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ADVANTAGES OF A SET OF SECOND HARMONIC RF CAVITIES*

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I. INTRODUCTION

The addition of higher harmonics to the accelerating wave is not new. Allen and Rees at SLAC have studied the addition of a set of high harmonic cavities driven by the beam to control the bunch length in electron-positron storage rings. Klaus at LRL considered the addition of several higher harmonics to improve the bunching factor. People at CERN have also studied the use of both the first and higher harmonic cavities to accelerate the beam in order to reduce the power requirements. Symon at Wisconsin has investigated the use of a second harmonic in order to double the number of bunches.

The purpose of this paper is to investigate some of the advantages of including a separate set of rf cavities operating at twice the frequency of the main accelerating cavities. The results show that the maximum value of the phase oscillation frequency ω_s can be reduced between 30 and 50%, the bunching factor B can be increased by 20 to 40%, the total power can be reduced up to 40%, the variation of the space charge betatron frequency shift with longitudinal position is substantially reduced, and the possibility of separately controlling the phase focusing and the energy gain may be useful in treating the problem of accelerating through transition energy.

In this paper the relative phase between the two sets of cavities is always chosen such that the synchronous particle arrives at the main accelerating cavities

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when the phase of the accelerating field is ϕ_s and arrives at the auxiliary cavities when the accelerating field is zero. Significant improvements may be made if the relative phase angle is varied; however, this is left for future study.

II. NOTATION

The notation that is used in this report is given below:

$v_1 V$ is the total peak voltage of the main cavities.

$v_2 V$ is the total peak voltage of the auxiliary cavities.

ϕ_s is the synchronous phase angle of the main cavities.

h is the harmonic number of the main cavities.

f is the revolution frequency.

p is the momentum.

$\eta = p/f \quad df/dp = \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_t^2}\right)$.

γ_t is the value of γ at transition energy.

E_o = rest energy of particle.

$W = \left(\frac{E - E_s}{f}\right)$ is the canonical momentum.

ω_s is the phase oscillation frequency.

a subscript o refers to the value of parameter when $v_1 = 1.0$ and $V_2 = 0.0$,

a subscript m refers to the maximum value of a parameter, and

a subscript ℓ refers to the linear value of a parameter.

The energy gain per turn for a particle which crosses the main accelerating cavities at a phase angle of ϕ is thus given by

$$\frac{dE}{dn} = eV\left[v_1 \sin\varphi + v_2 \sin 2(\varphi - \varphi_s)\right]. \qquad (1)$$

The phase motion of a particle is described by the following Hamiltonian:

$$\mathcal{H} = \left(\frac{2\pi h f^2 \eta}{\beta^2 \gamma E_0}\right) \frac{W^2}{2} + eV\,F(\varphi) \,, \tag{2}$$

with

$$F(\varphi) = v_1 \left[\cos\varphi_s - \cos\varphi - (\varphi - \varphi_s)\sin\varphi\right] + \\ + \frac{v_2}{2}\left[1 - \cos 2(\varphi - \varphi_s)\right]. \tag{3}$$

III. REDUCTION OF LINEAR SYNCHROTRON OSCILLATION FREQUENCY

The linear synchrotron oscillation frequency $\omega_{s\ell}$ is given by

$$\omega_{s\ell} = \left(\frac{2\pi h f^2 \eta eV}{\beta^2 \gamma E_0}\right)^{1/2} \sqrt{F''(\varphi_s)} \\ = \left(\frac{2\pi h f^2 \eta eV}{\beta^2 \gamma E_0}\right)^{1/2} \sqrt{(v_1 \cos\varphi_s + 2v_2)} \,. \tag{4}$$

Thus, in order to decrease $\omega_{s\ell}$ by use of second harmonic, v_2 must be negative. However there is a limit to how much we can decrease v_2 before $\omega_{s\ell}$ becomes imaginary. From now on we will always assume that v_1 is positive and

$$v_2 \geq -\frac{v_1}{2}\cos\varphi_s \,, \tag{5}$$

so that $\omega_{s\ell}$ is always real. We see that in principle we can always make $\omega_{s\ell}$ go to zero. However, when $\omega_{s\ell} = 0$ the motion is completely nonlinear, and the value of $\omega_{s\ell}$ is irrelevant, since none of the particles oscillate at this frequency.

So far, nothing has been discussed as to how the voltage v_1V and the synchronous phase ϕ_s must vary as the voltage v_2V changes. In order for the investigation to be meaningful, as v_2 is varied, it is necessary for the maximum stable phase area A_m (called the bucket area) and the voltage gain per turn of the synchronous particle to be held constant. The bucket area of one of the h buckets is given by

$$A_m = 2\left(\frac{\beta^2 \gamma e V E_0}{\pi h \Omega^2 \eta}\right)^{1/2} \int_{\varphi_{1m}}^{\varphi_{2m}} \sqrt{K - F(\varphi)}\, d\varphi , \tag{6}$$

where K_m is the maximum value for which there are two roots to the equation $F(\phi) = K_m$ and the limits are determined by $F(\phi_{1m}) = F(\phi_{2m}) = K_m$. The voltage gain per turn of the synchronous particle is

$$\Gamma = e v_1 V \sin \varphi_s . \tag{7}$$

The constraints that $\Gamma = \Gamma_o$ and $A_m = A_{mo}$ determine $v_1(v_2, \phi_{so})$ and $\phi_s(v_2, \phi_{so})$. These functions of v_2 are shown in Figs. 1 and 2 and the corresponding linear synchrotron oscillation frequency $\omega_{s\ell}$ is shown in Fig. 3 for several different values of ϕ_{so}. Note that for angles of $\phi_{so} < 30^o$ as v_2 decreases (becomes more negative) v_1 decreases and ϕ_s increases to satisfy the above constraints.

IV. NONLINEAR SYNCHROTRON FREQUENCY BEHAVIOR

Since the nonlinear motion of particles enclosed in the rf bucket becomes more important as the linear frequency $\omega_{s\ell}$ decreases, it is necessary to determine how the frequency ω_s changes with particle oscillation amplitude for various values of v_2. We will characterize the amplitude of a particle by the area that its phase trajectory encloses; for example, if a particle's phase trajectory encloses an area A, then we define its phase oscillation frequency to be $\omega_s(A)$. The phase oscillation frequency is given by

$$\omega_s = \omega_{so} \pi \sqrt{\frac{2}{\cos \varphi_{so}}} \left\{ \int_{\varphi_1}^{\varphi_2} \frac{d\varphi}{\sqrt{K - F(\varphi)}} \right\}^{-1}, \tag{8}$$

where

$$\omega_{s\ell o} = f \left(\frac{2\pi h \eta e V \cos \varphi_{so}}{\beta^2 \gamma E_o} \right)^{1/2}, \tag{9}$$

K is a constant that varies between zero and K_m, and the limits of ϕ_1 and ϕ_2 are given by $K = F(\phi_1) = F(\phi_2)$. The corresponding phase area is given by

$$A = 2 \left(\frac{\beta^2 \gamma e V E_o}{\pi h f \eta} \right)^{1/2} \int_{\varphi_1}^{\varphi_2} \sqrt{K - F(\varphi)}\, d\varphi. \tag{10}$$

The dependence of $\omega_s(A)$ upon the value of v_2 is shown in Figs. 4 through 7 for various values of ϕ_{so} with again the constraints that the bucket area and synchronous energy gain per turn be held constant. Note that for small values of negative v_2 the maximum value of ω_s is governed by the linear value while for larger values of negative v_2 the maximum value of ω_s is governed by the nonlinear values. Thus, while the linear value of ω_s can be reduced to zero, there is a limit to how far the maximum values of ω_s can be reduced. Attempts to further reduce the maximum value of ω_s by varying the relative phase of the two rf systems have not been made, and it is possible that such attempts might be more successful.

V. INCREASE OF BUNCHING FACTOR

The bunching factor B entering space charge calculations is the ratio of average to peak particle density. The variation of B with v_2 is studied in this section. If we assume that the phase area A within a phase trajectory is uniformly filled with particles, then the number of particles within an element $d\phi$ is proportional to $W(\phi)d\phi$. The average particle density is proportional to $\frac{1}{2\pi}\int W d\phi = \frac{1}{4\pi} A$, and the bunching factor is

$$B = \frac{A}{4\pi W_m} , \tag{10}$$

where

$$W_m = \sqrt{\left(\frac{\beta^2 \gamma e V E_o}{\pi h f^3 \eta}\right) \left[K - F(\phi_s)\right]} \tag{11}$$

and K is the constant used in Eq. (7). The dependence of B(A) upon v_2 is shown in Figs. 8 through 11 for various values of ϕ_{so}. For the case where the phase space occupied by the particles is 70% of the bucket area, the bunching factor is shown as a function v_2 in Fig. 12 for several values of ϕ_{so}.

VI. TOTAL POWER REQUIREMENT

The total amount of power required by a system of rf cavities is dependent upon many factors, and a complete description of the system must be known before definite answers can be obtained. However, it is still interesting to make some estimates of how the addition of a set of second harmonic cavities will change the power requirements.

The power delivered to the beam will be independent of the rf system and so will not change by adding additional second harmonic cavities.

The power losses in the ferrite and walls account for the major portion of the power requirement and is considered here by assuming that the shunt impedance per unit length is constant, independent of frequency. If the shunt impedance of the first harmonic cavity is equal to S, then the shunt impedance of the second harmonic cavity (which is one-half of the length of the first harmonic cavity) is equal to $\frac{S}{2f}$, where f is equal to one when the tuning range of the second harmonic cavity is equal to the tuning range of the first harmonic cavity, and $f \ll 1$ for a narrow tuning range of the second harmonic cavity. The total power losses can be written as

$$P = \frac{N_1}{2S}\left(\frac{v_1 V}{N_1}\right)^2 + \frac{N_2}{2(S/2f)}\left(\frac{v_2 V}{N_2}\right)^2 \qquad (12)$$

where N_1 and N_2 are the number of first and second harmonic cavities, respectively, and $|v_1 V/N_1|$ and $|v_2 V/N_2|$ are the peak voltages per cavity. By defining P_o as the power loss for the case where $v_1 = 1.0$ and $v_2 = 0.0$, we have

$$P/P_o = \left(v_1^2 + 2\,\frac{f\,N_1}{N_2}\,v_2^2\right) \qquad (13)$$

with

$$P_0 = \left(V^2 / 2 N_1 S \right) \tag{14}$$

For the case where we have an equal number of first and second harmonic cavities ($N_1 = N_2$), the power loss ratio is given by

$$P/P_0 = \left(v_1^2 + 2 f v_2^2 \right) \tag{15}$$

while for the case where we have equal gap fields on both sets of cavities (the voltage on a second harmonic cavity is one-half the voltage on a first harmonic cavity, since the gap dimensions are in the ratio 0.5:1 and thus $N_2 = 2\left|\frac{v_2}{v_1}\right| N_1$) the power loss ratio is given by

$$P/P_0 = \left(v_1^2 + f \, |v_1 v_2| \right) \tag{16}$$

For the case where the second harmonic cavity has a narrow tuning range, the power loss ratio is approximately given by

$$P/P_0 = v_1^2 \tag{17}$$

The power requirements for these various cases are shown as a function of v_2 in Figs. 13 through 16 for various values of ϕ_{so}.

