FORMULAS FOR RESONANCES OF TRANSVERSE OSCILLATIONS IN A CIRCULAR ACCELERATOR

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The transverse oscillations of a charged particle in a circular accelerator have been studied extensively. This report gives a handy compilation of useful formulas derived by standard procedures.

Reference Curve and Coordinate System

We consider the motion of a particle with charge e and momentum p moving in a static magnetic field having an approximate midplane. We choose as the reference curve a closed plane curve lying in the midplane and having radius of curvature $\rho = \rho\ (z)$ where z is the coordinate along the curve in the direction of motion of a positively charged particle. The reference curve, or $\rho\ (z)$, is so chosen that deviations of particle orbits from the reference curve are small. The x coordinate is along the outward normal and in the plane of the reference curve, and the y coordinate is perpendicular to the plane of the reference curve and in the direction of the main magnetic field, thus, forming a righthanded coordinate system. The circumference of the reference curve is written as $2\pi R$.

Magnetic Field

The scalar potential of the static magnetic field expanded about the reference curve has the form

$$\Phi(x,y,z) = \sum_{n=o}^{\infty} \Phi_n(x,z) \frac{y^n}{n!}$$

$$= \left(a_o + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \ldots\ldots\right)$$

$$+ \left(b_1 + b_2 x + b_3 \frac{x^2}{2!} + \ldots\ldots\right) y$$

$$+ \left(c_2 + c_3 x + \ldots\ldots\right) \frac{y^2}{2!}$$

$$+ \left(d_3 + \ldots\ldots\right) \frac{y^3}{3!} + \ldots\ldots \qquad (1)$$

Where Φ_o is small so that the $y = 0$ plane (plane of the reference curve) is the approximate magnetic midplane. The magnetic field components are

$$\left\{\begin{aligned}
B_x &= \frac{\partial\Phi}{\partial x} = \left(a_1 + a_2 x + a_3 \frac{x^2}{2!} + \ldots\right) \\
&\quad + \left(b_2 + b_3 x + \ldots\right) y \\
&\quad + \left(c_3 + \ldots\right) \frac{y^2}{2!} + \ldots \\
B_y &= \frac{\partial\Phi}{\partial y} = \left(b_1 + b_2 x + b_3 \frac{x^2}{2!} + \ldots\right) \\
&\quad + \left(c_2 + c_3 x + \ldots\right) y \\
&\quad + \left(d_3 + \ldots\right) \frac{y^2}{2!} + \ldots \\
B_z &= \frac{1}{1+kx} \frac{\partial\Phi}{\partial z} = \frac{1}{1+kx} \left(a_o' + a_1' x + a_2' \frac{x^2}{2!} + \ldots\right) \\
&\quad + \frac{1}{1+kx} \left(b_1' + b_2' x + \ldots\right) y \\
&\quad + \frac{1}{1+kx} \left(c_2' + \ldots\right) \frac{y^2}{2!} + \ldots
\end{aligned}\right. \qquad (2)$$

where $k = k(z) \equiv \frac{1}{\rho(z)}$ and prime means $\frac{d}{dz}$. The coefficients c's and d's are related to the a's and b's by the Laplace equation

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial y^2} + \frac{1}{1+kx}\left\{\frac{\partial}{\partial x}\left[(1+kx)\ \frac{\partial\Phi}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{1}{1+kx}\ \frac{\partial\Phi}{\partial z}\right]\right\} = 0 \qquad (3)$$

and are

$$\begin{cases} c_2 = -a_o'' - ka_1 - a_2 \\ c_3 = 2ka_o'' + k'a_o' - a_1'' + k^2a_1 - ka_2 - a_3 \\ c_4 = \ldots\ldots \\ d_3 = -b_1'' - kb_2 - b_3 \\ d_4 = \ldots\ldots \end{cases} \qquad (4)$$

$$\ldots\ldots$$

In terms of the field on the reference curve (x=y=0) the coefficients a's and b's are

$$\begin{cases} a_o = \int B_z dz, \ (a_o' = B_z \equiv a) \\ a_1 = B_x \\ a_2 = \frac{\partial B_x}{\partial x} \\ a_3 = \frac{\partial^2 B_x}{\partial x^2} \\ \ldots\ldots \end{cases} \qquad \begin{cases} b_1 = B_y = \text{dipole field} \\ b_2 = \frac{\partial B_y}{\partial x} = \text{quadrupole field} \\ b_3 = \frac{\partial^2 B_y}{\partial x^2} = \text{sextupole field} \\ \ldots\ldots \end{cases} \qquad (5)$$

Since only a_o' (= B_z on reference curve) and never a_o will appear in orbit equations we will substitute the letter a for a_o'.

