provide the particular parameters setting scale and duration.

The kinds of particle distribution used for the full range of interesting problems include considerable in addition to elliptical or gaussian bunch distributions and uniform or gaussian coasting beams. Various special distributions based on regular grids, bunch outlines, correlated bunch displacements, *etc.* become very useful at some times. A program should be open-ended in this respect because any new problem may introduce a new requirement. The generation of a matched contour of specified emittance is a primitive that can be shared by a number of different routines which populate matched regions according to different distribution functions. The contour is best provided by using the difference equations with static parameters because there is no assurance that preconceived analytic formulas will apply to arbitrary rf waveforms. The use of weighted distributions to reduce the time spent tracking many particles near the center of a peaked distribution has limited generality and does not seem very important or even safe for a general purpose program. However, applications tracking tens of thousands of particles for thousands of turns certainly invite reconsideration of this point.

A calculation may follow, for example, a simple capture process with one rf frequency or a complex piece of gymnastics with two or more independent systems. There must be some means to preselect interesting material for output so that for the simple process one need not be deluged with the irrelevant. To provide this selectivity by option switches is trivial but of great importance if a general code is to be used productively for simple situations. Default output should be limited to items of significance to most applications.

Tracking consists of executing the single-turn map on each particle in the distribution and calculating some selection of distribution-dependent numbers for output and for calculating the rhs of the energy equation of the map for the following turn. The simplest cases may require no end-of-turn calculation or perhaps just the calculation of mean and width of the bunch projections. Because of the indefinite number of possibilities it is ungainly to include the end-of-turn calculations in the mapping kernel; they are better modularized according to what subsystems are involved and called as needed according to option switches in the kernel. In this context "subsystem" includes not only identifiable technical subsystems like feedback loops but also processes like spacecharge force calculation. Because the kernel is the major focus of computation, a highly optimized code might implement the option switches as preprocessor flags for conditional compilation specific to a given run or data set. This removes conditional code from the tightest loops, but it would be painful for general use.

For widest applicability the map given in eqs. 4 & 5 should be used. However, if the problems of interest are restricted to bounded motion in buckets far from transition, the map can be simplified by using the same β and γ_T for all particles so that the same difference equations can be used for every particle on a given turn to excellent approximation. This simplification results in a major reduction in the amount of computing required for the map. A completely separate version of the kernel containing this simplified map would be useful for appropriate applications and for trial runs in largescale calculations.

Because the mapping proceeds turn-by-turn it is natural to adopt turn number as the running parameter. However, it is likely to be easier to prepare data and interpret output with respect to a time base. The correspondence between the two bases is so simple that the internal mechanics are little changed by using time for input and output conventions.

Most of the substantive problems of numerical analysis involve the treatment of the distribution for fourier analysis and closely related questions of optimum azimuthal binning and number of particles required. The particulars depend in detail on which collective effects are calculated and the impedances present. These matters are typically resolved by trial, but the unwary can mistake an artifact of numerical noise for a beam instability. This subject is discussed at greater length elsewhere[6] in a treatment of spacecharge.

In general, tabular output should be produced infrequently and consist of commonly understood quantities. The output of large tables or numbers expressing the working of special processes should depend on a request for that specific information. Graphical output visible during execution is desirable in a testing stage and as a protection against misdirected running. Furthermore, it may be particularly effective in highlighting the sequence of events in a complicated process. Many large computer systems make execution-time graphics difficult or impossible, however. Thus, it can be efficient to start a problem on a friendly system and migrate to a cruncher when the demands warrant and the data have been tested. The desirability of implementation for more than one computer even for a single problem is one of the arguments in favor of post-processed graphics which could be handled on the most convenient machine while the computation is moved around as required by the scope of the calculation.

The effectiveness of a tracking calculation as a source of insight into process and technique rests heavily on intuitively direct graphics. Plotting distributions at sample times is a good start, but also useful are reference contours like boundaries of buckets or matched regions, projections in energy and phase, curves of calculated quantities *vs.* time or *vs.* each other with time as a parameter, fourier spectra of beam current and feedback error signals, and so on. The most important results of a calculation will usually be expressed graphically, and code for generating graphics will constitute one of the major portions of a general program. There are strong arguments for doing most or all of the graphics in an independent post-processing program. One reason is of course is to a tracking code minimally restricted by local graphics facilities and standards. Probably more important to many is the facility to look at the results of a calculation in several different formats without repeating a phasespace calculation which may require hours of processor time.