The above treatment for power losses in the dual cavity system is by no means complete. For example, by removing the restriction that the auxiliary cavity does not accelerate the synchronous particle, it should be possible to further reduce the total power loss. Also, for the case where the tuning range of the main cavity is close to 2:1, it may be possible to use the auxiliary cavity in the latter part of the tuning range as a fundamental cavity to aid in the acceleration.*

VII. VARIATION OF BETATRON FREQUENCY WITH LONGITUDINAL POSITION

Since the variation of the space charge betatron frequency shift with longitudinal position can introduce synchrobetatron resonances, it is desirable to reduce this variation. The variation of the betatron frequency arises from the fact that the particle density varies with longitudinal position. For the case where the variation of the transverse beam dimensions is small over the length of a bunch, the vacuum chamber height is small compared to the bunch length, and image effects are negligible, the betatron frequency shift $\Delta\nu$ is proportional to the particle density per unit length λ. If we assume that the phase area within a phase trajectory is uniformly filled with particles, then $\Delta\nu$ is proportional to the canonical momentum W. Figure 17 shows the variation of W and hence $\Delta\nu$ with longitudinal position for two values of v_2 with $\phi_{so} = 30°$ and $A/A_{mo} = 0.75$. The betatron frequency shift $\Delta\nu$ can also be written as

$$\Delta\nu = K \sum_{n=0}^{\infty} a_n \cos\left(2\pi n \frac{z}{L} + \alpha_n\right) \tag{18}$$

where K is a constant and L the bunch length.

*The authors would like to thank Dr. M. A. Allen for pointing out this possibility.

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Several values of a_n are given in the table below for the two curves in Fig. 17.

n	a_n	
	$v_2/v_1 = 0.0$	$v_2/v_1 = -.436$
0	.558	.435
1	.232	.118
2	.079	.073
3	.043	.040
4	.028	.026

The value of the average betatron frequency shift is reduced by including a set of second harmonic cavities, but more important (as can be seen from Fig. 17 or by comparing the values of a_1) the variation of $\Delta\nu$ with z is reduced by approximately a factor of 2.

VIII. USE OF HIGHER HARMONICS IN ACCELERATING THROUGH TRANSITION

The problem of bunch length oscillations due to space charge forces at transition energy has been pointed out by Sørenseen and Hereward[1-4] and a method of correcting it called the "triple switch" has been proposed. The introduction of a higher harmonic set of cavities could also be useful in correcting the bunch length oscillations.

The introduction of the higher harmonic cavities, phased such that the synchronous particle arrives when the voltage on these cavities is zero, can be used to cancel the defocusing space charge force. For the higher harmonic cavities, let the harmonic number be p times the fundamental frequency and the total voltage

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be $v_p V$. Then Eq. (3.1) of Ref. 4, which describes the linear phase motion, is modified as given below:

$$\frac{d}{dt}\left(\frac{R\gamma E_0}{h c^2 \eta}\frac{d\bar{\varphi}}{dt}\right) + \left[\frac{eV}{2\pi}\left(|\cos\varphi_{s0}|\,\mathrm{sgn}(t_0 - t) + p\, v_p\right) + \right.$$

$$\left. - \frac{3 h g_0 c^2 N}{2\gamma^2 R \bar{\varphi}^3_{max}}\right]\bar{\varphi} \tag{19}$$

where $\bar{\phi} = \phi - \phi_s$

N = total number of particles

$2\bar{\phi}_{max}$ is the maximum bunch length for a parabolic bunch

R is the radius of the accelerator

g_o is a geometrical factor

t_o is time at transition energy

R is the radius of the accelerator

(Note that in this equation η is positive below transition and that the phase is jumped only on the fundamental cavity.)

Thus, in order to cancel the space charge defocusing, we must have

$$v_p = \frac{3\pi h g_0 E_0 r_0 N}{p e V \gamma^2 R \varphi^3_{max}} \tag{20}$$

with r_o the classical particle radius. We see that it is inversely proportional to the harmonics factor p; however, if p becomes too large, the linear approximation for the focusing due to the higher harmonic cavities is invalid. Thus, we must restrict p by

$$p < \frac{\pi}{2\varphi_{max}} \tag{21}$$

We take, for example, a booster with the following parameters:

$h = 84$	$eV = 0.8$ MeV
$r_o = 1.53 \times 10^{-16}$ cm	$\gamma = 7$
$g_o = 2.5$	$R = 75$ m
$N = 4 \times 10^{12}$	$\phi_{max} = 0.3$ rad
$E_o = 0.938$ MeV	$p = 4$

and obtain $v_p = 0.0358$. This is a rather small voltage and should be easy to obtain in practice.

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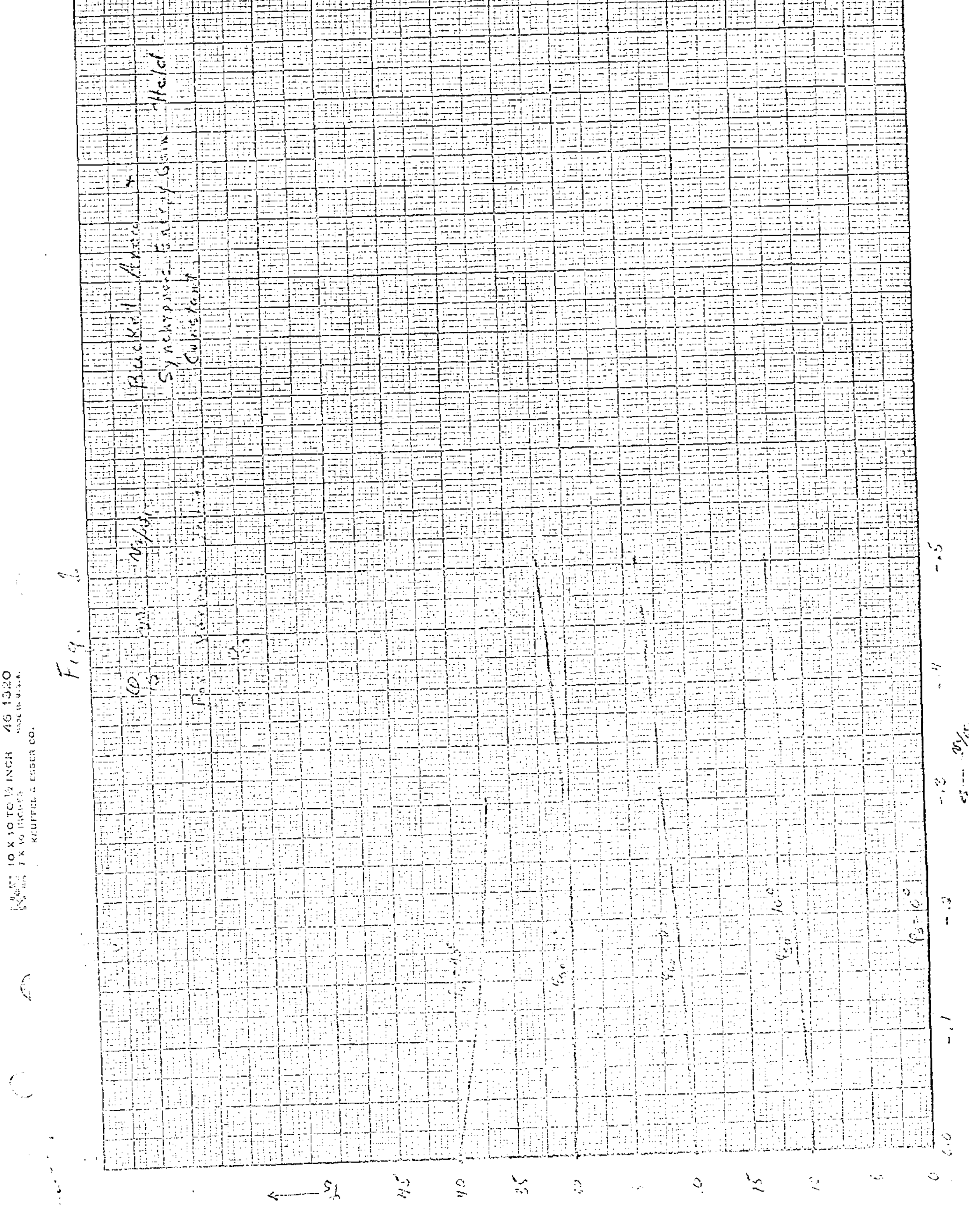
Fig. 2
Bucket Area +
Synchrotron Energy Gain Held
Constant
0
5
10
15
25
30
35
40
45
0.0
-.1
-.2
-.3
-.4
-.5
KEUFFEL & ESSER CO.