Orbit Hamiltonian

The orbit Hamiltonian in x, p_x, and y, p_y with z as the independent variable is

$$H\ (x,\ p_x,\ y,\ p_y;\ z)$$
$$= -(1+kx)\left[\sqrt{1-\left(p_x-\varepsilon A_x\right)^2-\left(p_y-\varepsilon A_y\right)^2}+\varepsilon A_z\right] \qquad (6)$$

where $\varepsilon \equiv \frac{e}{pc}$, c = velocity of light in vacuum, and A_x, A_y, A_z are components of the vector potential of the field and can be expressed in terms of the scalar potential Φ. When expanded in powers of x, p_x, and y, p_y we get

$$H = H^{(o)} + H^{(1)} + H^{(2)} + H^{(3)} + \ldots. \qquad (7)$$

with

$$\left\{\begin{aligned}
H^{(o)} &= -1 \text{ (irrelevant)} \\
H^{(1)} &= (\varepsilon b_1 - k)x - \varepsilon a_1 y \\
H^{(2)} &= \frac{1}{2}\left(p_x^{\ 2} + p_y^{\ 2}\right) + \frac{\varepsilon a}{2}\left(yp_x - xp_y\right) \\
&\quad + \frac{\varepsilon}{2}\left(kb_1 + b_2 + \frac{\varepsilon a^2}{4}\right)x^2 - \frac{\varepsilon}{2}\left(2ka_1 + 2a_2 + a'\right)xy + \frac{\varepsilon}{2}\left(-b_2 + \frac{\varepsilon a^2}{4}\right)y^2 \\
H^{(3)} &= \frac{1}{2}kx\left(p_x^{\ 2} + p_y^{\ 2}\right) + \frac{\varepsilon}{3}\left[\left(a_1' + \frac{ka}{2}\right)x + b_1' y\right]\left(yp_x - xp_y\right) \qquad (8) \\
&\quad + \frac{\varepsilon}{6}\left[2kb_2 + b_3 + \varepsilon a\left(a_1' - \frac{ka}{4}\right)\right]x^3 \\
&\quad - \frac{\varepsilon}{6}\left[2a_1'' + 6ka_2 + 3a_3 - 2ka' - 2k'a - \varepsilon ab_1'\right]x^2 y \\
&\quad - \frac{\varepsilon}{6}\left[b_1'' + 3kb_2 + 3b_3 - \varepsilon a\left(a_1' - \frac{ka}{4}\right)\right]xy^2 \\
&\quad + \frac{\varepsilon}{6}\left[a_1'' - k^2 a_1 + ka_2 + a_3 - 2ka' - k'a + \varepsilon ab_1'\right]y^3 \\
H^{(4)} &= \ldots\ldots
\end{aligned}\right.$$

Remembering that both k and εB have the dimension of $(\text{length})^{-1}$ we see that p_x, p_y, and H are all dimensionless. We can further express all length in units of the equivalent radius R of the reference curve, and redefine

$$\varepsilon \equiv \frac{eR}{pc} \quad \text{and} \quad k \equiv \frac{R}{\rho} \tag{9}$$

The variables x and y are, then, dimensionless, and in units of R the indepèndent variable z is the angle θ along the reference curve advancing 2π for each circuit around the curve.

Transformations and Approximations

Generally the desired magnetic field is one for which the only non-vanishing coefficients are b_1 and b_2 and these have sector periodicity. However, because of design and construction imperfections all coefficients are present as small errors. Write

$$b_1 = b_{10} + \bar{b}_1 \quad \text{and} \quad b_2 = b_{20} + \bar{b}_2 \tag{10}$$

where b_{10} and b_{20} have exact sector periodicity and $\bar{b}_1$ and $\bar{b}_2$ are small errors. The reference curve is so chosen that $k = \varepsilon b_{10}$ and, hence, also has exact sector periodicity.

