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TWO-DIMENSIONAL RESONANCE EFFECTS DUE TO A LOCALIZED BI-GAUSSIAN CHARGE DISTRIBUTION

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Introduction

The bi-dimensional effect of a nonlinear kick supplied by one beam to the other in a storage and colliding device had already been investigated by E. Keil[1] and J. LeDuff[2] in the strong beam-weak beam approximation. The strong beam had elliptical cross-section and bi-gaussian distribution of the charge. The nonlinear kick was multiple analyzed and a single resonance was considered, but only the average term and the lowest Fourier mode driving the resonance were taken into account. More recently, A.G. Ruggiero and L. Smith[3] approached the problem again, but with a different technique. They found it possible to describe a single resonance for the case of a round beam taking the exact analytical expression of the nonlinear kick, which means considering the contribution of all multiples of any order. Also it has been possible to take into account the contribution of all the higher Fourier modes driving the same resonance. Nevertheless, their calculation was limited to the one-dimensional case.

The purpose of this paper is to extend this kind of calculation to the bi-dimensional case. We shall still assume a round beam with bi-gaussian distribution. The nonlinear kick is taken to occur over a zero length interval, namely it is represented by a delta-function. The kick is centered to the equilibrium orbit, $x = o$ and $y = o$.

The main application is the calculation of the motion in proximity of a single, isolated and weak resonance; the calculation of the resonances width; and of the stochasticity limit.

Our result for the stochastic limit is higher than the one obtained by Keil.

Equations of Motion

We assume that the strong beam is round, has zero length and that the equilibrium orbit of the test particle is centered on the strong beam.

We shall consider both degrees of freedom. The equations of motion are, then,

$$x'' + k_x(s)\ x = -4\pi\ \frac{\xi_x}{\beta_x^*}\ \frac{1 - e^{-u^2}}{u^2}\ x\ \delta_{int}(s) \tag{1}$$

$$y'' + k_y(s)\ y = -4\pi\ \frac{\xi_y}{\beta_y^*}\ \frac{1 - e^{-u^2}}{u^2}\ y\ \delta_{int}(s) \tag{2}$$

where $' \equiv d/ds$, and

$$u^2 = \frac{x^2 + y^2}{2\sigma^2}\ . \tag{3}$$

The r.h. side of the above equations has been calculated by taking a gaussian of standard deviation σ for the particle distribution. Both beams are considered ultrarelativistic; i.e. $v \sim c$. k_x and k_y are the two unperturbed linear focusing functions. $\delta_{int}(s)$ is the periodic delta function which represents a kick every revolution of circumference $2\pi R$. Also, it is

$$\xi_x/\beta_x^* = \xi_y/\beta_y^* = \xi/\beta^* = Nr_o/4\pi\sigma^2\gamma \tag{4}$$

N = number of particles in the strong bunch

r_o = classical radius of the test particle, to be taken as positive for charges of equal sign and negative for charges of opposite sign

γ = ratio of the total energy of the test particle to its own rest energy.

β_x^* and β_y^* are constant and denote the values of the beta-functions at the crossing point. ξ_x and ξ_y are the usual linear tune shifts per interaction (Amman's notation).

We assume we can solve the homogeneous equations associated to (1) and (2). The solution of these equations is described by the beta-functions β_x and β_y and by the numbers ν_x and ν_y of betatron oscillations per turn.

Transformation to Angle-Action Variables

The transformation to the two pairs of angle-action variables ψ_x, I_x and ψ_y, I_y is accomplished by introducing the following generator

$$S(x,\psi_x,y,\psi_y) = \frac{x^2}{2\beta_x}\left[\text{cotg}\left(\psi_x - \frac{\nu_x}{R}s + \int\frac{ds}{\beta_x}\right) + \frac{\beta_x'}{2}\right] + \frac{y^2}{2\beta_y}\left[\text{cotg}\left(\psi_y - \frac{\nu_y}{R}s + \int\frac{ds}{\beta_y}\right) + \frac{\beta_y'}{2}\right]. \tag{5}$$

The details of the transformation are found in Appendix A. We obtain the following first order differential equations

$$\psi_x' = \nu_x + 4\pi\ \xi_x \frac{1 - e^{-u^2}}{u^2} \sin^2\psi_x\ \delta_{int}(\theta) \tag{6}$$

$$I_x' = -4\pi\ \xi_x \frac{1 - e^{-u^2}}{u^2} I_x \sin 2\psi_x\ \delta_{int}(\theta) \tag{7}$$

and similarly for ψ_y and I_y replacing the index x by y.

The angle $\theta = s/R$ is now the independent variable and prime denotes, from now on, derivative with respect to θ. Also, it is

$$u^2 = \frac{\beta_x^* I_x \sin^2 \psi_x + \beta_y^* I_y \sin^2 \psi_y}{\sigma^2} . \tag{8}$$

Fourier Expansions

The r.h. side of equations (6) and (7) and of the respective equations for ψ_y and I_y are periodic functions of the angles ψ_x, ψ_y and θ with the same period of 2π. By performing a triple Fourier expansion we obtain

$$\psi_x' = \nu_x + 2\xi_x \sum_{nm\ell} f_{nm} (I_x, I_y) e^{i(n\psi_x + m\psi_y - \ell\theta)} \tag{9}$$

$$I_x' = -2\xi_x I_x \sum_{nm\ell} g_{nm} (I_x, I_y) e^{i(n\psi_x + m\psi_y - \ell\theta)} \tag{10}$$

$$\psi_y' = \nu_y + 2\xi_y \sum_{nm\ell} f_{mn} (I_y, I_x) e^{i(n\psi_x + m\psi_y - \ell\theta)} \tag{11}$$

$$I_y' = -2\xi_y I_y \sum_{nm\ell} g_{mn} (I_y, I_x) e^{i(n\psi_x + m\psi_y - \ell\theta)} \tag{12}$$

where

$f_{nm} (I_x, I_y) =$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} \frac{1 - e^{-u^2}}{u^2} \sin^2 \psi_x \, e^{-in\,\psi_x} \cdot e^{-im\,\psi_y} \, d\psi_x \, d\psi_y \tag{13}$$

$g_{nm}\ (I_x, I_y) =$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} \frac{1 - e^{-u^2}}{u^2} \sin 2\psi_x \, e^{-in\psi_x} \cdot e^{-im\psi_y} \, d\psi_x \, d\psi_y \tag{14}$$

and we used

$$\delta_{int}\ (\theta) = \frac{1}{2\pi} \sum_{\ell} e^{-i\ell\theta} \quad . \tag{14a}$$

The Fourier transforms f_{nm} and g_{nm} are calculated in Appendix B.

All the summations extend from $-\infty$ to $+\infty$.

Single Isolated Resonance

We now define a resonance by choosing three integer numbers, with no common divisor, such that the quantity,

$$X = p\psi_x + q\psi_y - r\theta,$$

can be considered as slowly varying, and retain in the triple sum only terms of the form,

$$e^{is(p\psi_x + q\psi_y - r\theta)} \; ,$$

where s is any integer, positive, negative or zero. Equations (9) through (12) then become

$$\psi_x' = \nu_x + 4\xi_x \sum_{s=o}^{\infty} \varepsilon_s \, f_{sp,sq}\ (I_x, I_y) \cos(sX) \tag{15}$$

$$I_x' = -4i\xi_x \; I_x \sum_{s=1}^{\infty} g_{sp,sq} \; (I_x,I_y) \; \sin(sX) \tag{16}$$

$$\psi_y' = \nu_y + 4\xi_y \sum_{s=o}^{\infty} \varepsilon_s \; f_{sq,sp} \; (I_y,I_x) \; \cos(sX) \tag{17}$$

$$I_y' = -4i\xi_y \; I_y \sum_{s=1}^{\infty} g_{sq,sp} \; (I_y,I_x) \; \sin(sX) \tag{18}$$

where $\varepsilon_s = 1$ except $\varepsilon_o = 1/2$, and we have used the relations

$$f_{n,m} = f_{-n,-m} \quad , \quad g_{n,m} = - g_{-n,-m} \quad , \; g_{oo} = 0.$$

In particular, we also have

$$X' = p\nu_x + q\nu_y - r + \sum_{s=o}^{\infty} \left\{ 4p \; \xi_x \; f_{sp,sq} \; (I_x,I_y) + \right.$$

$$\left. + \; 4q \; \xi_y \; f_{sq,sp} \; (I_y, \; I_x) \right\} \varepsilon_s \; \cos(sX) \; . \tag{19}$$

From equations (B9) and (B11) in appendix B and equations (16) and (18) above, it is easily seen that one invariant of the motion is

$$\boxed{W_1 = qI_x - pI_y = \text{constant}} \tag{20}$$

The other invariant is the hamiltonian W_2 which relates the action variables I_x and I_y to the new angle variables

$$\alpha_x = \psi_x - a\theta \;\; , \;\; \alpha_y = \psi_y - \frac{r - pa}{q} \theta$$

with a an arbitrary real number.

Observe that

$$X = p\alpha_x + q\alpha_y$$

and the equations of motion now are

$$\alpha_x' = \nu_x - a + 4\xi_x \sum_{s=o}^{\infty} f_{sp,sq} (I_x, I_y)\, \varepsilon_s \cos s(p\alpha_x + q\alpha_y) \qquad (21)$$

$$I_x' = -4i\, \xi_x I_x \sum_{s=1}^{\infty} g_{sp,sq} (I_x, I_y) \sin s(p\alpha_x + q\alpha_y) \qquad (22)$$

$$\alpha_y' = \nu_y - \frac{r - pa}{q} + 4\xi_y \sum_{s=o}^{\infty} f_{sq,sp} (I_y, I_x)\, \varepsilon_s \cos s(p\alpha_x + q\alpha_y) \qquad (23)$$

$$I_y' = -\, 4i\, \xi_y I_y \sum_{s=1}^{\infty} g_{sq,sp} (I_y, I_x) \sin s(p\alpha_x + q\alpha_y). \qquad (24)$$

Introducing the function $g^*_{sp,sq} (I_x, I_y)$ which is defined by equation (B10) of Appendix B and has the property (B11), we easily derive the hamiltonian W_2 from (22) and (24) above

$$\boxed{\begin{aligned} W_2 = {} & (\nu_x - a) I_x + (\nu_y - \frac{r - pa}{q}) I_y + \\ & - 2i\sigma^2 (\frac{\xi}{\beta^*}) \sum_{s=o}^{\infty} \varepsilon_s\, g^*_{sp,sq} (I_x, I_y) \cos s(p\alpha_x + q\alpha_y). \end{aligned}} \qquad (25)$$

Take note of the quantities

$$\tau_x = \frac{\beta^*_x I_x}{2\sigma^2} \quad , \qquad \tau_y = \frac{\beta^*_y I_y}{2\sigma^2}$$

that have been introduced in Appendix B to short the writing.