The remarks on programs are necessarily superficial for reasons of space; perhaps they seem simpleminded as well. However, of all the things that might be said about programs as opposed to concepts, those mentioned at least allude to matters of major impact on informativeness, ease of use, and general utility of code for longitudinal phasespace tracking. The experience with a particular code used at Fermilab and a few other places underlies much of what has been said. However, there are other programs and there will be additional ones written either because they are more appropriate to a particular range of uses or because the author needs to satisfy himself that the code he uses is correct and correctly understood by him. If such authors aspire to generality they will find that choices made in response to issues of the kind raised in this section are very important in determining the effectiveness of the product.

6 Applications

Some applications have been cited above as particularly illustrative of concepts underlying tracking calculations. In what follows examples are cited to illustrate the diversity of areas in which useful results have been obtained by these techniques. All of the cited material which is published or easily obtained is included in the references. However, not every useful calculation is reported formally. The techniques are applicable in a variety of modes from quick trial of a concept at one extreme to detailed modeling of observations of accelerator operation at the other. A simple check of an idea that can be formulated in a few lines of data and executed in a matter of minutes may be a vital link in a larger piece of work, but it is unlikely to be described in a writeup unless it has been elaborated. The somewhat involved and technical nature of some of the reported work should not discourage the uninitiated from testing ideas and understanding in idealized versions which can be defined by a few parameters.

A design simulation is probably the next qualitative step in realism and, therefore, generally in complexity also, beyond a simple concept test. There are several examples of this mode in the Tevatron I design work.[13] The key idea behind the Debuncher is a pair of complementary bunch rotations in mismatched buckets, one shortening bunches in the Main Ring for $\bar{p}$ production and the other in the Debuncher trading increased time spread for reduced momentum spread.[2] The simulation of these processes served to quantify the momentum reduction to be expected and its dependence on such quantities as γ_T in the Debuncher. Processes associated with the extraction of a $\bar{p}$ bunch in an isolated bucket, its division into small bunches for acceleration in the Main Ring, and its recombination into a single bunch for collision were illustrated in a series of simulations which showed the qualitative features of novel processes and tied down quantitatively the parameters for adiabatic debunching from high-frequency buckets into a low-frequency bucket under simultaneous action of the two rf systems.[1] A similar example where a third rf system is used at the same time to reduce dilution shows that multiple rf frequencies are easily accommodated.[14] An example in which six harmonic numbers appear, but never more than two at once, is the azimuthal charge redistribution scheme used in the CERN PS to prepare protons for antiproton production in the ACOL upgrade of the AA.[15][7] These examples do not involve the calculation of any collective effects, and the data to describe processes at this level is typically a few tens of lines.

The development of new systems for operating accelerators or detailed modeling to understand observations is very likely to include one or more collective effects and consequently is likely to require more extensive and detailed data to specify the problem. However, initial studies can often begin with generic models for coupling impedance, feedback loops, *etc.* which require only a few quantities to characterize the accelerator considered. A detailed study of injection and capture in the Fermilab Booster[16] includes spacecharge effects and 200 MHz microstructure produced by the linac injector. The development of a γ_T-jump system for this same accelerator included simulation of spacecharge, measured wall impedance, realistic phase feedback, lattice non-linearity, *etc.*[18]

There are examples of treatment of collective instability arising from both long-range and short-range wakefields. A study of coupled bunch instability induced by spurious cavity resonances in the Fermilab Booster[17] and demonstration of its amelioration by Landau damping from a higher harmonic cavity, by bunch-by-bunch longitudinal damping, and by bunch-to-bunch synchrotron tune spread by use of radiofrequency systems separated by a few units in harmonic number convincingly illustrates that multi-bunch effects can be treated realistically. Bunch coalescing in the Main Ring is a case where the microwave instability of a single bunch induced by the broadband impedance is a major concern. The effects on final coalesced bunch width of Z/n values within a plausible range go from scarcely detectable to serious.[10]

There have been several studies on details of bunch evolution in passing through transition, some many years old. The very slow passage of transition in the Fermilab accumulator for a gas jet target experiment and in the proposed RHIC heavy ion collider has renewed interest in this topic and may lead to new approaches. The major role played by lattice nonlinearity near transition complicates analytic treatment of the evolution of a phasespace region and is not typically considered in detail. Where customary aids to visualization like a conventional bucket become invalid the availability of a concrete model for phasespace motion is valuable even in the process of defining the problem.