The effects of the exactly periodic terms in $H^{(2)}$ have been studied extensively and were shown to lead to linear transverse oscillations (betatron oscillations) of the orbits about the reference curve which can be transformed (Floquet transformation) to harmonic oscillations with wave numbers ν_x and ν_y. We can, therefore, replace the exact sector-periodic terms in $H^{(2)}$ by harmonic oscillator terms. The Hamiltonian, now, becomes

$$H^{(1)} = \varepsilon\bar{b}_1 x - \varepsilon a_1 y$$

$$\begin{aligned} H^{(2)} &= \frac{1}{2}\left(p_x^{\,2} + p_y^{\,2}\right) + \frac{1}{2}\left(\nu_x^{\,2}x^2 + \nu_y^{\,2}y^2\right) + \frac{\varepsilon a}{2}\left(yp_x - xp_y\right) \\ &\quad + \frac{\varepsilon}{2}\left(\kappa\bar{b}_1 + \bar{b}_2 + \frac{\varepsilon a^2}{4}\right)x^2 - \frac{\varepsilon}{2}\left(2\kappa a_1 + 2a_2 + a'\right)xy + \\ &\quad + \frac{\varepsilon}{2}\left(-\bar{b}_2 + \frac{\varepsilon a^2}{4}\right)y^2 \end{aligned}$$

$$H^{(3)} = \text{unchanged} \tag{11}$$

$$H^{(4)} = \ldots\ldots$$

where we have kept the same letters x, y and p_x, p_y to denote the transformed coordinate and conjugate variables. The first two terms in $H^{(2)}$ give the linear (now, harmonic) oscillations.

Far away from resonances the effects of all other terms are small non-secular modifications of the linear oscillations. Close to a resonance certain of these terms produce large resonant (secular) modifications. For each resonance we can pick out the relevant terms (excitation terms) and transform the Hamiltonian under the adiabatic approximation to a form explicitly independent of the independent variable $\theta \left(= \frac{z}{R}\right)$. For values of ν_x and ν_y such that

$$\ell\nu_x + m\nu_y = n + \delta\sqrt{\ell^2 + m^2} \qquad \begin{pmatrix} \delta \text{ small} \\ \ell,\ m,\ n \text{ integers, } n \geq 0 \end{pmatrix} \tag{12}$$

the transformed canonically conjugate variables ϕ_x, J_x, and ϕ_y, J_y are related to the original variables x, p_x, and y, p_y by

$$\begin{cases} \phi_x = \psi_x + \nu_x\theta - \dfrac{\ell\delta}{\sqrt{\ell^2 + m^2}}\,\theta\ , \quad J_x = \nu_x A_x^{\,2} \\[2ex] \phi_y = \psi_y + \nu_y\theta - \dfrac{m\delta}{\sqrt{\ell^2 + m^2}}\,\theta\ , \quad J_y = \nu_y A_y^{\,2} \end{cases} \tag{13}$$

where

$$A_x e^{i\psi_x} \equiv x + i\,\frac{p_x}{\nu_x}\ , \qquad A_y e^{i\psi_y} \equiv y + i\,\frac{p_y}{\nu_y} \tag{14}$$

The motion is then given by the transformed Hamiltonian $K\ (\phi_x, J_x, \phi_y, J_y)$ through the canonical equations

$$\left\{\begin{aligned} \frac{d\phi_x}{d\theta} &= \frac{\partial K}{\partial J_x} \\ \frac{dJ_x}{d\theta} &= -\frac{\partial K}{\partial \phi_x} \end{aligned}\right. \qquad \left\{\begin{aligned} \frac{d\phi_y}{d\theta} &= \frac{\partial K}{\partial J_y} \\ \frac{dJ_y}{d\theta} &= -\frac{\partial K}{\partial \phi_y} \end{aligned}\right. \tag{15}$$

Transformed Adiabatic Hamiltonians

The transformed adiabatic Hamilton $K\ (\phi_x, J_x, \phi_y, J_y)$ could be written in a general form for the resonance $\ell\nu_x + m\nu_y = n + \delta\sqrt{\ell^2 + m^2}$, but such a form is rather complex and difficult to use. Since in most cases one is interested in resonances of low orders and since we have the explicit expanded form of H only up to $H^{(3)}$ we shall list here the transformed Hamiltonians separately for each resonance up to the third order.