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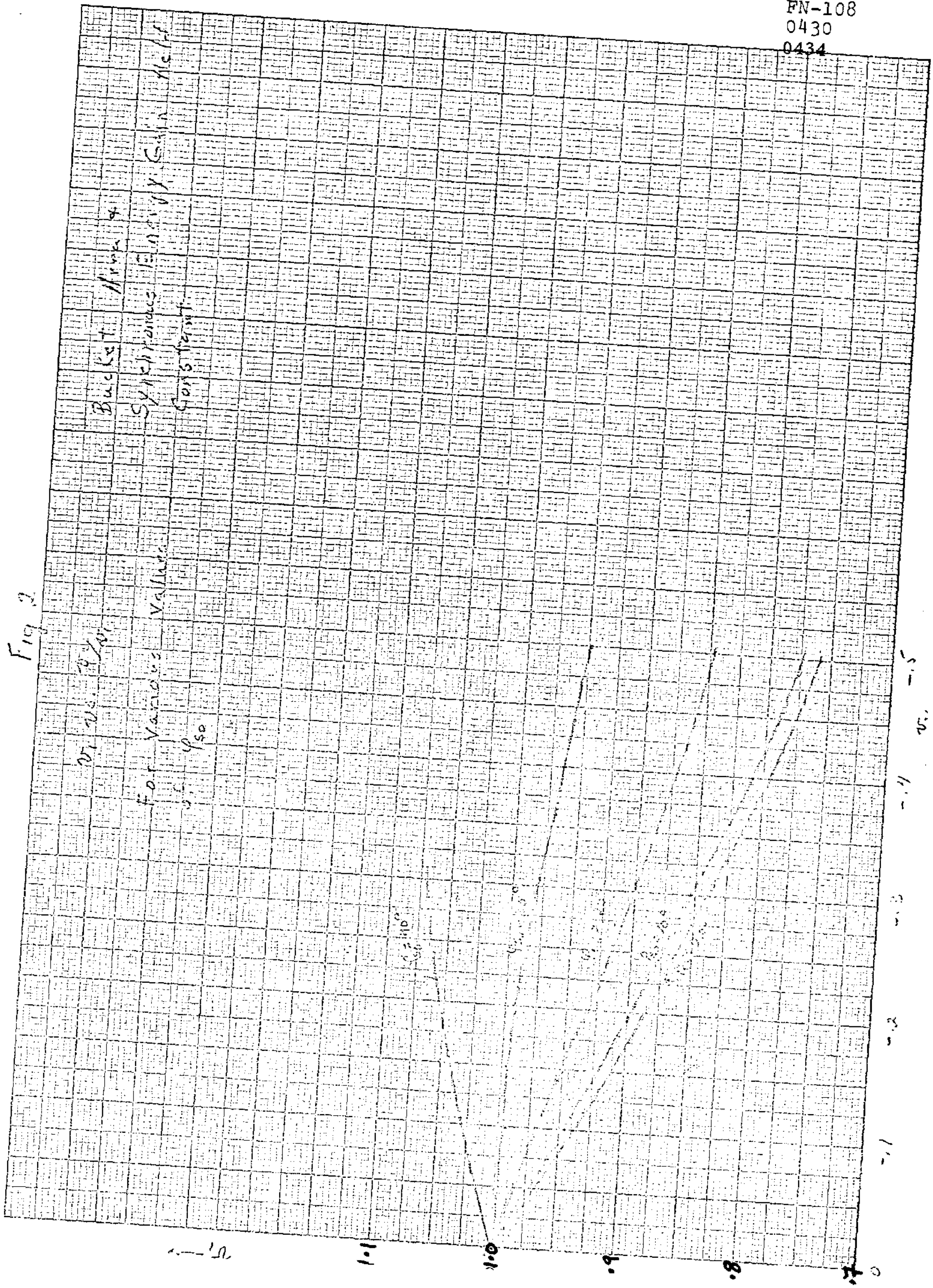
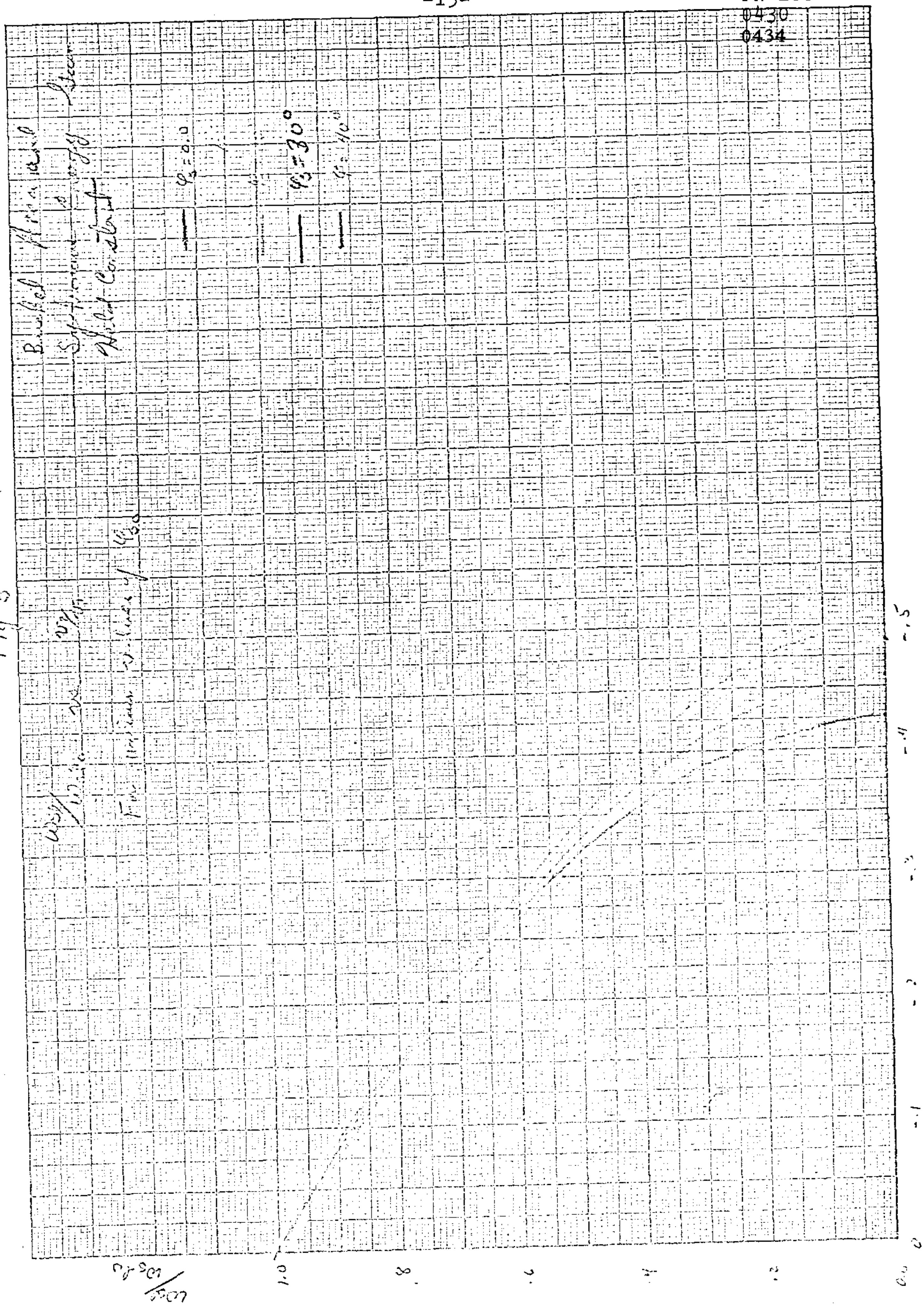
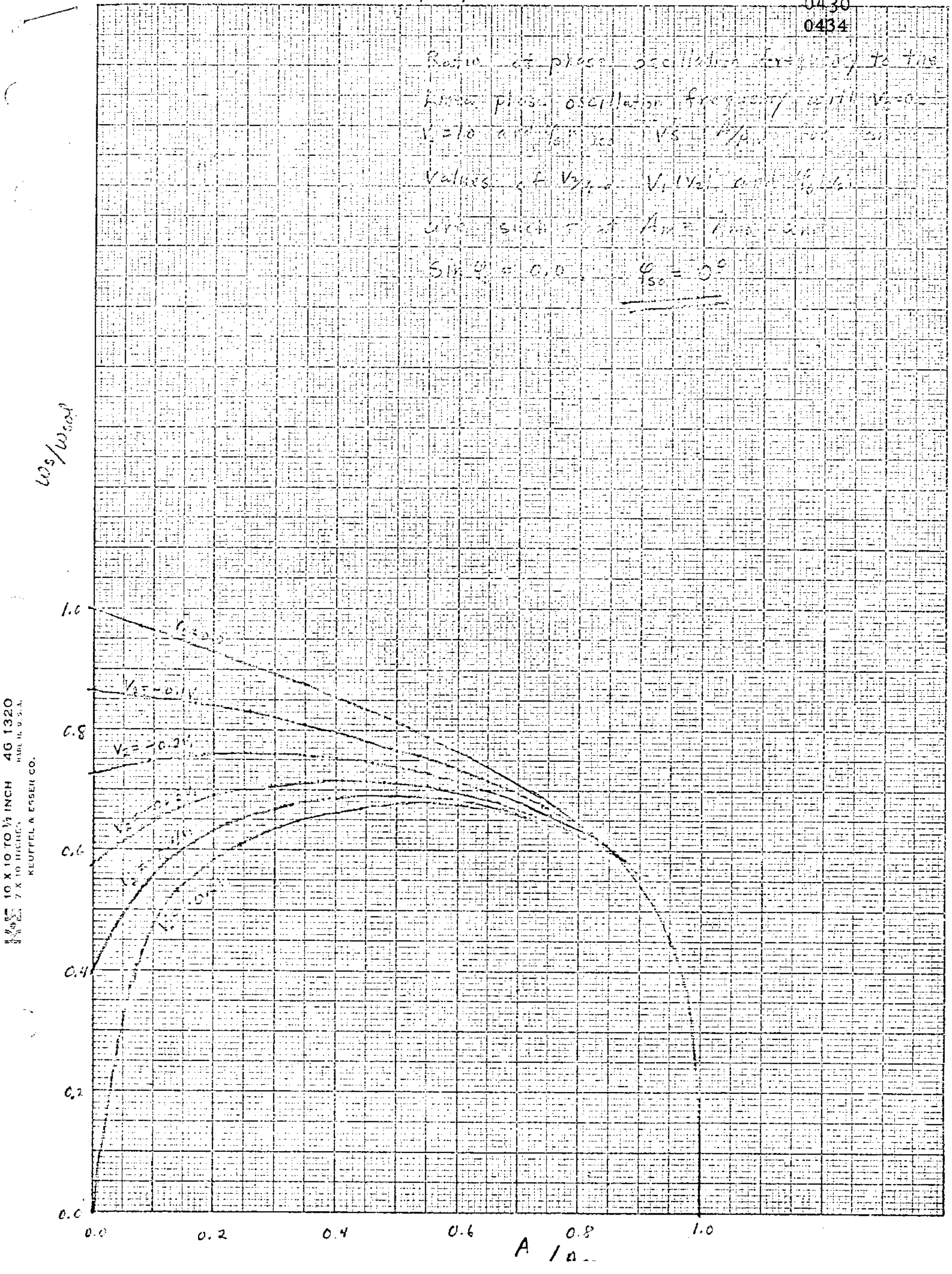
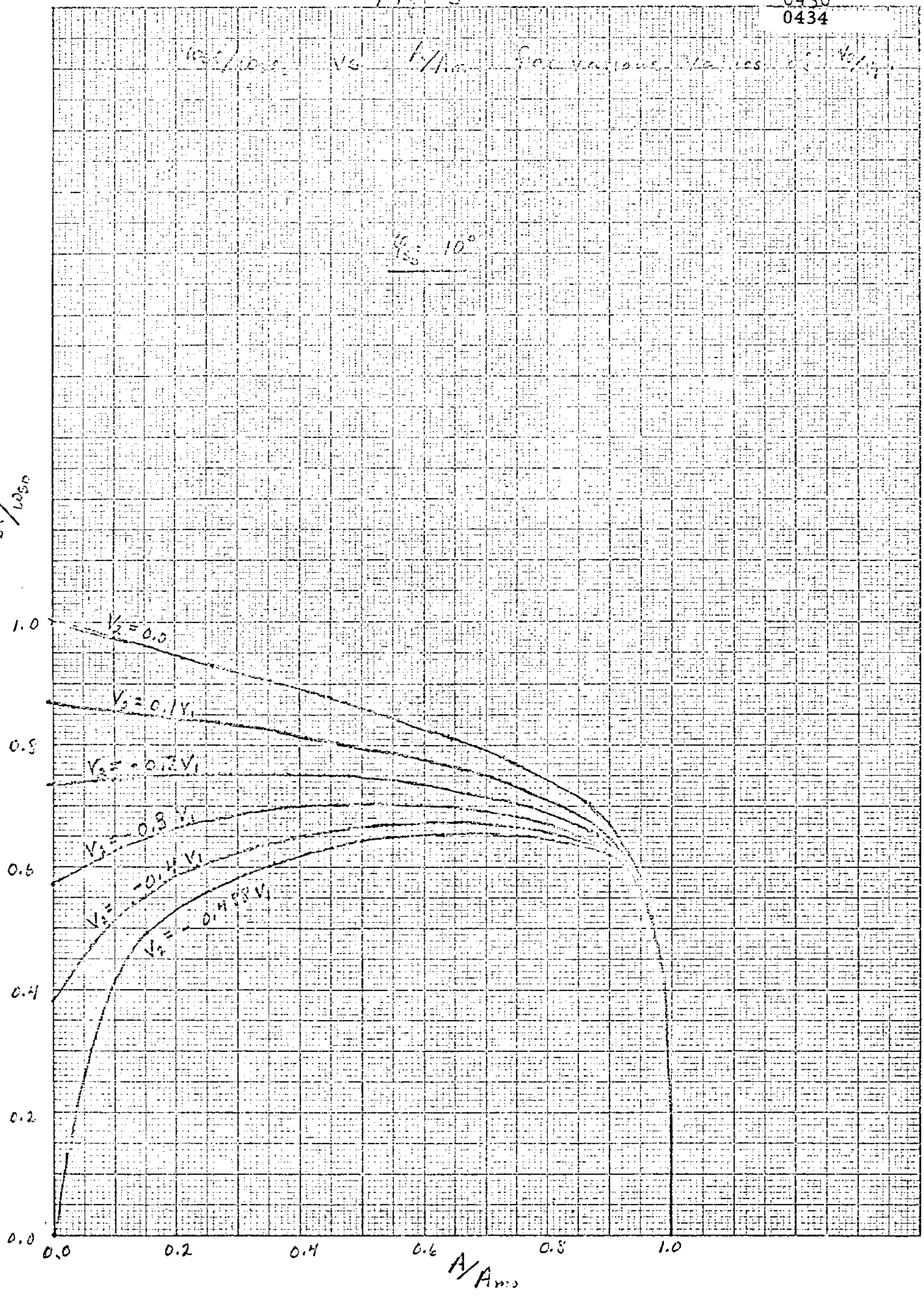