Acknowledgments

I am grateful to Steve Stahl for a close reading of the section on the single-turn mapping and his pointing out of errors in that presentation. I am indebted to Jim Griffin for repeated discussion of the effects on the beam of transients in resonant structures and his personal notes on the Laplace transform approach to the initial value problem.

[7] numerical technique not discussed

References

[1] J. E. Griffin, J. A. MacLachlan, Z. B. Qian," RF Exercises Associated with Acceleration of Intense Antiproton Bunches at Fermilab", IEEE Trans. Nucl. Sci. **30**#4(1983) 2627–2629

[2] J. E. Griffin, J. A. MacLachlan, A. G. Ruggiero, K. Takayama, "Time and Momentum Exchange for the Production and Collection of Intense Antiproton Beams at Fermilab", IEEE Trans. Nucl. Sci. **30**#4(1983) 2630–2632

[3] J. A. MacLachlan, "Particle Tracking in E - ϕ Space as a Design Tool for Cyclic Accelerators", Proc. 1987 IEEE Particle Accelerator Conference held at Washington, D.C.(March 16–19, 1987) 1087–1089

[4] J. A. MacLachlan, "ESME: Longitudinal Phasespace Particle Tracking — Program Documentation", Fermilab TM-1274(May 1984), unpublished (largely obsolete)

[5] A. A. Garren, A. S. Kenney, E. D. Courant, M. J. Syphers, "A Users Guide to Synch", Fermilab FN-420(June 1985), unpublished

[6] J. A. MacLachlan, "Longitudinal Phasespace Tracking with Spacecharge and Wall Coupling Impedance", Fermilab FN-446 (February 1987), unpublished

[7] V. K. Neil and A. M. Sessler, "Longitudinal Resistive Instabilities of Intense Coasting Beams in Particle Accelerators", Rev. Sci. Inst. **36**(1965), p429

[8] A. Hoffman, K. Hübner, and B. Zotter, "A Computer Code for the Calculation of Beam Stability in Circular Electron Machines", IEEE Trans. Nucl. Sci. **NS-26**#3(June 1979)p3514

[9] M. Month and R. Peierls, "Coupling Impedance Structure Above the Cutoff Frequency", Nucl. Inst. Meth. **137**(1976)p299

[10] R. Garoby, "Simulation of Bunches Coalescing in the Main Ring in the Presence of a High-Frequency Wideband Resonator", Fermilab TM-1495 & TM-1495-Rev.(4 December 1987), unpublished

[11] G. Hok, "Response of Linear Resonant Systems to Excitation of a Frequency Varying Linearly with Time", Jour. Appl. Phys. **19**(March 1948) p242

[12] K. Robinson, "Stability of Beam in Radiofrequency System", CEAL-1010(27 February 1964), unpublished

[13] *Design Report Tevatron I Project*, Fermilab(September 1984)

[14] J. A. MacLachlan and J. E. Griffin, "Debunching Into a Bucket of Lower Harmonic Number", Fermilab TM-1504(9 December 1987), unpublished

[15] R. Garoby, "New RF Exercises Envisaged in the CERN-PS for the Antiprotons Production Beam of the ACOL Machine", IEEE Trans. Nucl. Sci. **NS-32**#5(October 1985)pp 2332–2334

[16] S. Stahl and C. Ankenbrandt, "Simulation of the Capture Process in the Fermilab Booster", Proc. 1987 IEEE Particle Accelerator Conference held at Washington, D.C, (16–19 March 1987)p1117

[17] S. Stahl and S. A. Bogacz, "Coupled Bunch Instability in a Circular Accelerator and Possible Cures: Longitudinal Phasespace Simulation", Phys. Rev. **D37**#5(1 March 1988)1300–1306

[18] P. Lucas and J. MacLachlan, "Simulation of Spacecharge Effects and Transition Crossing in the Fermilab Booster", Proc. 1987 IEEE Particle Accelerator Conference held at Washington, D.C, (16–19 March 1987) p1114

Appendix: RLC Resonator Response via Laplace Transform

This appendix provides some of the details of a Laplace transform treatment of the response of the RLC resonator represented by the circuit diagram in fig. 2a. Write the current $i(t)$ from the generator as the sum of the currents in the three parallel branches:

$$i(t) = \frac{v}{R} + \frac{1}{L}\int_0^t v(t')\,dt' + C\frac{dv}{dt} .$$

Apply the Laplace transform assuming that the circuit is entirely quiescent at $t = 0$:

$$\left[\frac{1}{R} + \frac{1}{L}\frac{1}{s} + Cs\right] V(s) = I(s) \tag{27}$$

or

$$V(s) = \frac{1}{C}\frac{s}{s^2 + 2\alpha s + \omega_0^2} I(s) ,$$

where

$$\alpha = \frac{1}{2RC} = \frac{\omega_0}{2Q} .$$

By defining

$$\beta^2 = \omega_0^2 - \alpha^2$$

one can write

$$V(s) = \frac{1}{C}\frac{s + \alpha - \frac{\alpha}{\beta}\beta}{(s+\alpha)^2 + \beta^2} I(s) .$$