(A) Resonances excited by $H^{(1)}$ - Only first order resonances are excited by $H^{(1)}$

(1) $\nu_x = n + \delta$

$$K = -\frac{2C_n}{\sqrt{\nu_x}}\sqrt{J_x}\cos\left(\phi_x + \alpha_n\right) - \delta J_x \tag{16}$$

where

$$\sum_{n=-\infty}^{\infty} C_n e^{i(n\theta+\alpha_n)} = \varepsilon\bar{b}_1 \tag{17}$$

(2) $\nu_y = n + \delta$

$$K = -\frac{2C_n}{\sqrt{\nu_y}} \sqrt{J_y} \cos\left(\phi_y + \alpha_n\right) - \delta J_y \tag{18}$$

where

$$\sum_{n=-\infty}^{\infty} C_n e^{i(n\theta+\alpha_n)} = -\varepsilon a_1 \tag{19}$$

(B) Resonances excited by $H^{(2)}$ - Only second order resonances are excited by $H^{(2)}$

(1) $2\nu_x = n + 2\delta$

$$K = -\frac{C_n}{\nu_x} J_x \cos\left(2\phi_x + \alpha_n\right) - \delta J_x \tag{20}$$

where

$$\sum_{n=-\infty}^{\infty} C_n e^{i\left(n\theta+\alpha_n\right)} = \frac{\varepsilon}{2}\left(k\bar{b}_1 + \bar{b}_2 + \frac{\varepsilon a^2}{4}\right) \tag{21}$$

(2) $2\nu_y = n + 2\delta$

$$K = -\frac{C_n}{\nu_y} J_y \cos\left(2\phi_y + \alpha_n\right) - \delta J_y \tag{22}$$

where

$$\sum_{n=-\infty}^{\infty} C_n e^{i\left(n\theta+\alpha_n\right)} = \frac{\varepsilon}{2}\left(-\bar{b}_2 + \frac{\varepsilon a^2}{4}\right) \tag{23}$$

(3) $\nu_x + \nu_y = n + \sqrt{2}\ \delta$

$$K = -\frac{C_n}{\sqrt{\nu_x \nu_y}} \sqrt{J_x J_y} \cos\left(\phi_x + \phi_y + \alpha_n\right) - \frac{\delta}{\sqrt{2}}\left(J_x + J_y\right) . \tag{24}$$

where

$$\sum_{n=-\infty}^{\infty} C_n e^{i\left(n\theta+\alpha_n\right)} = -\frac{\varepsilon}{2}\left[\left(2ka_1+2a_2+a'\right)+ia\left(\nu_x-\nu_y\right)\right] \tag{25}$$

(4) $\pm\nu_x \mp \nu_y = n + \sqrt{2}\ \delta$

$$K = -\frac{C_n}{\sqrt{\nu_x\nu_y}}\sqrt{J_xJ_y}\cos\left(\pm\phi_x\mp\phi_y+\alpha_n\right) - \frac{\delta}{\sqrt{2}}\left(\pm J_x \mp J_y\right) \tag{26}$$

where

$$\sum_{n=-\infty}^{\infty} C_n e^{i\left(n\theta+\alpha_n\right)} = -\frac{\varepsilon}{2}\left[\left(2ka_1+2a_2+a'\right)\pm ia\left(\nu_x+\nu_y\right)\right] \tag{27}$$

(C) Resonances excited by $H^{(3)}$ - First and third order resonances are excited by $H^{(3)}$

(1) $3\nu_x = n + 3\delta$

$$K = -\frac{C_n}{2\nu_x^{3/2}} J_x^{3/2}\cos\left(3\phi_x+\alpha_n\right) - \delta J_x \tag{28}$$

where

$$\sum_{n=-\infty}^{\infty} C_n e^{i(n\theta+\alpha_n)} = \frac{\varepsilon}{6}\left[2kb_2+b_3+\varepsilon a\left(a_1' - \frac{ka}{4}\right)\right] - \frac{k\nu_x^2}{2} \tag{29}$$

(2) $3\nu_y = n + 3\delta$

$$K = -\frac{C_n}{2\nu_y^{3/2}} J_y^{3/2}\cos\left(3\phi_y+\alpha_n\right) - \delta J_y \tag{30}$$

where

$$\sum_{n=-\infty}^{\infty} C_n e^{i(n\theta+\alpha_n)} = \frac{\varepsilon}{6}\left(a_1''-k^2a_1+ka_2+a_3-2ka'-k'a+\varepsilon ab_1'\right) \tag{31}$$