As it is shown in Appendix C, we can also write

$$W_2 = (\nu_x - \frac{r}{p})\ I_x + \nu_y\ I_y + 2\sigma^2\ (\frac{\xi}{\beta^*}) \int_0^1 \frac{dt}{t} \Big\{ 1 +$$

$$- e^{-t(\tau_x + \tau_y)} \sum_{\ell=-\infty}^{-\infty} I_{\ell p_o}\ (t\tau_x)\ I_{\ell q_o}\ (t\tau_y)\ \cos\ (\ell X_o) \Big\} . \qquad (26)$$

where p_o, q_o and X_o are defined in Appendix C.

An important result already emerges from equation (26). In the case of a single, isolated and weak resonance the motion is bounded, because for large τ_x and τ_y the dependence of W_2 on X_o vanishes. This is connected to the form of the beam-beam interaction we used in equations (1) and (2), which has the property to decay rather fast as the particle moves further and further away from the origin of the interaction.

The equations of motion are easily derived from eq. (26)

$$I_x' = -\ 4p_o\ \sigma^2\ (\frac{\xi}{\beta^*}) \int_0^1 \frac{dt}{t}\ e^{-t(\tau_x + \tau_y)} \sum_{\ell=-\infty}^{+\infty} \ell\ I_{\ell p_o}(t\tau_x) \cdot$$

$$\cdot\ I_{\ell q_o}(t\tau_y)\ \sin(\ell X_o) \qquad (27)$$

$$I_y' = -\ 4q_o\ \sigma^2\ (\frac{\xi}{\beta^*}) \int_0^1 \frac{dt}{t}\ e^{-t(\tau_x + \tau_y)} \sum_{\ell=-\infty}^{+\infty} \ell\ I_{\ell p_o}(t\tau_x)$$

$$\cdot\ I_{\ell q_o}(t\tau_y)\ \sin(\ell X_o) \qquad (28)$$

$$\alpha_x' = (\nu_x - \frac{r}{p}) + \xi_x \int_0^1 dt\ e^{-t(\tau_x + \tau_y)} \sum_{\ell=-\infty}^{+\infty} \Big[I_{\ell p_o}(t\tau_x) +$$

$$- I_{\ell p_o}'(t\tau_x) \Big] I_{\ell q_o}(t\tau_y)\ \cos(\ell X_o) \qquad (29)$$

$$\alpha_y' = \nu_y + \xi_y \int_0^1 dt\ e^{-t(\tau_x + \tau_y)} \sum_{\ell=-\infty}^{+\infty} \Big[I_{\ell q_o}(t\tau_y) +$$

$$- I_{\ell q_o}'(t\tau_y) \Big] I_{\ell p_o}(t\tau_x) \cos(\ell X_o) \tag{30}$$

where $I_n'(x) \equiv dI_n(x)/dx$.

In particular from the last two equations we derive

$$X' = p\alpha_x' + q\alpha_y' = \varepsilon_{pq} +$$

$$+ p\xi_x \int_0^1 dt\ e^{-(\tau_x + \tau_y)} \sum_{\ell=-\infty}^{+\infty} \Big[I_{\ell p_o}(t\tau_x) - I_{\ell p_o}'(t\tau_x) \Big] \cdot$$

$$\cdot I_{\ell q_o}(t\tau_y) \cos(\ell X_o) + q\xi_y \int_0^1 dt\ e^{-t(\tau_x + \tau_y)} \sum_{\ell=-\infty}^{+\infty} \Big[I_{\ell q_o}(t\tau_y) +$$

$$- I_{\ell q_o}'(t\tau_y) \Big] I_{\ell p_o}(t\tau_x) \cos(\ell X_o) \tag{31}$$

where $\varepsilon_{pq} = p\nu_x + q\nu_y - r$.

Fixed Lines

These are defined as the ensembles of points in the four-dimensional phase space rotating with the resonance, that are unchanged under the action of the same resonance.

The equation of a fixed line is obtained by setting

$$I_x' = I_y' = o \qquad \text{and} \qquad X' = o.$$

From equations (27) and (28), we notice that there are two fixed lines, namely for

$$X_o = o \qquad \text{and} \qquad X_o = \pi .$$

From eq. (31), then, the equations of the two fixed lines are, respectively,

$$p\xi_x \int_o^1 dt\ e^{-t(\tau_x + \tau_y)} \sum_{\ell=-\infty}^{+\infty} \left[I_{\ell p_o}(t\tau_x) - I_{\ell p_o}{}'(t\tau_x) \right] I_{\ell q_o}(t\tau_y) +$$

$$+ q\xi_y \int_o^1 dt\ e^{-t(\tau_x + \tau_y)} \sum_{\ell=-\infty}^{+\infty} \left[I_{\ell q_o}(t\tau_y) - I_{\ell q_o}{}'(t\tau_y) \right] .$$

$$\cdot I_{\ell p_o}(t\tau_x) + \varepsilon_{pq} = o \qquad (32)$$

and

$$p\xi_x \int_o^1 dt\ e^{-t(\tau_x + \tau_y)} \sum_{\ell=-\infty}^{+\infty} \left[I_{\ell p_o}(t\tau_x) - I_{\ell p_o}{}'(t\tau_x) \right] I_{\ell q_o}(t\tau_y)\ (-1)^{\ell} +$$

$$+ q\xi_y \int_o^1 dt\ e^{-t(\tau_x + \tau_y)} \sum_{\ell=-\infty}^{+\infty} \left[I_{\ell q_o}(t\tau_y) - I_{\ell q_o}{}'(t\tau_y) \right] .$$

$$\cdot I_{\ell p_o}(t\tau_x)(-1)^{\ell} + \varepsilon_{pq} = o. \qquad (33)$$

The motion around the fixed lines can be investigated by expanding the r.h. side of eqs.(27), (28) and (31), respectively around $X_o = o$ and $X_o = \pi$. It is found that the $X_o = \pi$ fixed line (eq. 33 above) is a <u>stable</u> line, in the sense that the motion in the proximity is stable. At the same time the $X_o = 0$ fixed line is an unstable line, in the sense that a particle which happened to be in its proximity, leaves it without performing oscillations.

It has been proven that the condition for existence of the fixed lines is

$$- 1 < \frac{\varepsilon_{pq}}{p\xi_x + q\xi_y} < o \tag{34}$$

except for $q = o$ and $p_o = 1$ in which case the condition is

$$- 1 < \frac{p\nu_x - r}{2p\xi_x} < o \ . \tag{34a}$$

Let us consider the three-dimensional space with ε_{pq} on the abscissa and I_x, I_y on the other two axis. Eqs. (32) and (33) represent two surfaces in this space. For fixed I_x and I_y we can go from one surface to another moving parallel to the ε_{pq}-axis. The distance $\Delta\varepsilon_{pq}$ which separates the two surfaces defines the "width" of the resonance at amplitude I_x and I_y. This is obtained by simply subtracting (33) from (32). We have

$$\Delta\varepsilon_{pq} =$$

$$= 4\left| \sum_{\ell, \text{odd}} \left\{ p\xi_x \int_o^1 dt\, e^{-t(\tau_x + \tau_y)} \left[I_{\ell p_o}(t\tau_x) - I_{\ell p_o}{}'(t\tau_x) \right] \cdot \right.\right.$$

$$\left.\left. \cdot I_{\ell q_o}(t\tau_y) + q\xi_y \int_o^1 dt\, e^{-t(\tau_x + \tau_y)} \left[I_{\ell q_o}(t\tau_y) - I_{\ell q_o}{}'(t\tau_y) \right] I_{\ell p_o}(t\tau_x) \right\} \right| \ (\colon$$

where $\sum_{\ell,\text{odd}} \equiv \ell = 1, 3, 5, \ldots.$

To have a general idea of the motion let us observe, first of all, that it is sufficient to consider the case of positive $p \geqslant |q|$. The other case of $p < |q|$ can be reduced to this by exchanging the variables x and y in the equations of motion. Let us consider then the plane of coordinates I_x and I_y. We found that (20) is an invariant, then the motion must occur along straight lines as shown in Fig. 1a and 1b. Each straight line corresponds to one continuous set of initial conditions. For assigned ε_{pq}, eqs. (32) and (33) represent two curves, C_1 and C_2, which lie across the invariant straight lines. The case we show in Fig. 1a and 1b corresponds to $|q_o| > 1$. In this case the curves C_1 and C_2 cross each other at $I_x = o$ and $I_y = o$. The other case with $|q_o| \leqslant 1$ will be considered later. Finally a third curve, W_s, is shown.* The (I_x, I_y) plane is divided in 3 regions. The first region is bounded by the two axis and the curve C_1, and contains the origin 0 which is always a stable fixed point. This region contains all stable small amplitude oscillations with relatively small amplitude and frequency perturbation, the amount of which vanishes as the phase point is closer and closer to the origin 0. The second region is the region of "islands" of stable oscillations around the stable fixed line (curve C_2). The boundary of this region is formed by the curves C_1 and W_S together.

*We do not know really very much about this curve.