Ratio of phase oscillation frequency to the
linear phase oscillation frequency with
Values of
Sin φ = 0.0 ; φs0 = 0°
ωs/ωs0
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
A / a
10 X 10 TO 1/2 INCH 46 1320
7 X 10 INCHES
KEUFFEL & ESSER CO.


$\psi_0 = 10°$
$V_2 = 0.0$
$V_2 = 0.1 V_1$
0.0
0.2
0.4
0.6
0.8
1.0
A/A_m
$\omega_{3\ell}/\omega_{30}$

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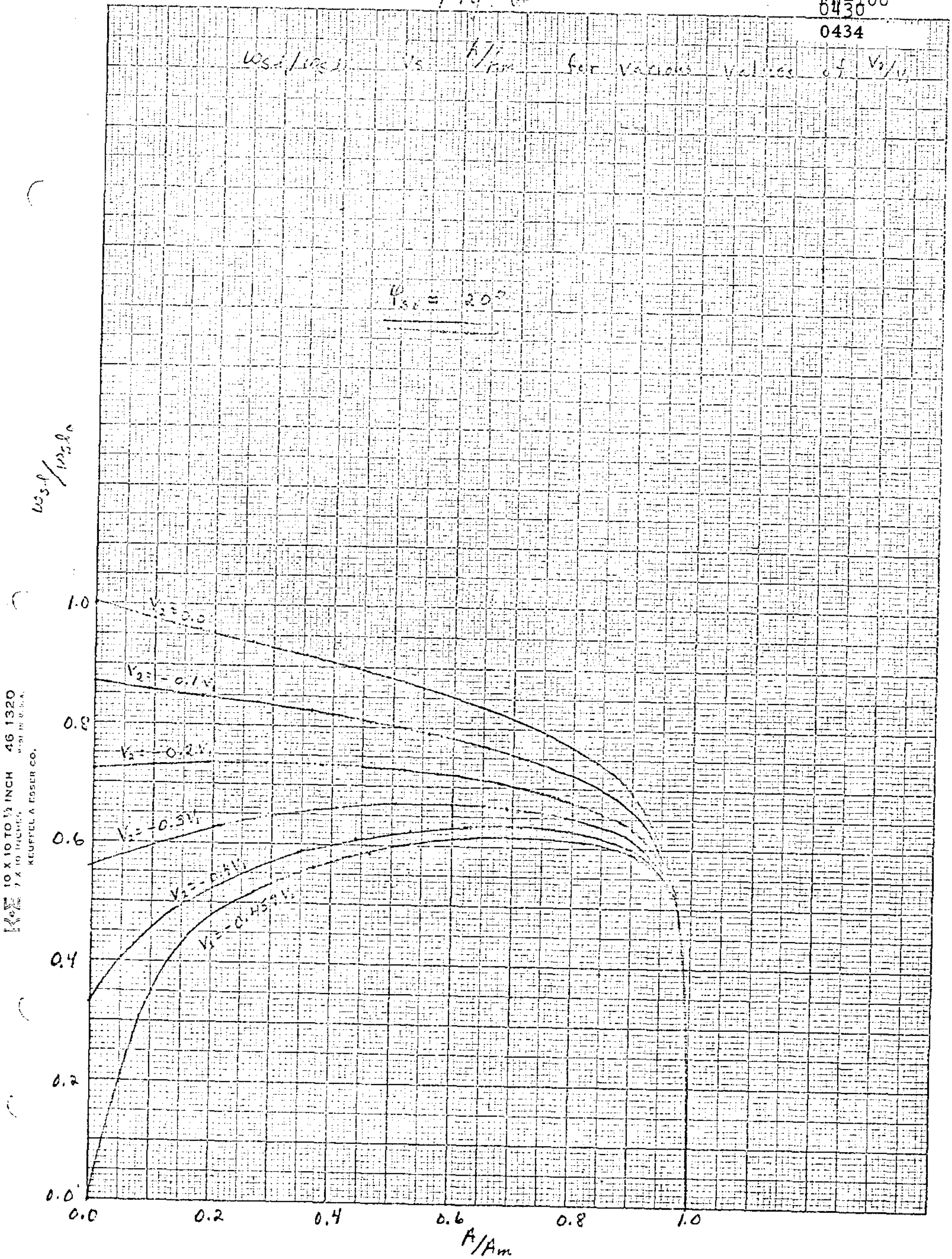