(3) $2\nu_x + \nu_y = n + \sqrt{5}\delta$

$$K = -\frac{C_n}{2\nu_x\sqrt{\nu_y}} J_x\sqrt{J_y}\cos\left(2\phi_x+\phi_y+\alpha_n\right) - \frac{\delta}{\sqrt{5}}\left(2J_x+J_y\right) \tag{32}$$

where

$$\sum_{n=-\infty}^{\infty} C_n e^{i(n\theta+\alpha_n)} = -\frac{\varepsilon}{6}\left(2a_1'' + 6ka_2 + 3a_3-2ka'-2k'a-\varepsilon ab_1'\right) -i\,\frac{\varepsilon}{3}\left(a_1' + \frac{ka}{2}\right)\left(\nu_x-\nu_y\right) \tag{33}$$

(4) $\pm 2\nu_x \mp \nu_y = n + \sqrt{5}\delta$

$$K = -\frac{C_n}{2\nu_x\sqrt{\nu_y}} J_x\sqrt{J_y}\cos\left(\pm 2\phi_x \mp \phi_y+\alpha_n\right) - \frac{\delta}{\sqrt{5}}\left(\pm 2J_x \mp J_y\right) \tag{34}$$

where

$$\sum_{n=-\infty}^{\infty} C_n e^{i(n\theta+\alpha_n)} = -\frac{\varepsilon}{6}\left(2a_1'' + 6ka_2 + 3a_3-2ka'-2k'a-\varepsilon ab_1'\right) \mp i\,\frac{\varepsilon}{3}\left(a_1' + \frac{ka}{2}\right)\left(\nu_x+\nu_y\right) \tag{35}$$

(5) $2\nu_y + \nu_x = n + \sqrt{5}\delta$

$$K = -\frac{C_n}{2\nu_y\sqrt{\nu_x}} J_y\sqrt{J_x}\cos\left(2\phi_y+\phi_x+\alpha_n\right) - \frac{\delta}{\sqrt{5}}\left(2J_y+J_x\right) . \tag{36}$$

where

$$\sum_{n=-\infty}^{\infty} C_n e^{i\left(n\theta+\alpha_n\right)} = -\frac{\varepsilon}{6}\left[b_1'' + 3kb_2 + 3b_3 - \varepsilon a\left(a_1' - \frac{ka}{4}\right)\right] - \frac{k\nu_y^2}{2}$$

$$- i\,\frac{\varepsilon}{3}\,b_1'\left(\nu_x - \nu_y\right) \qquad (37)$$

(6) $\pm 2\nu_y \mp \nu_x = n + \sqrt{5}\delta$

$$K = -\frac{C_n}{2\nu_y\sqrt{\nu_x}}\,J_y\sqrt{J_x}\,\cos\left(\pm 2\phi_y \mp \phi_x + \alpha_n\right) - \frac{\delta}{\sqrt{5}}\left(\pm 2J_y \mp J_x\right) \qquad (38)$$

where

$$\sum_{n=-\infty}^{\infty} C_n e^{i\left(n\theta+\alpha_n\right)} = -\frac{\varepsilon}{6}\left[b_1'' + 3kb_2 + 3b_3 - \varepsilon a\left(a_1' - \frac{ka}{4}\right)\right] - \frac{k\nu_y^2}{2}$$

$$\pm i\,\frac{\varepsilon}{3}\,b_1'\left(\nu_x + \nu_y\right) \qquad (39)$$

(7) $\nu_x = n + \delta$

$$K = -\frac{C_n}{2\nu_x^{3/2}}\,J_x^{3/2}\cos\left(\phi_x + \alpha_n\right)$$

$$-\frac{D_n}{2\nu_y\sqrt{\nu_x}}\,J_y\sqrt{J_x}\,\cos\left(\phi_x + \beta_n\right) - \delta J_x \qquad (40)$$

where

$$\left\{\begin{aligned} &\sum_{n=-\infty}^{\infty} C_n e^{i\left(n\theta+\alpha_n\right)} = \frac{\varepsilon}{2}\left[2kb_2 + b_3 + \varepsilon a\left(a_1' - \frac{ka}{4}\right)\right] + \frac{k\nu_x^2}{2} \\ &\sum_{n=-\infty}^{\infty} D_n e^{i\left(n\theta+\beta_n\right)} = -\frac{\varepsilon}{3}\left[b_1'' + 3kb_2 + 3b_3 - \varepsilon a\left(a_1' - \frac{ka}{4}\right)\right] + k\nu_y^2 \\ &\qquad - i\,\frac{2\varepsilon}{3}\,b_1'\nu_x \end{aligned}\right. \qquad (41)$$

