FN-89
2040

IS IT POSSIBLE TO ACHIEVE A MATRIX OF +1 BY MEANS OF A QUADRUPOLE QUADRUPLET?

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October 5, 1967

I. Introduction

In a previous paper[1] we have examined the possibility of achieving a matrix -1, i.e., an optical turn-over system, by means of a quadrupole quadruplet; analytic conditions, relating the focusing parameters of the quadrupoles to the drift lengths located between the quadrupoles, have been formulated for obtaining this transformation. We now attempt to treat the similar problem for the case of reproduction, i.e., when a total transfer matrix +1 is wanted.

We use exactly the same notations as before and assume in all cases θ [illegible] $/2$.

II. General Properties of a Reproduction Quadruplet

The conditions for achieving a matrix +1 can generally be written for the case of a quadrupole[2]

$$\left(L_{12} + \frac{a_1}{c_1} + \frac{a_2}{c_2}\right)\left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3}\right) = \frac{1}{c_2^{\,2}} - \frac{c_4}{c_1 c_2 c_3} \tag{1}$$

$$\left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3}\right)\left(L_{34} + \frac{a_3}{c_3} + \frac{a_4}{c_4}\right) = \frac{1}{c_3^{\,2}} - \frac{c_1}{c_2 c_3 c_4} \tag{2}$$

When these equations are satisfied, the transfer matrix of the base quadruplet is

$$M = \begin{Vmatrix} 1 & B \\ 0 & 1 \end{Vmatrix} \tag{3}$$

with

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$$B = \frac{c_2}{c_1}\frac{c_3}{c_4}\left(L_{23} + \frac{a_2}{c_2} + \frac{a_3}{c_3}\right) + \frac{a_1}{c_1} + \frac{a_4}{c_4} \tag{4}$$

and the object-image conjugation relation becomes

$$p + q = -B \tag{5}$$

For real images we must have

$$B < 0 \tag{6}$$

We shall now apply these conditions to an alternating-gradient structure.

III. Quadruplet with Symmetric Focusing

We call the focusing of the quadruplet symmetric if the quadrupoles are laid out following the sequence cddc in one plane and dccd in the other.

Eq. (4) gives then for the element B of the transfer matrix

$$B_{cddc} = \frac{k_2 k_3 \mathrm{sh}\ \theta_2 \mathrm{sh}\ \theta_3}{k_1 k_4 \sin\theta_1 \sin\theta_4}\left(L_{23} + \frac{\cot h\theta_2}{k_2} + \frac{\cot h\theta_3}{k_3}\right) - \frac{\cot\theta_1}{k_1} - \frac{\cot\theta_4}{k_4} \tag{7}$$

in the cddc plane and

$$B_{dccd} = \frac{k_2 k_3 \sin\theta_2 \sin\theta_3}{k_1 k_4 \mathrm{sh}\ \theta_1 \mathrm{sh}\ \theta_4}\left(L_{23} - \frac{\cot\theta_2}{k_2} - \frac{\cot\theta_3}{k_3}\right) + \frac{\cot h\theta_1}{k_1} + \frac{\cot h\theta_4}{k_4} \tag{8}$$

For obtaining real images in both planes, we must require $B_{cddc} < 0$ and $B_{dccd} < 0$, i.e.,

$$\frac{k_2 k_3 \mathrm{sh}\ \theta_2 \mathrm{sh}\ \theta_3}{k_1 k_4 \sin\theta_1 \sin\theta_4}\left(L_{23} + \frac{\cot h\theta_2}{k_2} + \frac{\cot h\theta_3}{k_3}\right) < \frac{\cot\theta_1}{k_1} + \frac{\cot\theta_4}{k_4} \tag{9}$$

$$\frac{k_2 k_3 \sin\theta_2 \sin\theta_3}{k_1 k_4 \mathrm{sh}\ \theta_1 \mathrm{sh}\ \theta_4}\left(L_{23} - \frac{\cot\theta_2}{k_2} - \frac{\cot\theta_3}{k_3}\right) < -\frac{\cot h\theta_1}{k_1} - \frac{\cot h\theta_4}{k_4} \tag{10}$$