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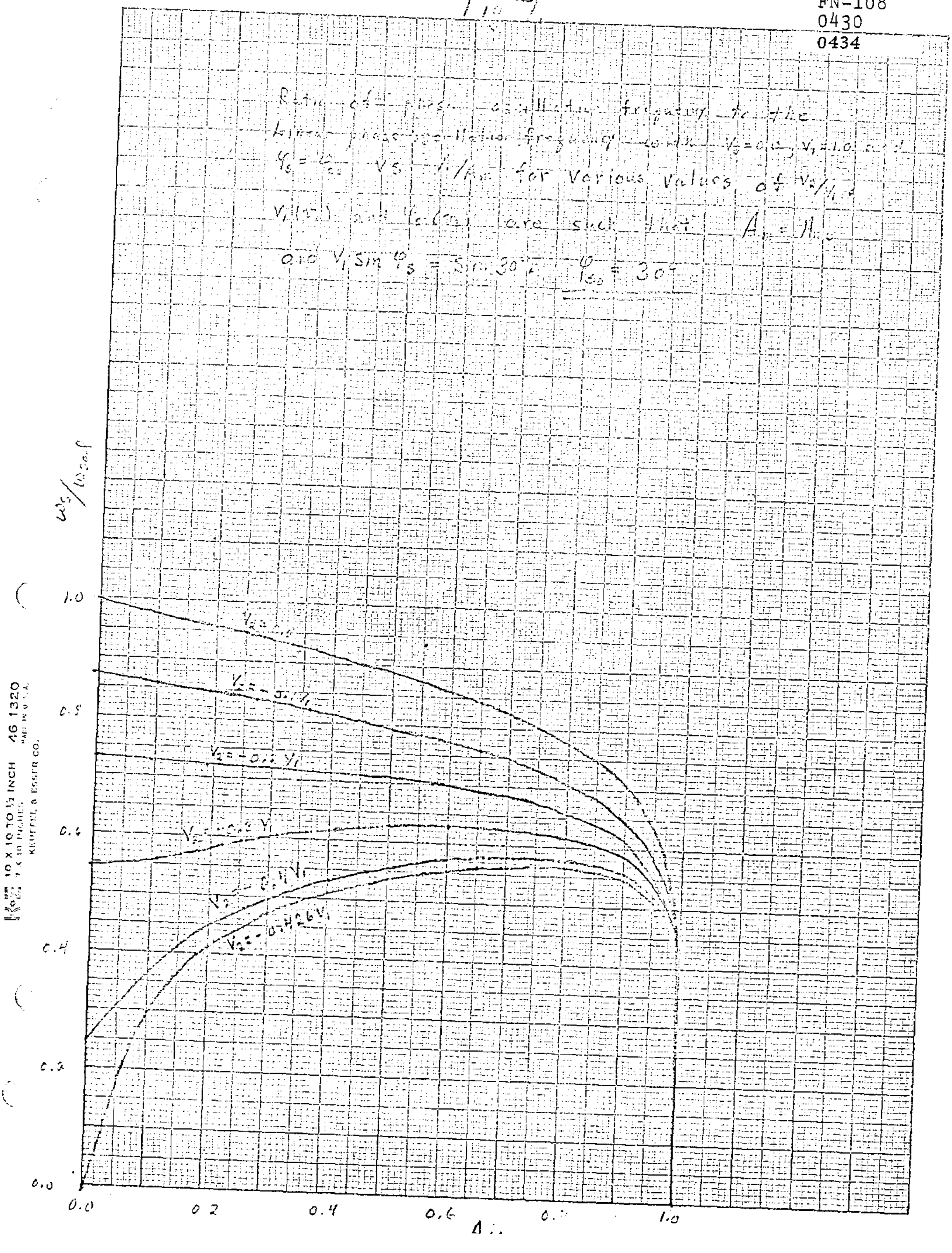
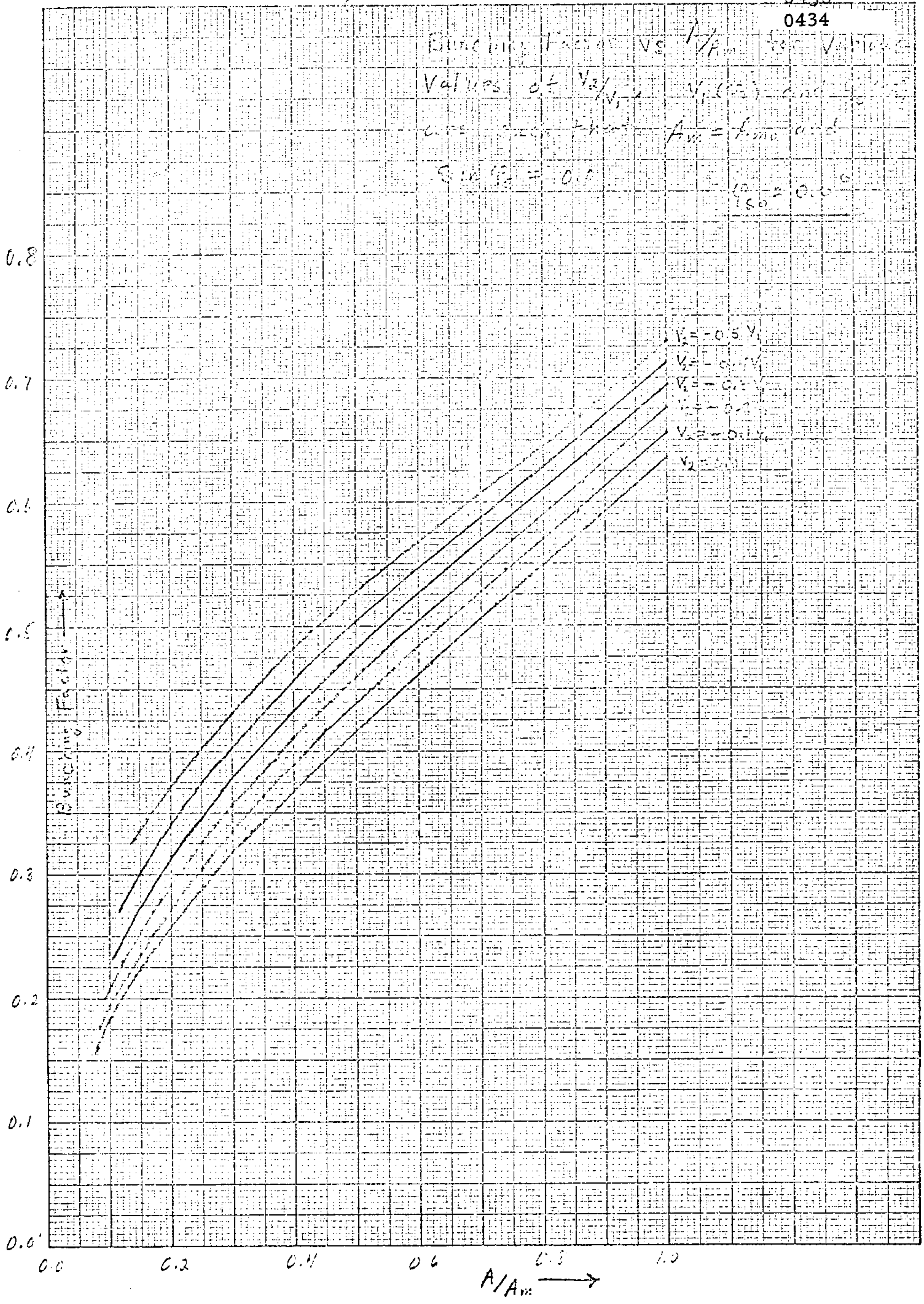
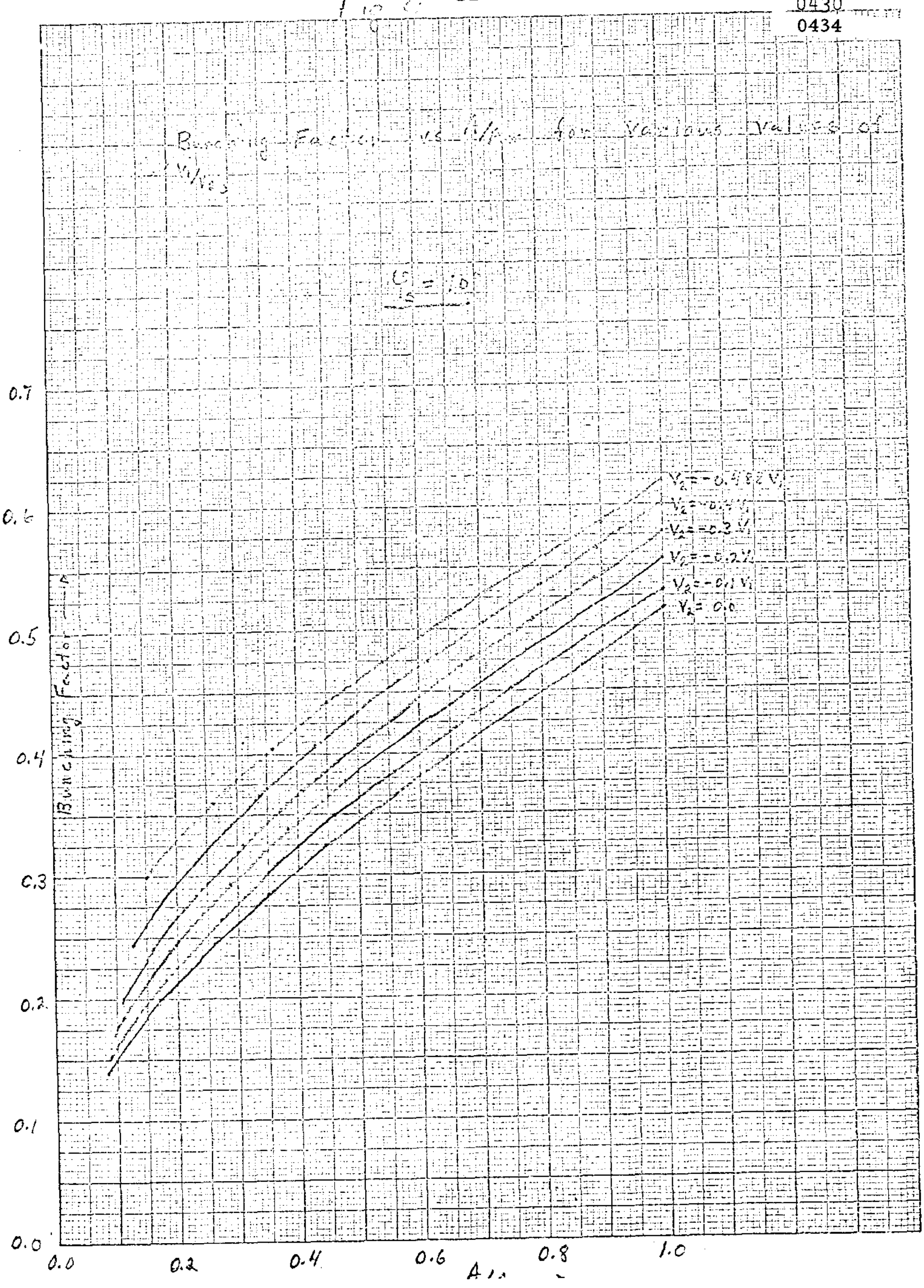



Bunching Factor vs A/A_m for Various
Values of V_2/V_1
$A_m = A_{mo}$ and
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0
A/A_m
Bunching Factor


Bunching Factor vs
for Various Values of
$V_2 = -0.482 V_1$
$V_2 = -0.3 V_1$
$V_2 = -0.2 V_1$
$V_2 = -0.1 V_1$
$V_2 = 0.0$
Bunching Factor
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.0
0.2
0.4
0.6
0.8
1.0