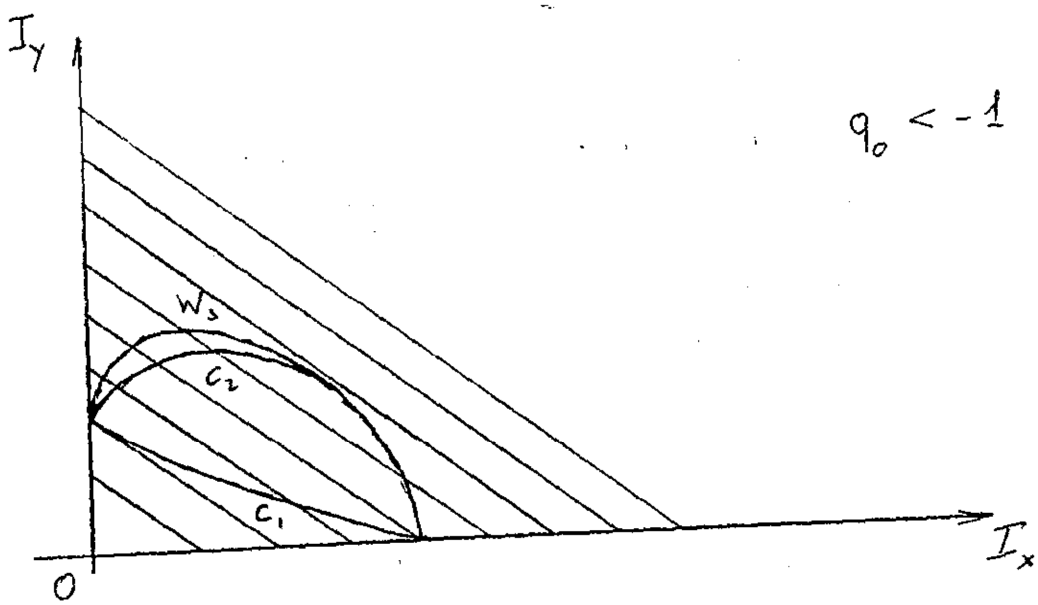

Figure 1a

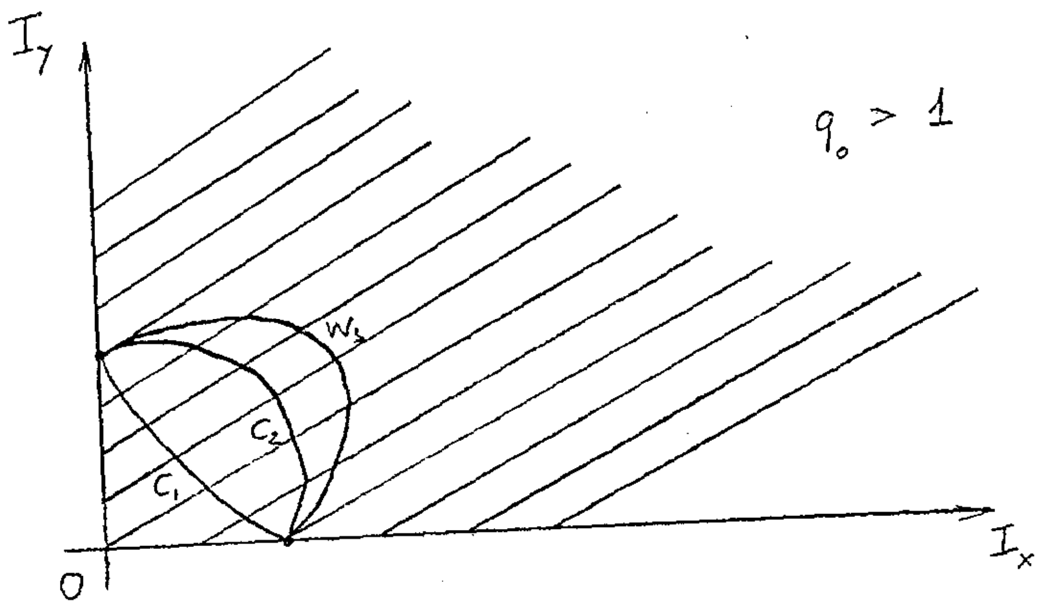

Figure 1b

The third region, which is the remaining of the (I_x, I_y) plane, contains stable large amplitude oscillations with amplitude and frequency modulation, the amount of which vanishes as the amplitude of the oscillation gets larger and larger.

This is pretty much all what we can say, from a general point of view, about a single resonance.

To have a better insight of the motion one should have a more compact expression for the summation

$$R = \sum_{\ell=-\infty}^{+\infty} I_{\ell p_o}(t\tau_x)\, I_{\ell q_o}(t\tau_y) \cos(\ell X_o)$$

which we have been able to derive only for some special cases such as (a) $q = o$ and (b) $p = |q|$, or $\tau_y = o$ (see ref. [3]).

In the following we shall describe these special cases but first we shall look at the so-called "first mode" approximation, where only the $\ell = o$ and $|\ell| = 1$ terms are retained in the Hamiltonian (26).

First Mode Approximation

In the previous paragraph we have calculated the width of a resonance taking into account all the higher Fourier modes. We have found that even as well odd order resonances are possible. This is in contrast to what we get by using the "first mode approximation" where only the average term $\ell = o$ and the next $\ell = |1|$ are retained. In this case, only even order resonances are possible. On the other hand, in the more exact approach, a resonance is defined only when the three integer numbers p, q and r do not have a common divisor. In contrast, in the "first mode approximation", the (p,q,r)-resonance is considered independent of the (ℓp, ℓq, ℓr)-resonance, where ℓ is any integer.

In the "first mode approximation" the width of a resonance is

$$\Delta\varepsilon_{pq} = \left| 4p\ \xi_x \int_0^1 dt\ e^{-t(\tau_x + \tau_y)} \left[I_{\frac{p}{2}}(t\tau_x) - I_{\frac{p}{2}}'(t\tau_x) \right] I_{\frac{q}{2}}(t\tau_y) + \right.$$

$$\left. + 4q\ \xi_y \int_0^1 dt\ e^{-t(\tau_x + \tau_y)} \left[I_{\frac{q}{2}}(t\tau_y) - I_{\frac{q}{2}}'(t\tau_y) \right] I_{\frac{p}{2}}(t\tau_x) \right| \quad (36)$$

Numerical calculation of the ratio of the width as calculated according to eq. (36) to the width calculated taking into account all the Fourier modes, is made difficult by the problem of accuracy. Nevertheless we found that this ratio is substantially different from unity only for very low order of the resonance and large τ_x and τ_y.

A resonance is described also by another parameter, the nonlinear tune shift. This is the distance of the center of the resonance from the linear tune. It is obtained by taking the arithmetic average of (32) and (33). In the "first mode approximation" the tune shift is

$$2p\,\xi_x \int_0^1 dt\, e^{-t(\tau_x + \tau_y)} \left[I_o(t\tau_x) - I_1(t\tau_x) \right] I_o(t\tau_x) \; +$$

$$+ \; 2q\,\xi_y \int_0^1 dt\, e^{-t(\tau_x + \tau_y)} \left[I_o(t\tau_y) - I_1(t\tau_y) \right] I_o(t\tau_x) \qquad (37)$$

which is independent of the order of the resonance. Similarly to eq. (36), also eq. (37) is accurate enough if τ_x and τ_y are not very large.

One-Dimension Resonances

These are defined by setting q = o. From eq. (C7) of Appendix C we have the following Hamiltonian

$$W_2 = (\nu_x - \frac{r}{p})\, I_x + \nu_y\, I_y \; +$$

$$+ \; 2\,\frac{\sigma^2}{p_o}\,(\frac{\xi}{\beta^*}) \sum_{s=1}^{p_o} \int_o^1 \frac{dt}{t} \left[1 - I_o(t\tau_y)\, e^{-t(\tau_x + \tau_y - \tau_x \cos \frac{X_o - 2\pi s}{p_o})} \right] .$$

In this case, I_y is an invariant of the motion. But the vertical tune is not a constant; it is modulated by the motion on the horizontal plane.

Reminding that it is

$$X_o = 2p_o\, \alpha_x$$

the phase equation is

$$X' = (p\nu_x - r) + \frac{p}{p_o}\,\xi_x \sum_{s=1}^{p_o} (1 - \cos \frac{X_o - 2\pi s}{p_o})$$

$$\cdot \int_o^1 dt\, I_o(t\tau_y)\, e^{-t(\tau_x + \tau_y - \tau_x \cos \frac{X_o - 2\pi s}{p_o})}$$

$$= (p\nu_x - r) + \frac{p}{p_o} \xi_x \sum_{s=1}^{p_o} \left(1 - \cos \frac{X_o - 2\pi s}{p_o}\right) e^{-(\tau_x + \tau_y - \tau_x \cos \frac{X_o - 2\pi s}{p_o})}$$

$$\cdot \left\{ I_o(\tau_y) + \frac{I_1(\tau_y)\, \tau_y}{\tau_x + \tau_y - \tau_x \cos \frac{X_o - 2\pi s}{p_o}} \right\} . \tag{38}$$

The fixed points are obtained by setting $X' = o$ with $X_o = o$ and $X_o = \pi$.

The analysis proceeds in the same way as outlined in another paper[3]. Here we have an extra parameter, the invariant I_y, to deal with. In the following we consider the following three cases: (a) p = 1, (b) p = 2, and (c) p = 4.

A. Case: p = 1

This resonance does not exist in the "first mode approximation", but it is real and can be found only by taking into account all the Fourier modes. In the (x, x')-plane there is one unstable fixed point, the origin, and two stable points, diametrically opposite, with coordinate given by the equation

$$r - \nu_x = 2\xi_x e^{-(\tau_y + 2\tau_x)} \left\{ I_o(\tau_y) + \frac{I_1(\tau_y)\tau_y}{\tau_y + 2\tau_x} \right\} . \tag{39}$$

The fixed points exist only when

$$\frac{2\xi_x}{r - \nu_x} > 1 \tag{40}$$

otherwise the motion is always stable, although violent distortions can also be expected.