Adding these two inequalities we obtain

$$L_{23}\left(\frac{\mathrm{sh}\,\theta_2\mathrm{sh}\,\theta_3}{\sin\theta_1\sin\theta_4}+\frac{\sin\theta_2\sin\theta_3}{\mathrm{sh}\,\theta_1\mathrm{sh}\,\theta_4}\right) <$$

$$-\frac{k_1k_4}{k_2k_3}\left[\frac{1}{k_1}(\cot h\theta_1-\cot\theta_1)+\frac{1}{k_4}(\cot h\theta_4-\cot\theta_4)\right]$$

$$-\frac{1}{k_2}\left(\frac{\mathrm{ch}\,\theta_2\mathrm{sh}\,\theta_3}{\sin\theta_1\sin\theta_4}-\frac{\cos\theta_2\sin\theta_3}{\mathrm{sh}\,\theta_1\,\mathrm{sh}\,\theta_4}\right)-\frac{1}{k_3}\left(\frac{\mathrm{sh}\,\theta_2\mathrm{ch}\,\theta_3}{\sin\theta_1\sin\theta_4}-\frac{\sin\theta_2\cos\theta_3}{\mathrm{sh}\,\theta_1\mathrm{sh}\,\theta_4}\right) \quad (11)$$

The l. h. s. of this inequality is always positive and the r. h. s. is always negative and therefore the inequality can never be satisfied.

As it is not possible to achieve simultaneously real images in both planes, the quadruplet with symmetric focusing is not appropriate for being used as a reproduction optics; this applies to the most general quadruplet, made of arbitrary quadrupoles, separated by arbitrary drift spaces.

IV. Quadruplet with Antisymmetric Focusing

We call the focusing of the quadruplet antisymmetric if the quadrupoles are laid out following the sequence cdcd in one plane and dcdc in the other.

We limit ourselves here to the consideration of two cases corresponding respectively to an antisymmetric and to a symmetric geometry.

i. Quadruplet with Antisymmetric Focusing and Antisymmetric Geometry

In this case we have

$$k_1 = k_3, \qquad k_2 = k_4$$

$$\theta_1 = \theta_3, \qquad \theta_2 = \theta_4 \quad (12)$$

From Eq. (4) we find

$$B_{cdcd} = L_{23} + \frac{2}{k_2} \cot h\theta_2 - \frac{2}{k_1} \cot \theta_1 \tag{13}$$

$$B_{dcdc} = L_{23} - \frac{2}{k_2} \cot \theta_2 + \frac{2}{k_1} \cot h\theta_1 \tag{14}$$

From the conditions $B_{cdcd} < 0$, $B_{dcdc} < 0$ we obtain

$$L_{23} < \frac{2}{k_1} \cot \theta_1 - \frac{2}{k_2} \cot h\theta_2 \tag{15}$$

$$L_{23} < \frac{2}{k_2} \cot \theta_2 - \frac{2}{k_1} \cot h\theta_1 \tag{16}$$

and adding these inequalities we find

$$L_{23} < -\frac{D_1}{k_1} - \frac{D_2}{k_2} \tag{17}$$

where we have put

$$D(\theta) = \cot h\theta - \cot \theta \tag{18}$$

Obviously it is not possible to satisfy the inequality (17) and therefore the quadruplet with antisymmetric focusing and antisymmetric geometry cannot be used to achieve a transfer matrix +1 with real images.

<u>ii. Quadruplet with Antisymmetric Focusing and Symmetric Geometry</u>

In this case we put

$$\begin{array}{ll} k_1 = k_4 = k_o & k_2 = k_3 = k_i \\ \theta_1 = \theta_4 = \theta_o & \theta_2 = \theta_3 = \theta_i \\ L_{12} = L_{34} = L_o & L_{23} = L_i \end{array} \tag{19}$$

Again from Eq. (4) we calculate

$$B_{cdcd} = B_{dcdc} = \left(\frac{k_i}{k_o}\right)^2 \frac{\mathrm{sh}\ \theta_i \sin\theta_i}{\mathrm{sh}\ \theta_o \sin\theta_o}\left(L_i + \frac{D_i}{k_i}\right) + \frac{D_o}{k_o} \tag{20}$$

so that stigmatism is automatically insured; however, in neither plane is the image real and therefore this structure is equally unable to satisfy the required conditions of reproduction and real images.

V. Conclusions

We have shown that neither the quadruplet with symmetric focusing and arbitrary geometry nor the quadruplet with antisymmetric focusing and symmetric or antisymmetric geometry are appropriate devices for achieving reproduction matrices. It seems unlikely that the quadruplet with antisymmetric focusing and arbitrary geometry could provide a useful solution.

VI. References

1. E. Regenstreif, CERN 77-22
2. E. Regenstreif, CERN 67-6