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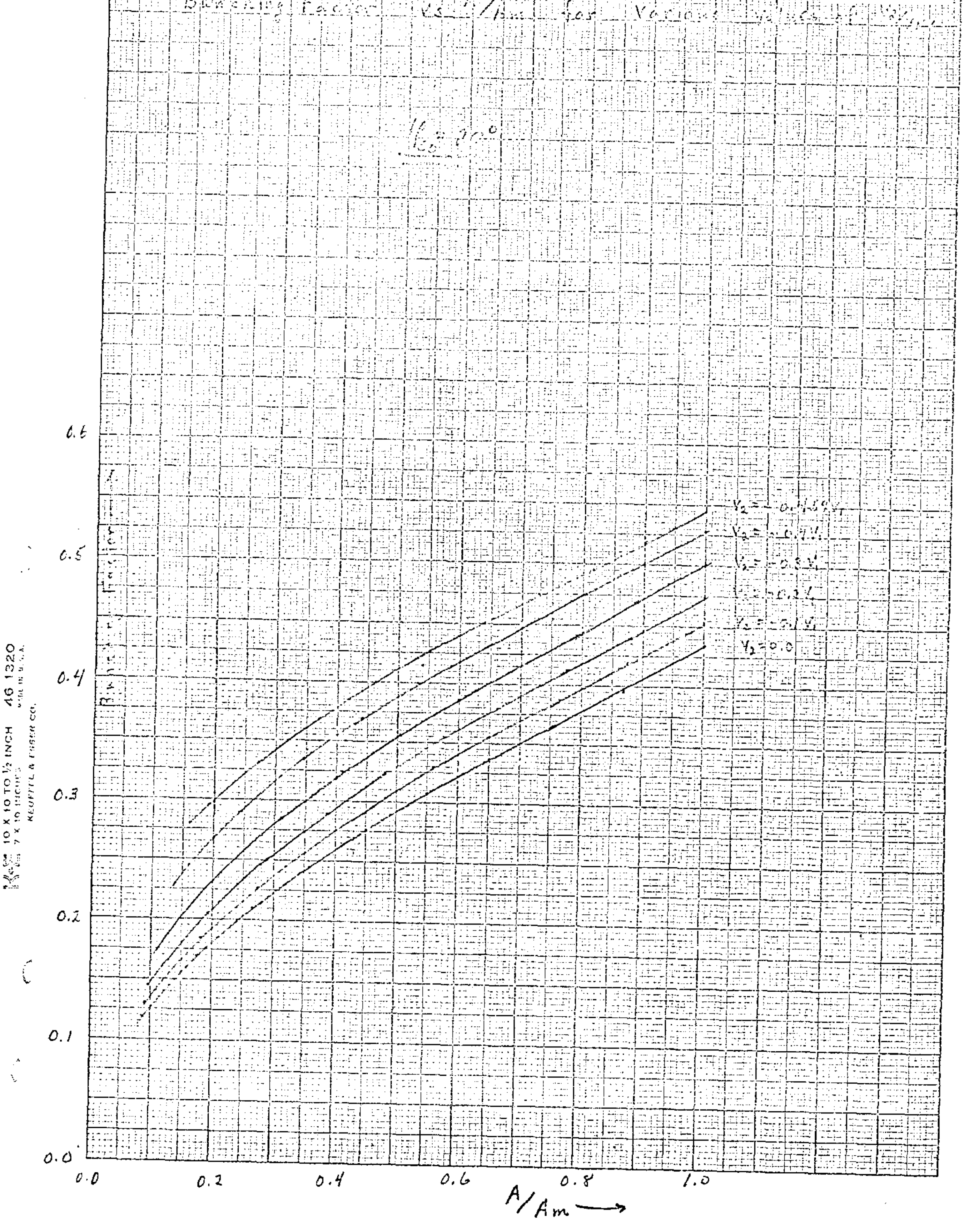

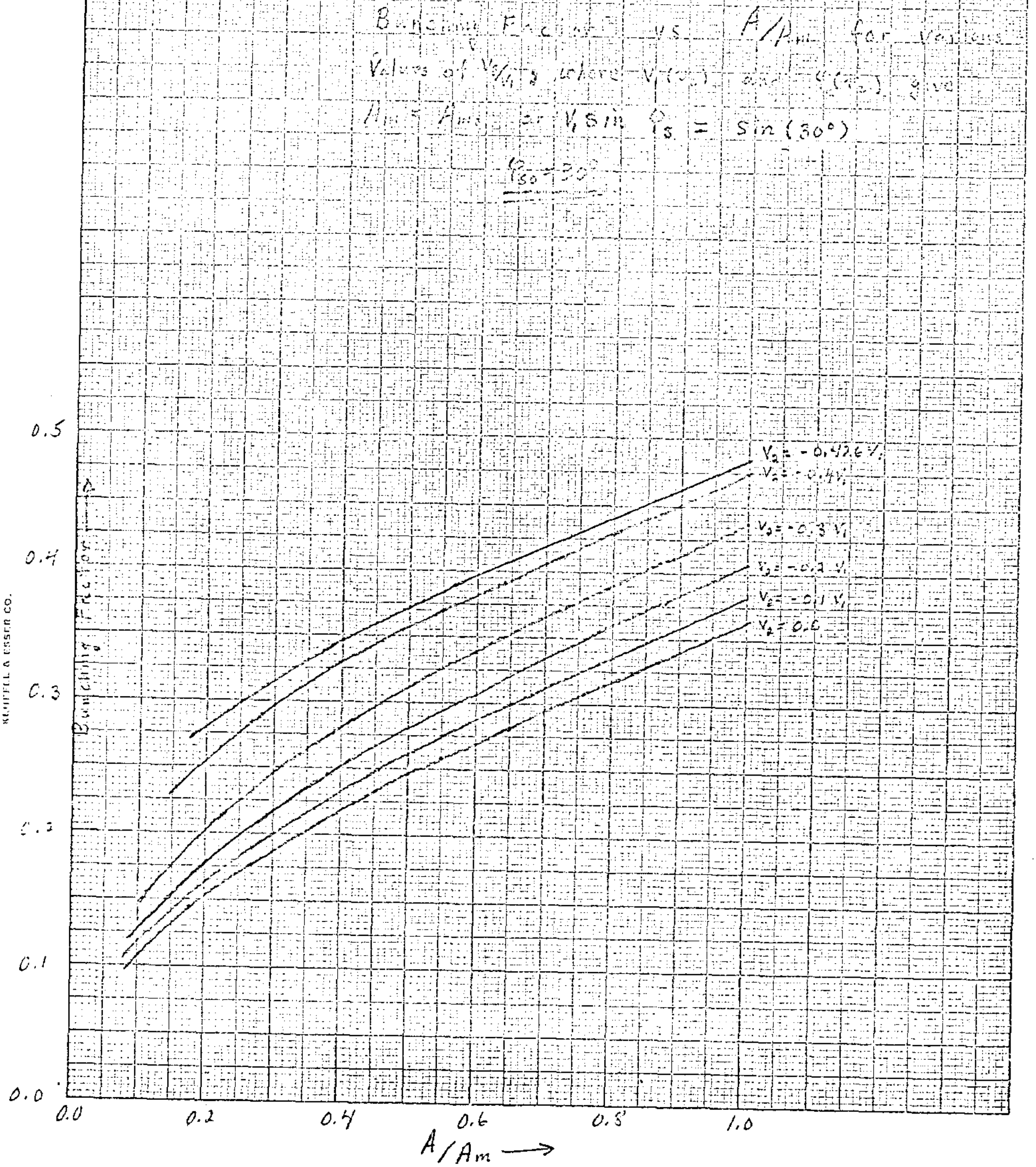
Values of V_2/V_1 where
$\varphi_{s0} = 30°$
$V_2 = -0.4V_1$
$V_2 = -0.3V_1$
$V_2 = -0.2V_1$
$V_2 = -0.1V_1$
$V_2 = 0.0$
Bunching Factor
A/Am
0.0
0.1
0.3
0.4
0.5
0.2
0.4
0.6
1.0