This resonance can certainly be responsible for beam loss. Particles injected in proximity of the origin can be spilled out along the separatrix as shown in fig. 2. Also, if the collision between two beams occurs in "adiabatic" way, the two beams would be split and locked each inside their own stable areas. In this case the separation of the two beams would be of the order of $2\tau_x$, where τ_x is the solution of (39).

B. Case: p = 2

The motion is generally stable, except when the relation (34a) is satisfied, in which case the flow diagram is still the one shown in fig. 2. The origin is again the unstable fixed point, and the other two points are stable and symmetric, their coordinate being still obtained by solving eq. (39) with r replaced by r/2.

This also, of course, can cause a beam growth and then a limitation on the luminosity achievable.

C. Case p = 4

The flow diagram is the one shown in fig. 3. There are 4 stable and 4 unstable fixed points which exist only when

$$\frac{\xi_x}{\frac{r}{4} - \nu_x} > 1$$

otherwise the all motion is stable.

The location τ_x of the stable and unstable fixed points are obtained by solving the following respective equations

$$(r - 4\nu_x) = 4\xi_x\, e^{-(\tau_x + \tau_y)} \left[I_o(\tau_y) + \frac{\tau_y\, I_1(\tau_y)}{\tau_x + \tau_y} \right]$$

$$(r - 4\nu_x) = 4\xi_x\, e^{-(2\tau_x + \tau_y)} \left[I_o(\tau_y) + \frac{\tau_y\, I_1(\tau_y)}{2\tau_x + \tau_y} \right].$$

The motion is always bounded when $p > 2$ and the origin is always a stable point. No catastrophic effect is then expected from higher order resonances ($p > 2$).

Coupling Resonances

For $q \neq o$, eq. (26) is a two-dimensional Hamiltonian. Nevertheless, because of the existence of the first invariant (20), it is possible to get a one-dimensional Hamiltonian with a proper rotation of the (x, x', y, y') four-dimensional phase-space around the origin. The rotation is accomplished by means of the following generating function

$$S = (p\alpha_x + q\alpha_y)\, W + \alpha_x\, W_1$$

which transforms the old variables ψ_x, I_x and ψ_y, I_y in the new variables X_1, W_1 and X, W through the relations

$$I_x = pW + W_1 \quad , \quad X = p\alpha_x + q\alpha_y$$

$$I_y = qW \quad , \quad X_1 = \alpha_x \quad .$$

We shall investigate here only the case

$$p = |q| \ .$$

From eq. (C7) of Appendix C we have

$$W_2 = (\nu_x - \frac{r}{p}) I_x + \nu_y I_y + \frac{2\sigma^2}{p_o} (\frac{\xi}{\beta^*}) \sum_{s=1}^{p_o} \int_o^1 \frac{dt}{t} \Big[1 +$$

$$- e^{-t(\tau_x + \tau_y)} I_o(t\sqrt{\tau_x^2 + \tau_y^2 + 2\tau_x \tau_y \cos \frac{X_o - 2\pi s}{p_o}}) \Big]$$

from which we derive the phase equation

$$X' = \varepsilon_{pq} + \frac{1}{2p_o} \sum_{s=1}^{p_o} \frac{p\xi_x \frac{\delta S}{\delta \tau_x} + q\xi_y \frac{\delta S}{\delta \tau_y}}{S} \Big[1 - e^{-(\tau_x + \tau_y)} I_o(\sqrt{S}) \Big] +$$

$$+ \frac{e^{-(\tau_x + \tau_y)}}{2p_o} \sum_{s=1}^{p_o} \Big[2p\xi_x + 2q\xi_y - \frac{\tau_x + \tau_y}{S} (p\xi_x \frac{\delta S}{\delta \tau_x} + q\xi_y \frac{\delta S}{\delta \tau_y}) \Big] \cdot$$

$$\cdot \Big[I_o(\sqrt{S}) + \frac{I_1(\sqrt{S})\sqrt{S}}{\tau_x + \tau_y} \Big]$$

where

$$S = \tau_x^2 + \tau_y^2 + 2\tau_x \tau_y \cos \frac{X_o - 2\pi s}{p_o} .$$

The fixed lines can be derived from this equation as usual, by setting $X' = o$ with either $X_o = o$ or $X_o = \pi$.

Let us consider the lowest orders $p = 1$ and $p = 2$ and the case that $\xi_x = \xi_y$, i.e. $\beta_x^* = \beta_y^*$.

A. <u>Difference Resonance, $q < o$</u>

We have the invariant

$$\tilde{W}_1 = \tau_x + \tau_y$$

and the new variable

$$\tilde{W} = \frac{1}{q}\, \tau_y \ .$$

On the $(X, \tilde{W})$-plane there are two fixed points. The origin $\tilde{W} = o$ is the unstable fixed point, and the stable fixed point is obtained by solving the following equation

$$1 + \frac{\varepsilon_{pq}}{2p\xi}(2p\tilde{W} + \tilde{W}_1) - e^{-\tilde{W}_1} (1 + \tilde{W}_1)\, I_o(2p\tilde{W} + \tilde{W}_1) +$$

$$- e^{-\tilde{W}_1} (2p\tilde{W} + \tilde{W}_1)\, I_1(2p\tilde{W} + \tilde{W}_1) = o \ .$$

Observe that the location of the unstable fixed point depends on the invariant $\tilde{W}_1$. The picture of the motion on the $(X, \tilde{W})$-plane is thus similar to the one shown in fig. 2.

B. Sum Resonance, $q > o$

We have the invariant

$$\tilde{W}_1 = \tau_x - \tau_y$$

and the new variable

$$\tilde{W} = \tau_y/q \ .$$

On the $(X, \tilde{W})$-plane there are two fixed points. The stable one has for coordinate the solution of

$$\varepsilon_{pq} + 2p\xi e^{-(2p\tilde{W} + \tilde{W}_1)} \left[I_o(\tilde{W}_1) + \frac{\tilde{W}_1 \; I_1(\tilde{W}_1)}{2p\tilde{W} + \tilde{W}_1} \right] = o$$

and the unstable one has for coordinate the solution of

$$\varepsilon_{pq} \; (2p\tilde{W} + \tilde{W}_1) + 2p\xi \Big[1 + $$

$$- e^{-(2p\tilde{W} + \tilde{W}_1)} \; I_o \; (2p\tilde{W} + \tilde{W}_1) \Big] = o \; .$$

Also here, the location of the fixed points depends on the invariant $\tilde{W}_1$. The picture of the motion of the $(X, \tilde{W})$-plane is similar to the one shown in fig. 4.

Several Crossings per Turn

We assume that there are n_c crossings per turn occurring at homologous locations, i.e. same β_x^* and the same β_y^*, and equally spaced. In this case eq. (14a) is replaced by

$$\delta_{int}(\theta) = \frac{n_c}{2\pi} \sum_{\ell} e^{\; i\ell n_c \theta} \; .$$

All the analyses remain unchanged except the following changes:

- ξ_x and ξ_y are now replaced respectively by $n_c\,\xi_x$ and $n_c\,\xi_y$.
- The only possible isolated resonances are those with the third integer number r that is an algebraic multiple of n_c.

Thus the strength of a resonance (if you want, the width) increases by a factor n_c, but the density of the resonances decreases also by the same amount.

Several Revolutions Between Crossings

In the case the particle receives the two-dimensional, nonlinear kick every n_r revolution, Eq. (14a) is replaced by

$$\delta_{int}(\theta) = \frac{1}{2\pi n_r} \sum_{\ell} e^{-i\frac{\ell}{n_r}\theta} .$$

The only changes are the following:

- ξ_x and ξ_y are now replaced respectively by ξ_x/n_r and ξ_y/n_r.
- A resonance is defined by the three numbers p q and r, where p and q are algebraic integers (p>0), and r is any algebraic multiple of $1/n_r$.

Thus the strength of a resonance is now decreased by the factor n_r, but the density of the resonances increases also, of the same amount.

The Stochasticity Limit

According to Chirikov[4] the stochasticity limit is reached when many nonlinear resonances overlap. As done by Keil, we take as criterion for resonance overlapping that the area covered by resonances in a square region in the (ν_x, ν_y) - plane of unit area becomes unity.

The extension of a resonance in the (ν_x, ν_y) - plane is given by the quantity $\Delta\varepsilon_{pq}$ we have calculated above. This quantity gives the range of $p\nu_x + q\nu_y - r$ which is locked to the resonance. The extension of the same resonance along the ν_x-axis is obviously given by $\Delta\varepsilon_{pq}/p$, and the extension along the ν_y-axis by $\Delta\varepsilon_{pq}/|q|$.

The sum of the areas occupied by resonances is obtained by summing all $\Delta\varepsilon_{pq}/p$ for resonances $p \geq |q|$, and all $\Delta\varepsilon_{pq}/|q|$. for resonances $p < |q|$, where, for obvious reasons, we assume $p > o$.

We first observe that the width of a resonance, $\Delta\varepsilon_{pq}$, does not depend on the number r, also in the "first mode approximation". For assigned p and q there are exactly p resonances all with the same width, in a square region of the (ν_x, ν_y)-plane of unit area, if $p > |q|$ and $|q|$ resonances if $p < |q|$.

Also, to calculate the sum of the area we should use eq. (35) for the width $\Delta\varepsilon_{pq}$, and we should sum over any p and q, because even and odd order resonances are possible.

Nevertheless, one should take into account in the sum only those triplets (p, q, r) that have no common divisor.

Encouraged by the fact that the "first mode" approximation gives an accurate estimate of the resonance width, we shall use eq. (36) instead of eq. (35) in our summation. But in this way, odd order resonances do not give any contribution. To balance this, we shall sum over all possible triplets (p, q, r) including those that do have common divisors.