This result is now combined with some particular driving currents i(t). If $i(t) = q\delta(t)$ the solution for $v(t)$ comes from the transfer function alone:

$$\check{v}(t) = \frac{q}{C}e^{-\alpha t}\left[\cos\beta t - \frac{\alpha}{\beta}\sin\beta t\right] .$$

This solution is the Green's function for the response of the quiescent resonator; the solution for an arbitrary driving current $i(t)$ is obtained by folding $\check{v}$ with $i(t)$:

$$v(t) = \frac{1}{C}\int_0^t i(t)e^{-\alpha(t-t')}\left[\cos\beta(t-t') - \frac{\alpha}{\beta}\sin\beta(t-t')\right] dt' .$$

The relation between R_{sh} and R_{series} in figs. 2a & 2b can be demonstrated by applying a dc current to the resonator at $t = 0$, *i.e.*, taking $i(t)$ to be a step-function $\bar{\imath}H(t-0)$. By folding

$$\begin{aligned}
\bar{v}(t) &= \frac{\bar{\imath}}{C}\int_0^t H(t')e^{-\alpha(t-t')}\left[\cos\beta(t-t') - \frac{\alpha}{\beta}\sin\beta(t-t')\right] dt' \\
&= e^{-\alpha t}\left\{\mathcal{R}e - \frac{\alpha}{\beta}\mathcal{I}m\right\} e^{i\beta t}\int_0^t e^{(\alpha - i\beta)t'}\,dt' \\
&= \left\{\mathcal{R}e - \frac{\alpha}{\beta}\mathcal{I}m\right\} \frac{1 - e^{-(\alpha - i\beta)t}}{\alpha - i\beta} .
\end{aligned}$$

Define $\zeta = \arctan(\beta/\alpha)$; note $\alpha = \omega_0 \cos\zeta$ and $\beta = \omega_0 \sin\zeta$. Then

$$\bar{v}(t) = \frac{\bar{\imath}}{\omega_0 C}\left\{\mathcal{R}e - \frac{\alpha}{\beta}\mathcal{I}m\right\}[e^{i\zeta} - e^{-\alpha t}e^{i(\beta t+\zeta)}] .$$

Taking the real and imaginary parts,

$$\bar{v}(t) = \frac{\bar{\imath}}{\omega_0 C}\left\{\cos\zeta - \frac{\alpha}{\beta}\sin\zeta - e^{-\alpha t}[\cos(\beta t + \zeta) - \frac{\alpha}{\beta}\sin(\beta t + \zeta)]\right\} .$$

There is a transient at the natural frequency of the damped resonator plus a dc voltage

$$\begin{aligned}\bar{v}_{dc} &= \frac{\bar{\imath}}{\omega_0 C}(\cos\zeta - \frac{\alpha}{\beta}\sin\zeta) \\ &= \frac{\bar{\imath}}{\omega_0 C}\left[\frac{\alpha}{\omega_0} - \frac{\alpha}{\beta}\left(\frac{-\beta}{\omega_0}\right)\right] \\ &= \frac{\bar{\imath}}{\omega_0 C}\frac{2\alpha}{\omega_0} \\ &= \bar{\imath}\frac{R}{Q^2} .\end{aligned}$$

R in this expression is R_{sh} of fig. 2a. The dc current through the fig. 2b version of the circuit is $\bar{\imath}R_{series}$ so that the two circuits are equivalent for dc if $R_{series} = R_{sh}/Q^2$. The complete equivalence of the circuits is established by the fact that the differential equation is the same if α is defined according to this relation between R_{sh} and R_{series}.