(8) $\nu_y = n + \delta$

$$K = -\frac{C_n}{2\nu_y^{3/2}} J_y^{3/2} \cos\left(\phi_y + \alpha_n\right) - \frac{D_n}{2\nu_x\sqrt{\nu_y}} J_x\sqrt{J_y}\, \cos\left(\phi_y + \beta_n\right) - \delta J_y \quad (42)$$

where

$$\begin{cases} \sum\limits_{n=-\infty}^{\infty} C_n e^{i\left(n\theta+\alpha_n\right)} = \frac{\varepsilon}{2}\left(a_1'' - k^2 a_1 + k a_2 + a_3 - 2ka' - k'a + \varepsilon a b_1'\right) \\ \sum\limits_{n=-\infty}^{\infty} D_n e^{i\left(n\theta+\beta_n\right)} = -\frac{\varepsilon}{3}\left(2a_1'' + 6ka_2 + 3a_3 - 2ka' - 2k'a - \varepsilon a b_1'\right) + i\,\frac{2\varepsilon}{3}\left(a_1' + \frac{ka}{2}\right)\nu_y \end{cases} \quad (43)$$

For ease of reference we list here, again, the definitions of the magnetic field coefficients

$a = B_z$ = field along reference curve

$$\left.\begin{aligned} a_1 &= B_x \\ a_2 &= \frac{\partial B_x}{\partial x} \\ a_3 &= \frac{\partial^2 B_x}{\partial x^2} \\ a_4 &= \ldots\ldots \end{aligned}\right\} \text{Midplane error field}$$

$b_1 = B_y$ = guide field (dipole component) = $b_{10} + \bar{b}_1$

$$\begin{cases} b_{10} = \text{exact sector-periodic guide field} \\ \bar{b}_1 = \text{guide field error} \end{cases}$$

$b_2 = \dfrac{\partial B_y}{\partial x}$ = guide field gradient (quadrupole component) = $b_{20} + \bar{b}_2$

$$\begin{cases} b_{20} = \text{exact sector-periodic guide field gradient} \\ \cancel{b_2 = \text{guide field gradient error}} \end{cases}$$

$$b_3 = \frac{\partial^2 B_y}{\partial x^2} = \text{guide field sextupole component}$$

$$b_4 = \ldots\ldots.$$

all evaluated on the reference curve (x=y=o). In addition we recall

$$\varepsilon \equiv \frac{eR}{pc} \quad \text{and} \quad k \equiv \frac{R}{\rho} = \varepsilon b_{10}$$

Example of Application

Take as an example the excitation of the 2nd order resonance

$$\nu_x + \nu_y = n + \sqrt{2}\,\delta \tag{44}$$

by the rotational misalignment of the quadrupoles in the main ring of the NAL accelerator. When an ideal quadrupole with field gradient G (in real units) is rotated along its axis by a small angle ω a midplane error field gradient $a_2 = \frac{\partial B_x}{\partial x} = -RG \sin 2\omega \approx - 2\omega RG$ is introduced and the guide field gradient is reduced to $b_2 = \frac{\partial B_y}{\partial x} = RG \cos 2\omega \approx RG$. Since we assume no other errors we have $a = a_1 = o$ and the transformed adiabatic Hamiltonian is given by (24), namely

$$K = - \frac{C_n}{\sqrt{\nu_x \nu_y}} \sqrt{J_x J_y} \cos\left(\phi_x + \phi_y + \alpha_n\right) - \frac{\delta}{\sqrt{2}} \quad J_x + J_y \tag{45}$$

where

$$\sum_{n=-\infty}^{\infty} C_n e^{i\left(n\theta + \alpha_n\right)} = -\varepsilon a_2 = 2\omega\varepsilon RG \tag{46}$$

The width of the resonance band can be obtained from (45) directly. Since the cosine function has to lie between -1 and +1 for an orbit to exist we must have

$$-\frac{C_n}{\sqrt{\nu_x\nu_y}} < \frac{K}{\sqrt{J_xJ_y}} + \frac{\delta}{\sqrt{2}}\left(\sqrt{\frac{J_x}{J_y}} + \sqrt{\frac{J_y}{J_x}}\right) < \frac{C_n}{\sqrt{\nu_x\nu_y}} \tag{47}$$

As J_x and J_y go to ∞, the K term goes to zero and the coefficient of $\frac{\delta}{\sqrt{2}}$ has a minimum value of 2 when $J_x = J_y$. Thus, in order for orbits with infinite J_x and J_y to exist we must have

$$-\frac{C_n}{\sqrt{2\nu_x\nu_y}} < \delta < \frac{C_n}{\sqrt{2\nu_x\nu_y}} \tag{48}$$

This, then, gives the width of the resonance.