Denoting the sum by S, we have

$$S = \sum_{p=2}^{\infty} \sum_{|q| \leq p} p \left(\frac{\Delta\varepsilon_{pq}}{p} \right) + \sum_{p=o}^{\infty} \sum_{|q|>p} |q| \left(\frac{\Delta\varepsilon_{pq}}{|q|} \right) \varepsilon_p \tag{41}$$

where p and q are all even integers, and $\varepsilon_p = 1$ but $\varepsilon_o = 1/2$.

By inserting (36) in (41), we obtain the upper limit

$$\frac{S}{4} \leq |\xi_x| \sum_{p=2}^{\infty} \sum_{q=-\infty}^{+\infty} p \left| \int_o^1 dt\, e^{-t(\tau_x + \tau_y)} \left[I_{\frac{p}{2}} (t\tau_x) + \right.\right.$$

$$\left.\left. - I_{\frac{p}{2}}' (t\tau_x) \right] I_{\frac{q}{2}} (t\tau_y) \right| + |\xi_y| \sum_{p=o}^{\infty} \sum_{q=-\infty}^{+\infty} \varepsilon_p |q| \cdot$$

$$\cdot \left| \int_o^1 dt\, e^{-t(\tau_x + \tau_y)} \left[I_{\frac{q}{2}} (t\tau_y) - I_{\frac{q}{2}}' (t\tau_y) \right] I_{\frac{p}{2}} (t\tau_x) \right| .$$

By using the relations

$$\sum_{n=-\infty}^{+\infty} I_n(x) = e^x$$

and

$$\sum_{n=1}^{\infty} n\, I_n(x) = \frac{x}{2}\left[I_o(x) + I_1(x)\right]$$

where, in both summations, n is any integer, odd or even, we finally have

$$S \leq 4\xi_x\, T(\tau_x) + 4\xi_y\, T(\tau_y) \tag{42}$$

where the function

$$T(x) = e^{-x}\left[I_o(x) + I_1(x)\right] \tag{43}$$

is plotted in fig. 5.

It is not difficult to see, by inspecting (41), that all the contributions to S come only from the one-dimensional resonances, namely the $q = o$ resonances contribute to the first term, in ξ_x, and the $p = o$ resonances to the second term, in ξ_y. No explanation is offered, at the moment, why the bi-dimensional resonances ($p \neq o$ and $q \neq o$) do not contribute to the sum S.

In the case one of β_x^* and β_y^* is much smaller than the other, the corresponding term at the r.h. side of (42) can be neglected. The stochasticity limit (S=1) is then reached for $\xi \sim 0.25$. Conversely, if $\beta_x^* = \beta_y^*$, the stochasticity limit is reached for $\xi \sim 0.125$. In both cases, the limit occurs at $x = y = o$.

We cannot avoid to observe that most of the contribution to the function T(x), plotted in fig. 6, comes mostly from the lowest order resonance $p = 2$ or $q = 2$. Indeed if the contribution of this resonance is ignored, T(x) is smaller and given by the lower curve in fig. 5. In this case the stochastic limit is reached at $\tau_x = \tau_y = 1.25$, for $\xi \sim 0.8$ if, say $\beta_x^* >> \beta_y^*$, and for $\xi \sim 0.4$ if $\beta_x^* \sim \beta_y^*$.

This result is in disagreement with Keil's results. The disagreement can be stated in the following way. We found that the contribution of the higher order resonance is smaller than the contribution of the few lowest order resonances. Keil found just the opposite. The discrepancy can be due to (a) the different definition of the resonance width, and/or (b) to the fact that Keil performs multiple expansion of the nonlinear kick and stops the summation to the order 30.

If we are to believe our result, (which, we believe, is in much better agreement with the experimental observations) we infer that the experimental beam-beam limit is mainly caused by few low-order resonances, and that it is rather below the stochastic limit.

In the case of one kick every n_r revolutions or n_c kicks every revolution, we would still obtain the same result if the summation of the resonance widths is taken over a square of area, respectively, $1/n_r^2$ and n_c^2.

Clearly, what is more important, especially in the second case, is a local summation of the widths. Likely the stochastic limit is a function a tune. To prove this, we limited ourselves to the one dimensional case ($\beta_y^* = o$ and $y = o$), then we summed the widths of those resonances that fall in a smaller interval of tune, let us say, between 0.1 and 0.2, or 0.2 and 0.3, and so on. The results are shown in the next table where the maximum tune shift ξ_{max} allowable is reported versus the tune. The amplitude τ_{max} is also shown in the table.

Range of the tune (incl.) - (excl.)	ξ_{max}	$2\tau_{max}$
0.0 - 0.1	0.050	0.0
0.1 - 0.2	0.984	4.9
0.2 - 0.3	0.473	1.4
0.3 - 0.4	0.984	4.9
0.4 - 0.5	2.884	19.0
0.5 - 0.6	0.050	0.0
0.6 - 0.7	0.984	4.9
0.7 - 0.8	0.473	1.4
0.8 - 0.9	0.984	4.9
0.9 - 1.0	2.884	19.0

References

1. E. Keil, CERN/ISR-TH/72-7 and CERN/ISR-TH/72-75
2. J. LeDuff, Laboratoire de l'Accélérateur Linéaire, Groupe Anneaux de Collisions, Rapporte Technique 6-72
3. A.G. Ruggiero and L. Smith, 1973 PEP Summer Study - PEP Note #52.
4. B.V. Cirikov, CERN Trans 71-40

Appendix A

The action variables are related to the angles and to the old variables through the operator (5) by means of the relations

$$I_x = -\frac{\partial S}{\partial \psi_x} \qquad \text{and} \qquad I_y = -\frac{\partial S}{\partial \psi_y} .$$

Similarly, the variables p_x and p_y canonically conjugated, respectively, to x and y, are given by

$$p_x = \frac{\partial S}{\partial x} \qquad \text{and} \qquad p_y = \frac{\partial S}{\partial y} .$$

This yields

$$x = \sqrt{2 I_x \beta_x} \; \sin \left(\psi_x - \frac{\nu_x}{R} s + \int \frac{ds}{\beta_x}\right) \tag{A1}$$

$$p_x = \sqrt{2 \frac{I_x}{\beta_x}} \left[\cos \left(\psi_x - \frac{\nu_x}{R} s + \int \frac{ds}{\beta_x}\right) + \right.$$

$$\left. + \frac{\beta_x'}{2} \sin \left(\psi_x - \frac{\nu_x}{R} s + \int \frac{ds}{\beta_x}\right) \right]$$

and similarly for y and p_y.

The equations of motion (1) and (2) are derived from the Hamiltonian

$$H = \frac{1}{2}\, p_x^{\;2} + \frac{1}{2}\, p_y^{\;2} + \frac{1}{2}\, k_x\,(s)\; x^2 + \frac{1}{2}\, k_y\,(s)\; y^2 +$$

$$+\; 4\pi \,\frac{\xi}{\beta^*}\, \delta_{int}\,(s)\; F\,(x,y)$$

where

$$\frac{\partial F}{\partial x} = \frac{1 - e^{-u^2}}{u^2}\; x \tag{A2}$$

$$\frac{\partial F}{\partial y} = \frac{1 - e^{-u^2}}{u^2}\; y \tag{A3}$$

u^2 being given by (3).

The new Hamiltonian derived by means of the generator (5) is

$$H_1 = H + \frac{\partial S}{\partial s}$$

$$= \frac{\nu_x}{R}\, I_x + \frac{\nu_y}{R}\, I_y + 4\pi\,\frac{\xi}{\beta^*}\,\delta_{int}\,(s)\; F\,(x,y) \tag{A4}$$

where, now, x and y are functions, respectively, of ψ_x, I_x and ψ_y, I_y, and of the independent variable s, as shown, for instance by (A1).

To obtain (A4), we have made use of the known relationship

$$k\beta - \frac{1 + \beta'^2/4}{\beta} = -\,\frac{\beta''}{2} \quad .$$

Observe that the function F(x, y) at the r.h. side of (A4) is multiplied by a delta-function. It is, then, possible to replace the explicit dependence on s with the value of F(x, y) at the location of the kicks. With a proper choice of the origin we have at every kick

$$\frac{\nu_x}{R} s - \int \frac{ds}{\beta_x} = \frac{\nu_y}{R} s - \int \frac{ds}{\beta_y} = o \ .$$

By taking $\theta \equiv s/R$ as independent variable instead of s, which is accomplished by multiplying the Hamiltonian by R, we have finally the new Hamiltonian

$$H_2 = RH_1 = \nu_x I_y + \nu_y I_y +$$

$$+ 4\pi \frac{\xi}{\beta^*} \delta_{int} (\theta) \ F(\sqrt{2I_x \ \beta_x^*} \sin \psi_x \ , \sqrt{2I_y \ \beta_y^*} \sin \psi_y)$$

where we have used the fact that β_x and β_y are both periodic functions of θ with period 2π.

The equations (6) and (7) (and the similar for ψ_y and I_y) are obtained from

$$\psi_x' = \frac{\partial H_2}{\partial I_x} \qquad \text{and} \qquad I_x' = - \frac{\partial H_2}{\partial \psi_x}$$

and by using (A2) and (A3).

Appendix B

Let us make the following expansion

$$\frac{1 - e^{-u^2}}{u^2} = \int_{-\infty}^{+\infty} \tilde{F}(\omega) e^{i\omega u^2} \, du^2 \tag{B1}$$

where

$$\tilde{F}(\omega) = \frac{1}{\pi} \int_0^{\infty} \frac{1 - e^{-x}}{x} \cos \omega x \, dx$$

$$= \frac{1}{2\pi} \log \frac{\omega^2 + 1}{\omega^2} \tag{B2}$$

and let us insert (B1) with (B2) in equations (13) and (14)

$$f_{nm} = \frac{1}{8\pi^3} \int_{-\infty}^{+\infty} d\omega \left(\log \frac{\omega^2 + 1}{\omega^2}\right) \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} e^{i\omega u^2} \sin^2 \psi_x \cdot$$

$$\cdot \, e^{-in\psi_x} e^{-im\psi_y} \, d\psi_x \, d\psi_y \tag{B3}$$

$$g_{nm} = \frac{1}{8\pi^3} \int_{-\infty}^{+\infty} d\omega \left(\log \frac{\omega^2 + 1}{\omega^2}\right) \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} e^{i\omega u^2} \sin 2\psi_x \cdot$$

$$\cdot \, e^{-in\psi_x} e^{-im\psi_y} \, d\psi_x \, d\psi_y \tag{B4}$$

where u^2 is given by equation (8).