If at $t = 0$ a harmonic excitation $i(t) = \bar{\imath}\cos\omega t$ is applied to the quiescent resonator, then

$$\begin{aligned}\bar{v}(t) &= \frac{\bar{\imath}}{C}\int_0^t \cos\omega t'\left[\cos\beta(t-t') - \frac{\alpha}{\beta}\sin\beta(t-t')\right]dt' \\ &= \frac{\bar{\imath}}{C}\frac{1}{2}\left\{\mathcal{R}e - \frac{\alpha}{\beta}\mathcal{I}m\right\}e^{-(\alpha-i\beta)t}\left[\int_0^t e^{i\omega t'}e^{(\alpha-i\beta)t'}dt' + \int_0^t e^{-i\omega t'}e^{(\alpha-i\beta)t'}dt'\right] \\ &= \frac{\bar{\imath}}{2C}\left\{\mathcal{R}e - \frac{\alpha}{\beta}\mathcal{I}m\right\}\left[\frac{e^{i\omega t} - e^{-(\alpha-i\beta)t}}{\alpha + i(\omega-\beta)} + \frac{e^{-i\omega t} - e^{-(\alpha-i\beta)t}}{\alpha - i(\omega+\beta)}\right] . \qquad (28)\end{aligned}$$

The first term in square brackets has the resonance denominator and will give the principal contribution for a high-Q resonance. Define $\delta\omega = \omega - \beta$ and use $\alpha = \omega_0/(2Q)$ to give the resonance denominator the form

$$\alpha + i\delta\omega = \alpha\left[1 + 2iQ\frac{\delta\omega}{\omega_0}\right] = \frac{e^{i\xi}}{\cos\xi} ,$$

where

$$\xi = \arctan 2Q\frac{\delta\omega}{\omega_0} .$$

Note that to first order in Q^{-1}

$$\frac{\alpha}{\beta} = \frac{\alpha}{\sqrt{\omega_0^2 - \alpha^2}} \approx \frac{\alpha}{\omega_0}\left(1 + \frac{1}{2}\frac{\alpha^2}{\omega_0^2}\right) \approx \frac{\omega_0/2Q}{\omega_0} = \frac{1}{2Q} .$$

To find an approximation valid near a high-Q resonance one can neglect all of the terms with α/β coefficients. Thus,

$$\begin{aligned} \tilde{v}(t) &= \frac{\tilde{\imath}}{2C\alpha}\mathcal{R}e\{[e^{i\omega t} - e^{-(\alpha-i\beta)t}]\cos\xi e^{-i\xi}\} + \mathcal{O}(Q^{-1}) \\ &= \tilde{\imath}R[\cos(\omega t - \xi) - e^{-\alpha t}\cos(\beta t - \xi)]\cos\xi . \end{aligned} \tag{29}$$

To first order both transient and steadystate solution are phase shifted by the tuning angle, and the magnitude of both is reduced by the cosine of the tuning angle. The result is substantially more complex for low-Q; only two of eight terms in eq. 28 have been retained in eq. 29.

To first order in Q^{-1} the the addition of an arbitrary phase to the harmonic driving term to give $i(t) = \tilde{\imath}\cos(\omega t + \chi)$ modifies the result simply by changing the phase of the oscillatory terms:

$$\tilde{v}(t) = \tilde{\imath}R[\cos(\omega t + \chi - \xi) - e^{-\alpha t}\cos(\beta t + \chi - \xi)]\cos\xi . \tag{30}$$

To provide for arbitrary initial conditions for the resonator it is simplest to put eq. 27 in the form of a second order differential equation by differentiating through with respect to t:

$$\frac{1}{C}\frac{di}{dt} = \ddot{v} + 2\alpha\dot{v} + \omega_0^2 v .$$

Apply the Laplace transform:

$$\frac{1}{C}[sI(s) - i_0] = s^2V(s) - sv_0 - \dot{v}_0 + 2\alpha[sV(s) - v_0] + \omega_0^2V(s) .$$

Therefore,

$$V(s) = \frac{s}{(s+\alpha)^2 + \beta^2}\left[\frac{1}{C}I(s) + v_0\right] + \frac{\dot{v}_0 + 2\alpha v_0 - i_0/C}{(s+\alpha)^2 + \beta^2} .$$

The first term is just the result of applying the excitation to the quiescent resonator; the next two terms embody the effects of the initial conditions. After combining v_0 parts of the second and third terms one gets to $O(Q^{-1})$

$$v(t) = \tilde{v}(t) + v_0e^{-\alpha t}\cos\beta t + \left[\dot{v}_0 - \frac{\tilde{\imath}}{C}\cos\chi\right]e^{-\alpha t}\frac{\sin\beta t}{\beta} ,$$

where $\tilde{v}$ is the expression given in eq. 30 for the undisturbed resonator. This is the expression suggested in the text for turn-by-turn calculation of the response of the resonator to excitation by a time dependent frequency.