Fig 24

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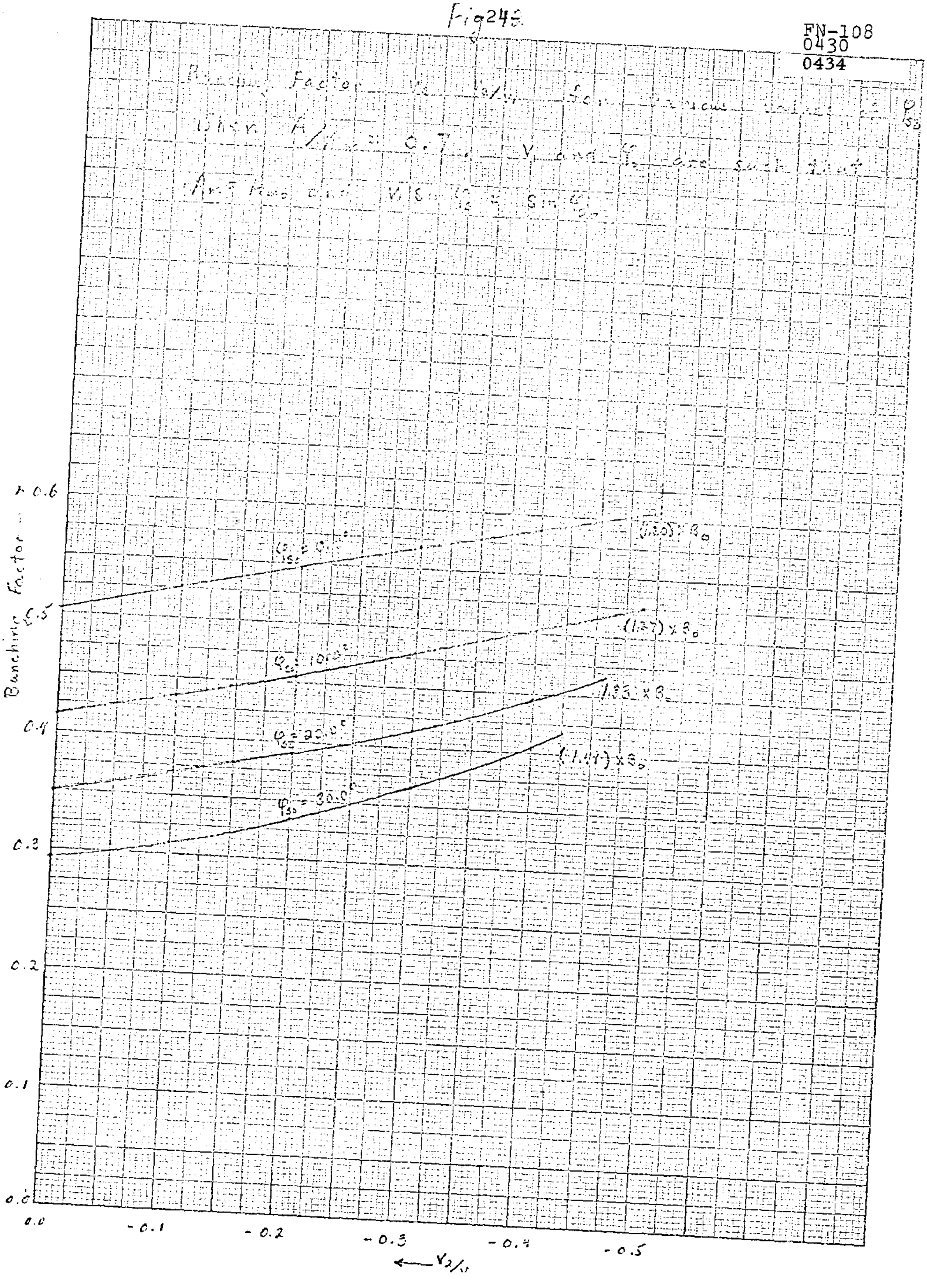
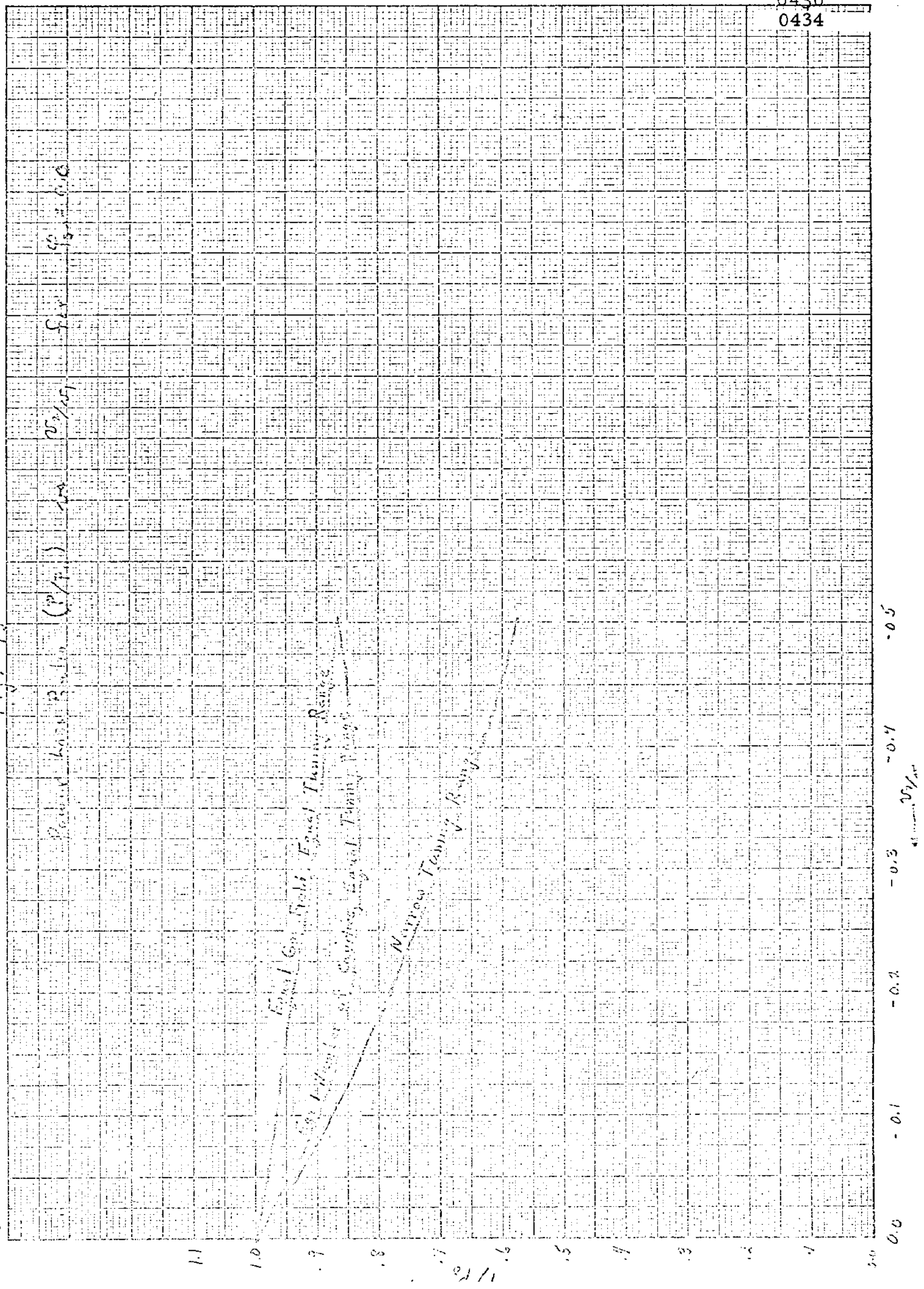


Fig. 13


Narrow Tuning Range
Equal Tuning Range
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1

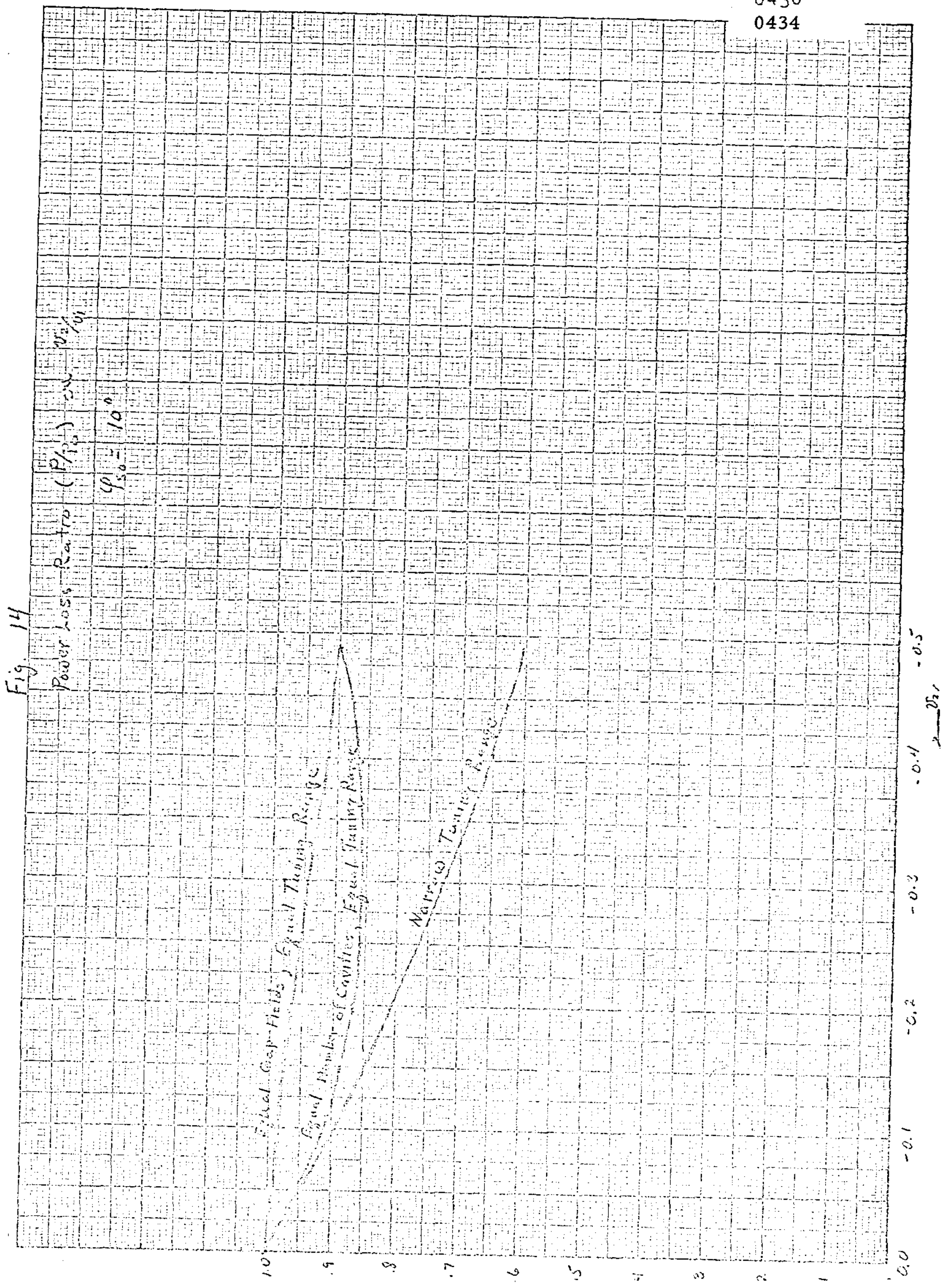


Fig. 14
Power Loss Ratio (P/P0) vs v2/v1
Equal Gap-Fields, Equal Tuning Range
Equal Number of Cavities, Equal Tuning Range
Narrow Tuning Range
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0.0
-0.1
-0.2
-0.3
-0.4
-0.5