Let us introduce the quantities

$$\tau_x = \frac{\beta_x^* I_x}{2\sigma^2} \quad \text{and} \quad \tau_y = \frac{\beta_y^* I_y}{2\sigma^2} \quad . \tag{B5}$$

We can write

$$e^{i\omega u^2} = e^{i\omega(\tau_x + \tau_y)} e^{-i\omega\tau_x \cos 2\psi_x} e^{-i\omega\tau_y \cos 2\psi_y}$$

$$= e^{i\omega(\tau_x + \tau_y)} \sum_{k,h} \Big\{ (-i)^{k+h} J_\kappa(\omega\tau_x) J_h(\omega\tau_y) \cdot$$

$$\cdot\, e^{2i(\kappa\psi_x + h\psi_y)} \Big\} \tag{B6}$$

where the double summation is from $-\infty$ to $+\infty$ and $J_\kappa(x)$ is the Bessel function of first kind and κ-th order.

In deriving (B6) we have made use of the following relation

$$e^{ix \cos\psi} = \sum_{\kappa=-\infty}^{+\infty} i^\kappa J_\kappa(x) e^{i\kappa\psi} \; .$$

By inserting (B6) in (B3) and (B4), by expanding

$$\sin 2\psi_x = \frac{e^{2i\psi_x} - e^{-2i\psi_x}}{2i}$$

$$\sin^2 \psi_x = \frac{2 - e^{2i\psi_x} - e^{-2i\psi_x}}{4}$$

and by making use of the relations

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

$$J_{n-1}(x) - J_{n+1}(x) = 2 J_n'(x)$$

where prime denotes derivation with respect to the argument, we finally obtain

$$f_{nm} = \frac{1}{4\pi} \int_{-\infty}^{+\infty} e^{i\omega(\tau_x + \tau_y)} (\log \frac{\omega^2 + 1}{\omega^2}) (-i)^{\frac{m+n}{2}} J_{\frac{m}{2}} (\omega\tau_y) \cdot$$

$$\cdot \left\{ J_{\frac{n}{2}} (\omega\tau_x) - i \, J'_{\frac{n}{2}} (\omega\tau_x) \right\} d\omega \qquad \text{(B7)}$$

$$g_{nm} = \frac{1}{4\pi} \int_{-\infty}^{+\infty} e^{i\omega(\tau_x + \tau_y)} (\log \frac{\omega^2 + 1}{\omega^2}) (-i)^{\frac{m+n}{2}} \cdot$$

$$\cdot \, J_{\frac{n}{2}} (\omega\tau_x) \, J_{\frac{m}{2}} (\omega\tau_y) \, \frac{n d\omega}{\omega\tau_x} \qquad \text{(B8)}$$

for n and m both even numbers, otherwise

$$f_{nm} = g_{nm} = o \ .$$

In particular equation (B8) can also be written as

$$g_{nm} (I_x, I_y) = \frac{n\sigma^2}{2\beta_x^* I_x} \; g^*_{nm} (I_x, I_y) \qquad \text{(B9)}$$

where

$$g^*_{nm} = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{i\omega(\tau_x + \tau_y)} (\log \frac{\omega^2 + 1}{\omega^2}) (-i)^{\frac{m+n}{2}} \cdot$$

$$\cdot \, J_{\frac{n}{2}} (\omega\tau_x) \, J_{\frac{m}{2}} (\omega\tau_y) \, \frac{d\omega}{\omega} \ . \qquad \text{(B10)}$$

This function has clearly the property

$$g^*_{nm}\,(I_x,\,I_y) = g^*_{mn}\,(I_y,\,I_x)\ . \tag{B11}$$

Appendix C

The Hamiltonian W_2 is given by eq. (25). Observe that as said in Appendix B, $g^*_{sp,sq}$ is not identically zero only when sp and sq are both, at the same time, even integer numbers. Thus, if p and q are both even, the summation at the r.h. side of (25) is over all $s \geqslant o$. On the other side, if at least one of p and q is odd, the summation is carried only over the even values of s, including s = o.

Inserting eq. (B10) of Appendix B in eq. (25) gives, with r = pa and p > o,

$$W_2 = (\nu_x - \frac{r}{p}) I_x + \nu_y I_y +$$

$$- 2i \frac{\sigma^2}{\pi} (\frac{\xi}{\beta^*}) \int_{-\infty}^{+\infty} e^{i\omega(\tau_x + \tau_y)} \log \frac{1 + \omega^2}{\omega^2} G(\omega) \frac{d\omega}{\omega} \tag{C1}$$

where

$$G(\omega) = \sum_{s=o}^{\infty} \varepsilon_s (-i)^{s(p_o + q_o)} J_{sp_o}(\omega\tau_x) J_{sq_o}(\omega\tau_y) \cos (sX_o) \tag{C2}$$

$$X_o = 2(p_o\alpha_x + q_o\alpha_y) \tag{C3}$$

and

$p = 2p_o$, $q = 2q_o$, if both p and q are even,

$p = p_o$, $q = q_o$, if at least one of p and q is odd.

By using the integral representation of the Bessel function

$$J_n(x) = \frac{i^{-n}}{2\pi} \int_{-\pi}^{+\pi} e^{ix\cos\theta} \cos(n\theta)\, d\theta$$

we have

$$\tilde{G}(\omega) = \frac{1}{4\pi^2} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} e^{i\omega(\tau_x \cos\theta' + \tau_y \cos\theta'')} \tilde{G}(\theta', \theta'', X_o)\, d\theta'\, d\theta'' \quad (C4)$$

$$\tilde{G}(\theta', \theta'', X_o) = \sum_{s=o}^{\infty} (-1)^{s(p_o + q_o)} \varepsilon_s \cos(sp_o\theta') \cos(sq_o\theta'') \cos(sX_o)$$

$$= \frac{\pi}{4} \sum_{n=-\infty}^{+\infty} \Big\{ \delta\left[p_o(\theta' + \pi) + q_o(\theta'' + \pi) + X_o + 2\pi n\right] +$$

$$+ \delta\left[p_o(\theta' + \pi) - q_o(\theta'' + \pi) + X_o + 2\pi n\right] +$$

$$+ \delta\left[p_o(\theta' + \pi) - q_o(\theta'' + \pi) - X_o - 2\pi n\right] +$$

$$+ \delta\left[p_o(\theta' + \pi) + q_o(\theta'' + \pi) - X_o - 2\pi n\right] \Big\} . \quad (C5)$$

Let us insert (C5) in (C4) and shift the integral variables θ' and θ'' by π. We obtain

$$G(\omega) = \frac{1}{16\pi p_o} \sum_{n=-\infty}^{+\infty} \int_o^{2\pi} \int_o^{2\pi} d\theta'\, d\theta''\, e^{-i\omega(\tau_x \cos\theta' + \tau_y \cos\theta'')} \quad .$$

$$\cdot \left\{ \delta(\theta' + \frac{q_o\theta'' + X_o + 2\pi n}{p_o}) + \delta(\theta' - \frac{q_o\theta'' - X_o - 2\pi n}{p_o} + \right.$$

$$\left. + \quad \delta(\theta'' - \frac{q_o\theta'' + X_o + 2\pi n}{p_o}) + \delta(\theta' + \frac{q_o\theta'' - X_o - 2\pi n}{p_o}) \right\} .$$

We perform first the integration over θ'. At this purpose we observe that the number of delta-functions falling in the interval between o and 2π is independent of the angles θ'' and X_o. This number is obviously p_o. Thus we have

$$G(\omega) = \frac{1}{8\pi p_o} \sum_{s=1}^{p_o} \int_0^{2\pi} d\theta \; e^{-i\omega\tau_y \cos\theta} \left\{ e^{-i\omega\tau_x \cos \frac{q_o\theta + X_o - 2\pi s}{p_o}} + \right.$$

$$\left. + \; e^{-i\omega\tau_x \cos \frac{q_o\theta - X_o + 2\pi s}{p_o}} \right\} . \tag{C6}$$

We insert now (C6) in (C1) and we remind that, denoting with R a constant,

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega R} \log \frac{\omega^2 + 1}{\omega^2} \frac{d\omega}{\omega} = i \int_0^1 \frac{1 - e^{-tR}}{t} dt.$$

We have

$$W_2 = (\nu_x - \frac{r}{p}) \; I_x + \nu_y \; I_y + \frac{\sigma^2}{2\pi p_o} \; (\frac{\xi}{\beta^*}) \sum_{s=1}^{p_o} \int_0^{2\pi} d\theta \int_0^1 \frac{2 - e^{-tH_s} - e^{-tH_s}}{t} dt$$

where

$$H_{s\pm} = \tau_x \left(1 - \cos \frac{q_o\ \theta \pm X_o + 2\pi s}{p_o}\right) + \tau_y\ (1 - \cos\theta).$$

Let us perform first the integration over θ

$$W_2 = (\nu_x - \frac{r}{p})\ I_x + \nu_y\ I_y + 2\sigma^2\ (\frac{\xi}{\beta^*}) \int_0^1 \frac{dt}{t}\left[1 - P_+(t) - P_-(t)\right] \quad \text{(C7)}$$

where

$$P_\pm(t) = \frac{e^{-t(\tau_x + \tau_y)}}{4\pi p_o} \sum_{s=1}^{p_o} \int_0^{2\pi} e^{t\tau_x \cos \frac{q_o\ \theta \pm X_o + 2\pi s}{p_o}}\ e^{t\tau_y \cos\theta}\ d\theta\ .$$

Let us remind the expansion

$$e^{x \cos \theta} = \sum_{\kappa=-\infty}^{+\infty} I_\kappa(x)\ e^{i\kappa\theta}$$

where $I_\kappa(x)$ is the modified Bessel function of first kind and the κ-th order. It should not be confused with the action variables I_x and I_y.