The canonical equations are

$$\begin{cases} \dfrac{d\phi_x}{d\theta} = \dfrac{\partial K}{\partial J_x} = -\dfrac{C_n}{2\sqrt{\nu_x\nu_y}}\sqrt{\dfrac{J_y}{J_x}}\cos\left(\phi_x+\phi_y+\alpha_n\right) - \dfrac{\delta}{\sqrt{2}} \\[2ex] \dfrac{dJ_x}{d\theta} = -\dfrac{\partial K}{\partial \phi_x} = -\dfrac{C_n}{\sqrt{\nu_x\nu_y}}\sqrt{J_xJ_y}\sin\left(\phi_x + \phi_y + \alpha_n\right) \end{cases} \tag{49}$$

$$\begin{cases} \dfrac{d\phi_y}{d\theta} = \dfrac{\partial K}{\partial J_y} = -\dfrac{C_n}{2\sqrt{\nu_x\nu_y}}\sqrt{\dfrac{J_x}{J_y}}\cos\left(\phi_x + \phi_y + \alpha_n\right) - \dfrac{\delta}{\sqrt{2}} \\[2ex] \dfrac{dJ_y}{d\theta} = -\dfrac{\partial K}{\partial \phi_y} = -\dfrac{C_n}{\sqrt{\nu_x\nu_y}}\sqrt{J_xJ_y}\sin\left(\phi_x + \phi_y + \alpha_n\right) \end{cases} \tag{50}$$

Taking the difference of the $\frac{dJ}{d\theta}$ equations we get immediately

$$J_x - J_y = \nu_xA_x^2 - \nu_yA_y^2 = \text{constant} \tag{51}$$

As a matter of fact since the phase variables ϕ_x and ϕ_y appear in K only in the combination $\ell\phi_x + m\phi_y$ we always have as part of the solution

$$mJ_x - \ell J_y = \text{constant} \tag{52}$$

for non-zero ℓ and m. This leads directly to the conclusion that sum resonances where ℓ and m have the same sign are unstable, namely both J_x and J_y can go to ∞ ; and difference resonances where ℓ and m have different signs are stable and represent only a coupling between x and y oscillations.

Further manipulations of (49) and (50) for $\delta = 0$ give the equation

$$\frac{d^2(J_x + J_y)}{d\theta^2} - \frac{C_n^{\ 2}}{\nu_x \nu_y}\left(J_x + J_y\right) = 0 \tag{53}$$

with the solution

$$J_x + J_y = \nu_x A_x^2 + \nu_y A_y^2 = e^{\pm \frac{C_n}{\sqrt{\nu_x \nu_y}}\theta} \tag{54}$$

This, then, gives the on-resonance growth rate.

If the rotational misalignment is assumed to be uniform within each quadrupole and uncorrelated between quadrupoles we have,

$$\left(C_n\right)_{rms} = \frac{2F}{\sqrt{Q}}\,\varepsilon RG(\omega)_{rms}, \left(\begin{array}{l}\text{independent of n up to the}\\ \text{cutoff value of } n = \frac{Q}{2}\end{array}\right) \tag{55}$$

where Q is the total number of quadrupoles in the ring and F is the fraction of the ring circumference occupied by quadrupoles. For the main ring of the NAL accelerator the parameters are (using the new lattice with two quadrupole lengths):

$$\begin{cases} R = 1000 \text{ m} & \nu_x \approx \nu_y \approx 20.25 \\ Q = 240 & \\ F = 0.075 & \\ \varepsilon RG = \dfrac{eR^2}{pc} G = 1.87 \times 10^4 & \end{cases}$$

For a large misalignment error of $(\omega)_{rms} = 1$ mrad. This gives $C_n = 0.18$ and

$$\begin{cases} \text{Half width of resonance band} = \dfrac{C_n}{\sqrt{\nu_x \nu_y}} = 0.009 \\ \text{On-resonance growth rate} = \dfrac{\sqrt{\nu_x \nu_y}}{2\pi C_n} = 17.7 \text{ rev/e-fold} \end{cases}$$