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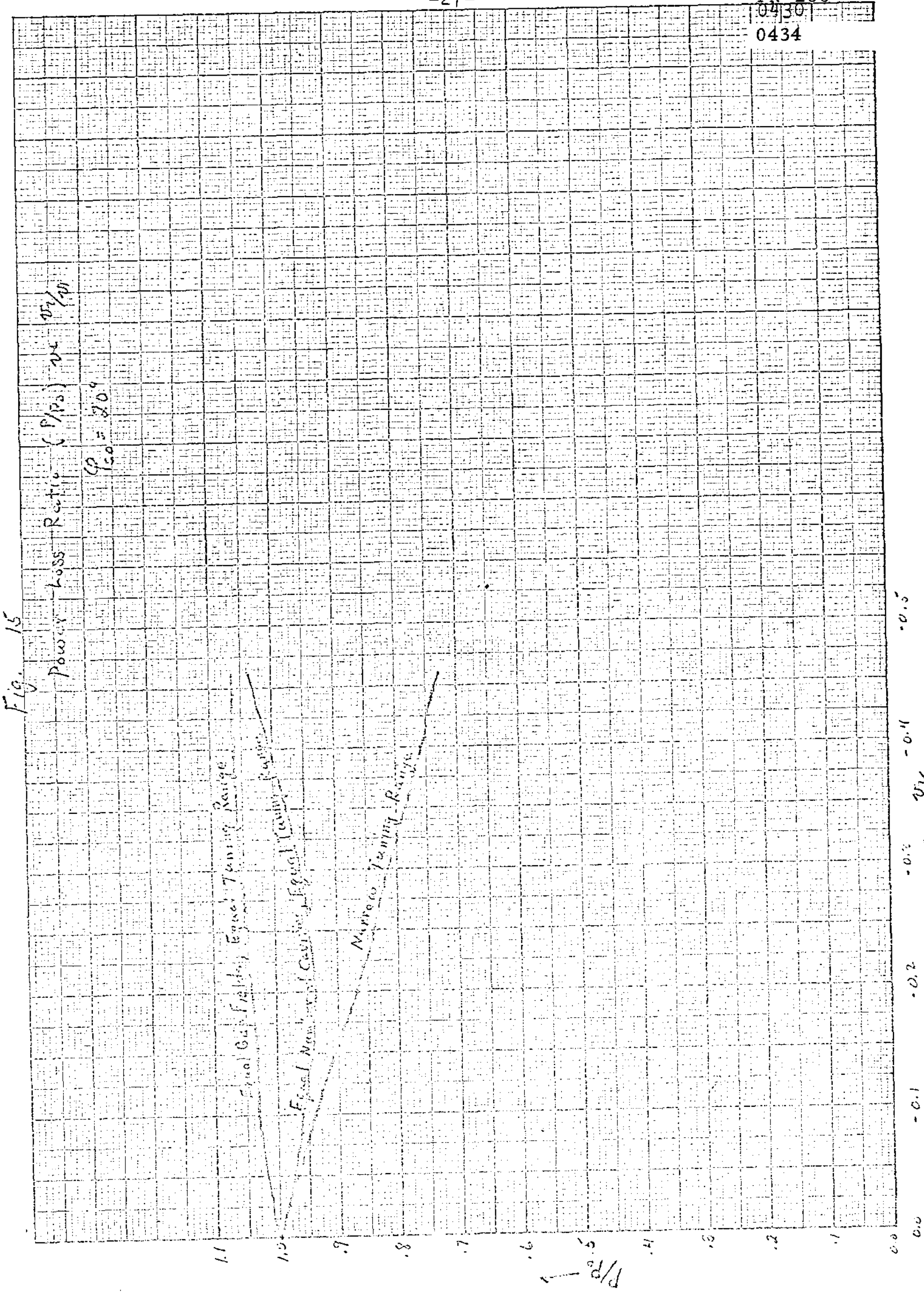
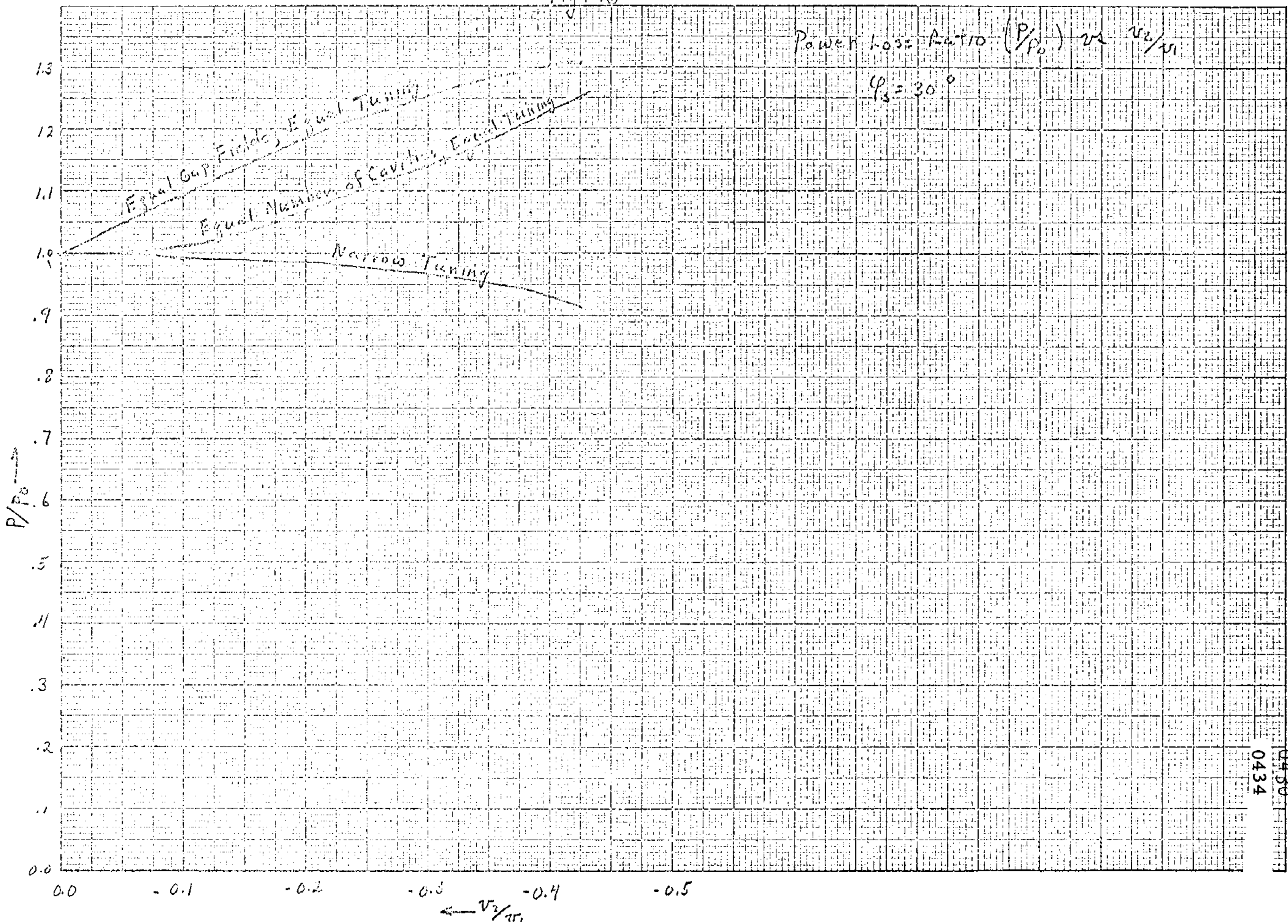


Fig. 16


Power Loss Ratio (P/P_0) vs v_2/v_1
$\varphi_s = 30°$
Equal Gap Fields, Equal Tuning
Equal Number of Cavities, Equal Tuning
Narrow Tuning
P/P_0
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0.0
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
v_2/v_1

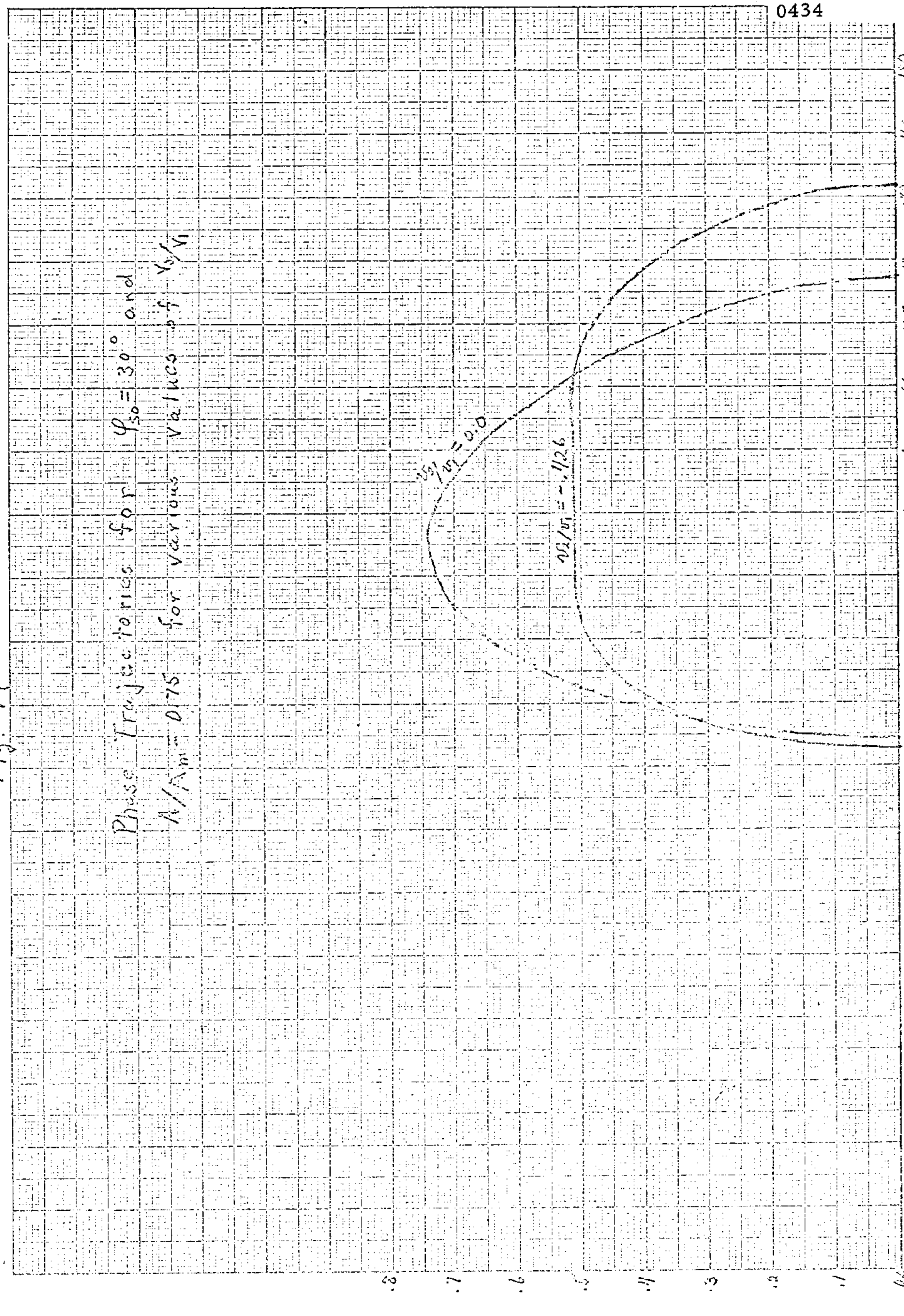


Fig. 17

Phase Trajectories for $\varphi_{s0} = 30°$ and $A/A_m = 0.75$ for various values of v_s/v_1