



Fermi National Accelerator Laboratory

FN-365
1500.000
(Submitted to
Lettere al Nuovo Cimento)

A NEW METHOD FOR POTENTIAL
OF A 3-DIMENSIONAL NON-UNIFORM CHARGE DISTRIBUTION

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May 1982

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for

Potential of a 3-Dimensional Non-Uniform Charge Distribution

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Summary

A new simple method for obtaining the expression of the electrical potential for a 3-dimensional non-uniform charge distribution is given. It is shown how the new method can be used to obtain the expression for a 3-dimensional gaussian distribution.

The traditional method for obtaining the potential of a three-dimensional non-uniform charge distribution has relied on its spatial symmetry⁽¹⁾⁽²⁾. In this paper, a simple method is given which is independent of such a spatial symmetry.

We consider the Poisson's equation

$$(1) \quad \nabla^2 \phi(\mathbf{r}) = -4\pi \rho(\mathbf{r}),$$

with the charge distribution $\rho(\mathbf{r})$,

where \mathbf{r} denotes (x, y, z) .

The Green's function corresponding to (1) has been well-known in the form

$$(2) \quad G(\mathbf{r}, \boldsymbol{\xi}) = \frac{1}{4\pi |\mathbf{r} - \boldsymbol{\xi}|},$$

which satisfies the equation

$$(3) \quad \nabla^2 G(\mathbf{r}, \boldsymbol{\xi}) = -\delta(\mathbf{r} - \boldsymbol{\xi}),$$

where $\boldsymbol{\xi}$ denotes (ξ_1, ξ_2, ξ_3) . Now we rewrite the Green's function (2) by an integral representation

$$(4) \quad \frac{1}{4\pi |\mathbf{r} - \boldsymbol{\xi}|} = \frac{1}{2\pi^{3/2}} \int_0^\infty e^{-|\mathbf{r} - \boldsymbol{\xi}|^2 \eta^2} d\eta,$$

which is well-known as the integration formula. Note that couplings among three coordinates (x, y, z) appearing in the

left-hand side of (4) are separated in the integrand. For later application, we make a change of the integration variable

$$(5) \quad t = \frac{1}{r},$$

then

$$(6) \quad G(r, \xi) = \frac{1}{4\pi^{3/2}} \int_0^{\infty} \frac{e^{-\frac{|r-\xi|^2}{t}}}{t^{3/2}} dt.$$

Using (6), we obtain a formal solution of the Poisson's equation

$$(7) \quad \begin{aligned} \phi(r) &= 4\pi \iiint_{-\infty}^{\infty} d\xi G(r, \xi) \rho(\xi) \\ &= \frac{1}{\pi^{1/2}} \int_0^{\infty} \frac{dt}{t^{3/2}} \iiint_{-\infty}^{\infty} d\xi \rho(\xi) e^{-\frac{|r-\xi|^2}{t}} \end{aligned}$$

As an example, we consider a three-dimensional gaussian charge distribution

$$(8) \quad \rho(r) = \frac{1}{(2\pi)^{3/2} abc} \exp\left(-\frac{x^2}{2a^2} - \frac{y^2}{2b^2} - \frac{z^2}{2c^2}\right),$$

where a, b, c are the standard deviations.

Substitution of (8) into (7) yields

$$(9) \quad \phi(r) = \frac{1}{(2\pi)^{3/2} abc \pi^{1/2}} \int_0^{\infty} \frac{dt}{t^{3/2}} \iiint_{-\infty}^{\infty} d\xi \exp\left(-\frac{|r-\xi|^2}{t} - \frac{\xi_1^2}{2a^2} - \frac{\xi_2^2}{2b^2} - \frac{\xi_3^2}{2c^2}\right).$$

Integrals of the form

$$(10) \quad \int_{-\infty}^{\infty} \exp \left[-\frac{(s-\xi)^2}{\tau} - \frac{\xi^2}{2l^2} \right] d\xi,$$

is easily obtained from the integration formula

$$(11) \quad \int_{-\infty}^{\infty} e^{-\alpha^2 \xi^2 + \beta \xi} d\xi = \frac{\sqrt{\pi}}{\alpha} e^{\left(\frac{\beta}{2\alpha}\right)^2} \quad (\beta > 0)$$

Introducing these results into (9), we therefore have

$$(12) \quad \phi(r) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{\exp\left(-\frac{x^2}{2a^2+t} - \frac{y^2}{2b^2+t} - \frac{z^2}{2c^2+t}\right)}{\sqrt{(2a^2+t)(2b^2+t)(2c^2+t)}} dt.$$

The potential (12) is identical with the expression ⁽³⁾ obtained by application of the traditional method, which makes use of ellipsoidal coordinates.

Separation of variables in the Green's function by introducing the auxiliary parameter q has led to the present success, and its exponential form makes it easy to perform the later integrations. For other charge distributions, choice of a suitable integral representation, as the present case may also help us to obtain analytical expressions of their potentials.

References

- (¹) O.D.Kellog : Foundations of Potential Theory (Springer-Verlag ,1967), p.184.
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- (³) B.W.Montague, Fourth-Order Coupling Resonances Excited By Space -Charge Forces In A Synchrotron, CERN/ISR/68-38 (1968).