Using the fact that

$$\sum_{s=1}^{p_o} e^{2\pi i\kappa \frac{s}{p_o}} = \begin{cases} p_o & \text{for } \kappa = \ell p_o\ ,\ \ \ell \text{ integer} \\ o & \text{, otherwise} \end{cases}$$

we have

$$P_\pm(t) = \frac{e^{-t(\tau_x + \tau_y)}}{4\pi} \sum_{\ell=-\infty}^{+\infty} I_{\ell p_o}(t\tau_x) e^{\pm i\ell X_o} \int_0^{2\pi} e^{t\tau_y \cos\theta}\ e^{i\ell q_o \theta}\ d\theta.$$

But it is

$$I_{\ell q_o}(t\tau_y) = \frac{1}{2\pi} \int_0^{2\pi} e^{t\tau_y \cos\theta} e^{i\ell q_o \theta} \, d\theta$$

then we have

$$P_{\pm}(t) = \frac{1}{2} e^{-t(\tau_x + \tau_y)} \sum_{\ell=-\infty}^{+\infty} I_{\ell p_o}(t\tau_x) \, I_{\ell q_o}(t\tau_y) \, e^{\pm i\ell X_o}$$

and

$$P_+(t) + P_-(t) = e^{-t(\tau_x + \tau_y)} \sum_{\ell=-\infty}^{+\infty} I_{\ell p_o}(t\tau_x) \, I_{\ell q_o}(t\tau_y) \cos(\ell X_o) \quad \text{(C8)}$$

Inserting (C8) in (C7) yields eq.(26). We believe there is no way to get a more simplified, general form for the Hamiltonian W_2.

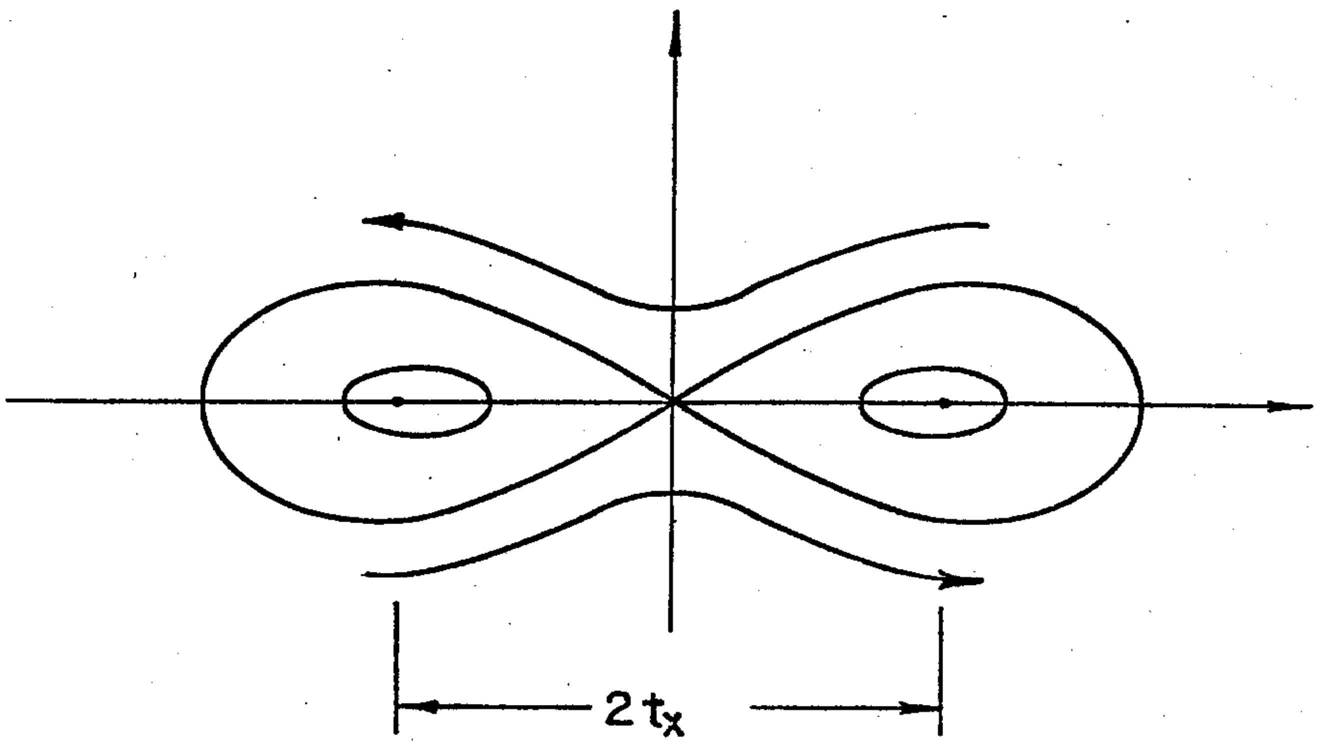

FIG. 2

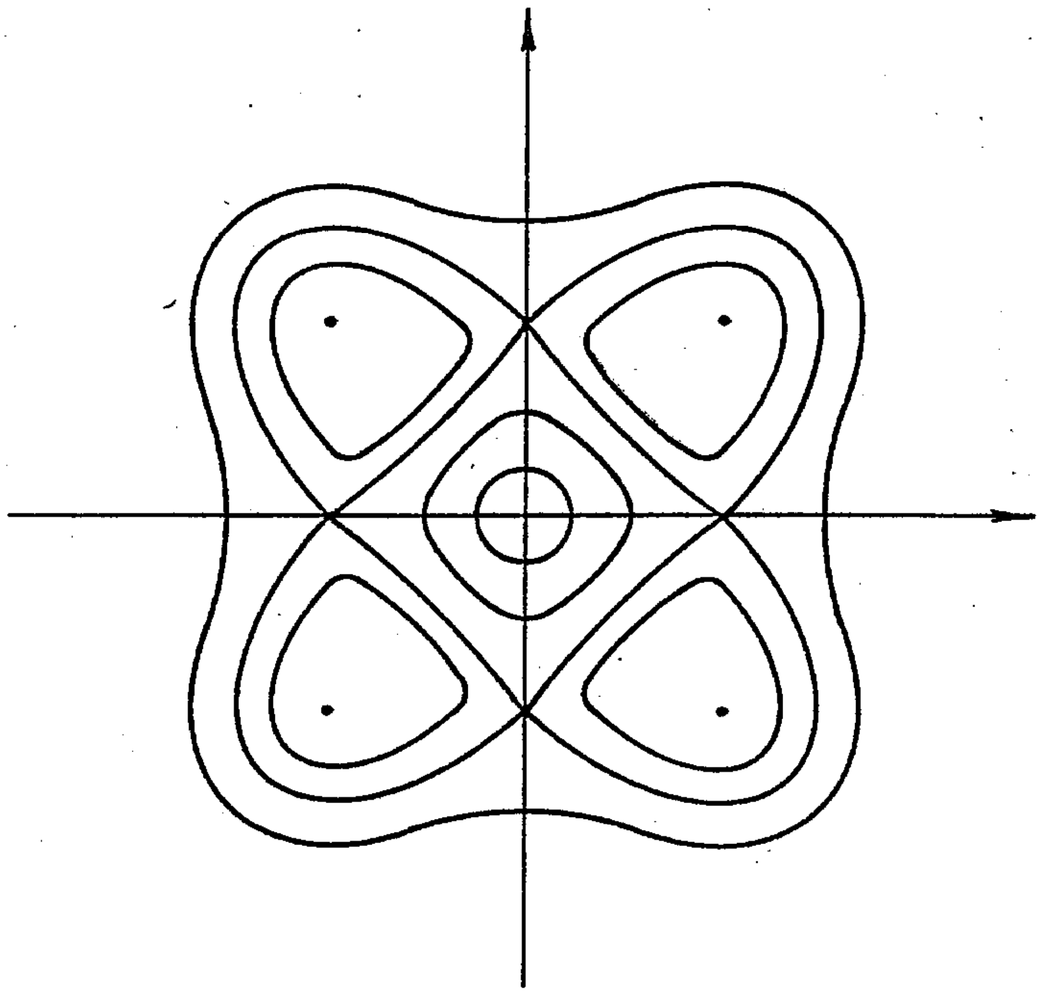

FIG. 3

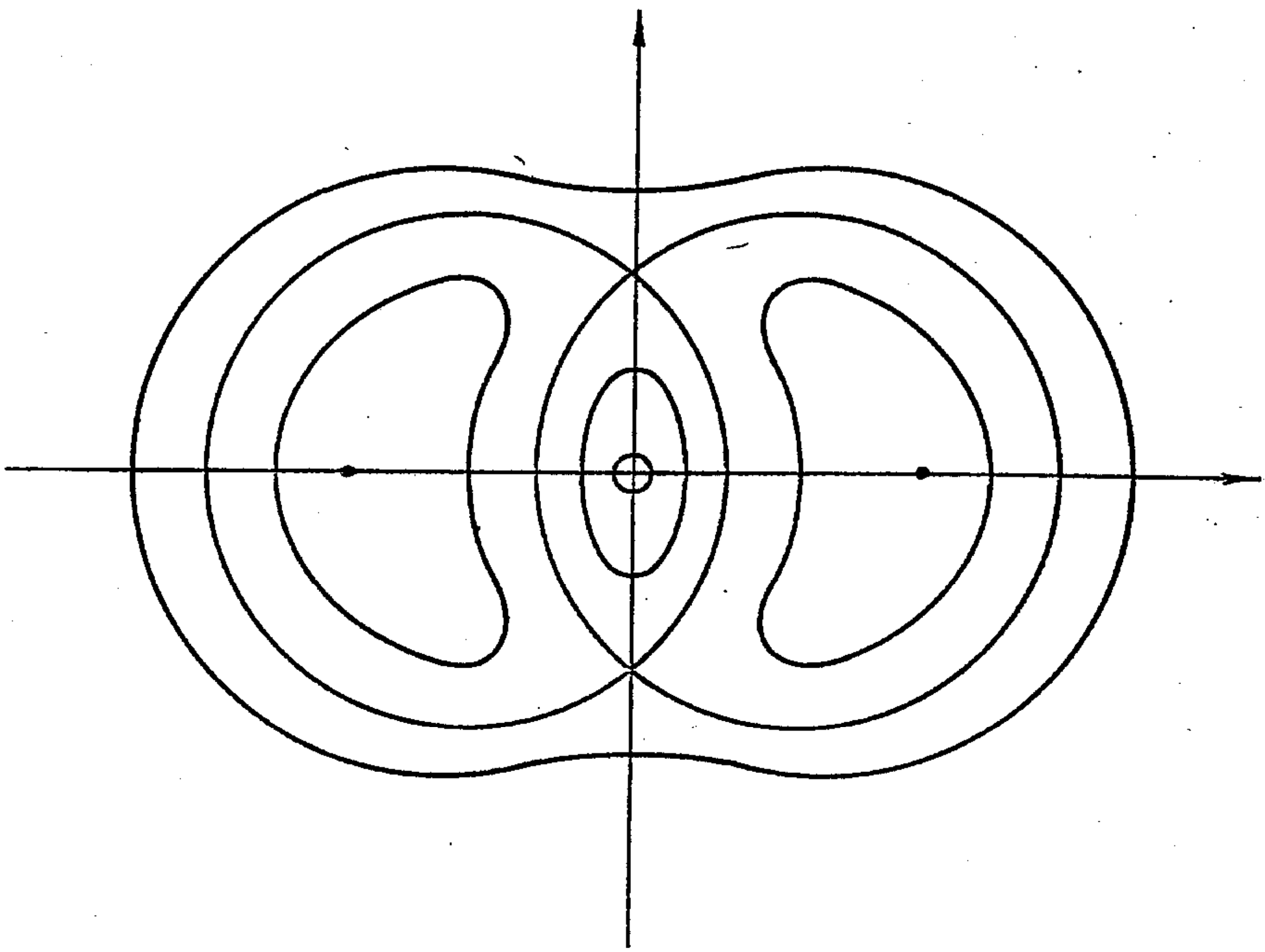

FIG. 4

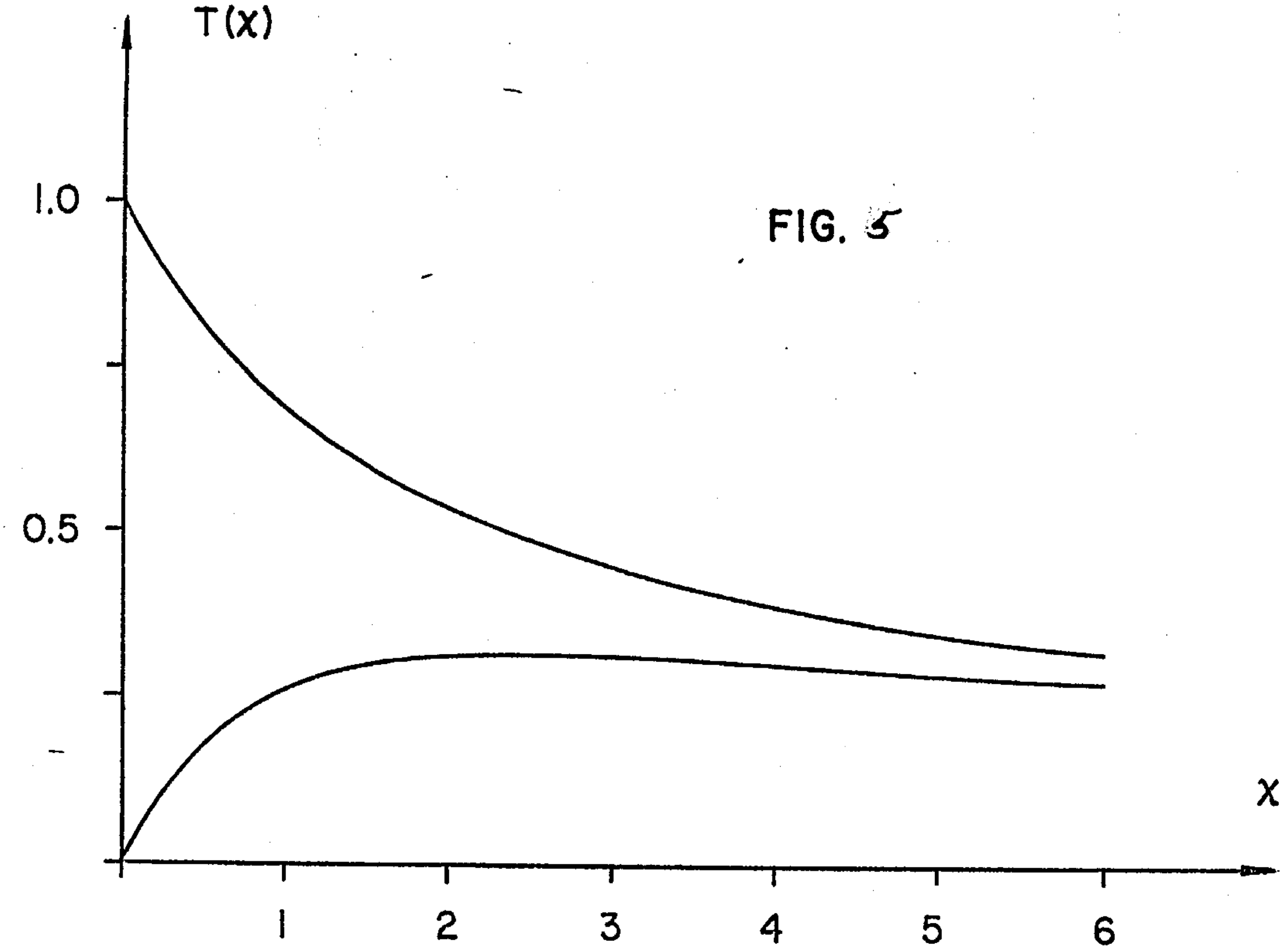

FIG. 5

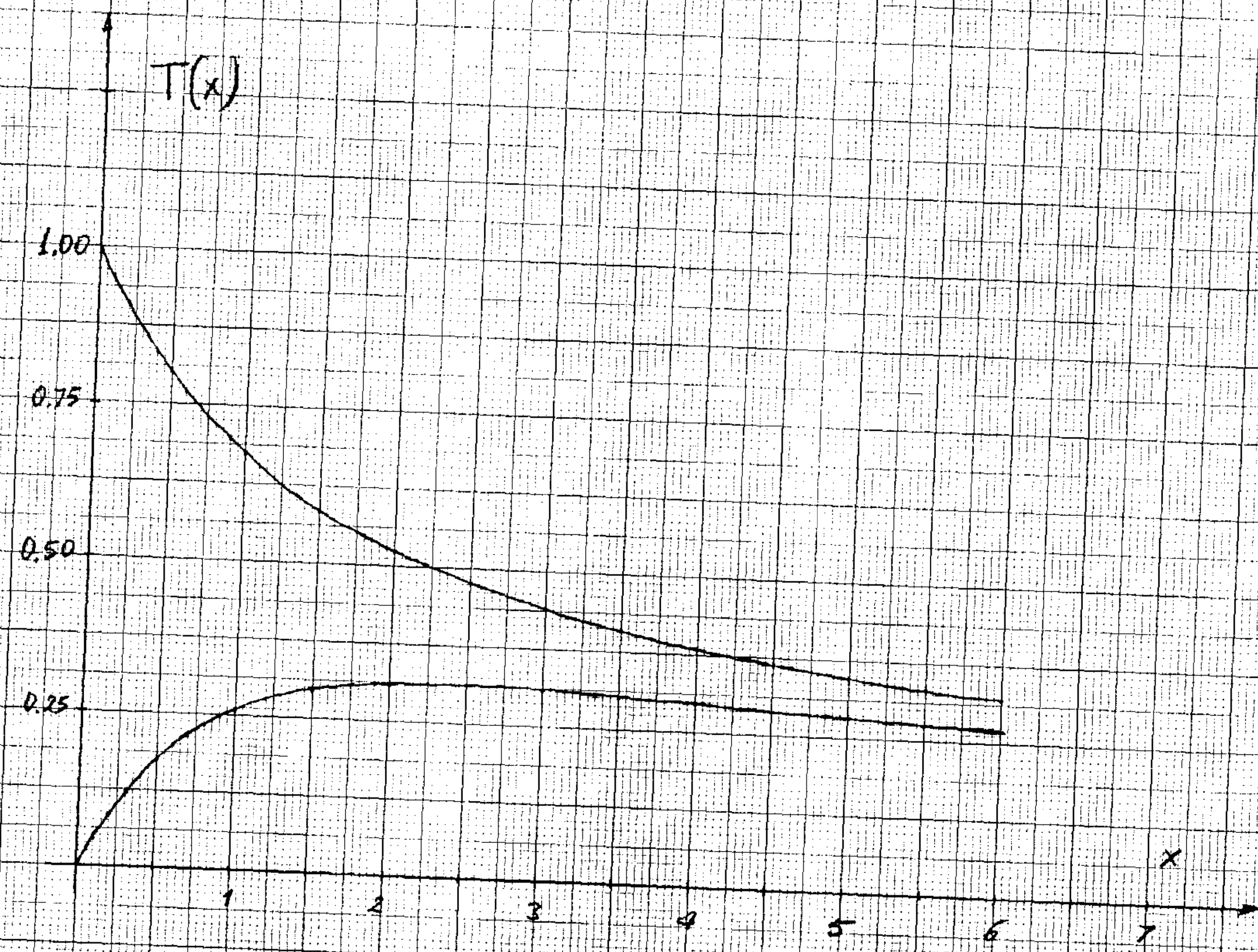

Fig. 6