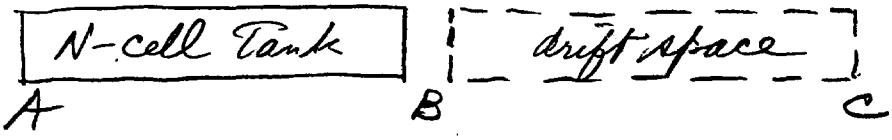
RCT 4/5/89 ①

ΔT Method of Measuring $\Delta\phi$ and $\frac{\Delta v}{v}$ $\left(\frac{\Delta W}{W} = \gamma(\gamma+1)\frac{\Delta v}{v}\right)$

(A) Basic Equations



To measure $\Delta\phi_A$ and $\left(\frac{\Delta v}{v}\right)_A$ from δt_{AB} and δt_{BC}

(Δ = deviation from synchronous value
δ = difference with tank AB off and on)

① For AB

$$\delta t_{AB} = \frac{L_{AB}}{v_{A0} + \Delta v_A} - \frac{2\pi N + \Delta\phi_B - \Delta\phi_A}{\omega_0}$$

$$= \underbrace{\left(\frac{L_{AB}}{v_{A0}} - \frac{2\pi N}{\omega_0}\right)}_{\equiv\, \delta t_{AB0}} - \underbrace{\frac{L_{AB}}{v_{A0}}}_{\equiv\, \tau_A}\left(\frac{\Delta v}{v}\right)_A + \frac{1}{\omega_0}\Delta\phi_A - \frac{1}{\omega_0}\Delta\phi_B$$

Then

$$\delta t_B \equiv \delta t_{AB} - \delta t_{AB0} = -\tau_A\left(\frac{\Delta v}{v}\right)_A + \frac{1}{\omega_0}\Delta\phi_A - \frac{1}{\omega_0}\Delta\phi_B \quad (1)$$

② For BC – To get $\left(\frac{\Delta v}{v}\right)_B$ we need to drift beam after B.

$$\delta t_{BC} = \frac{L_{BC}}{v_{A0} + \Delta v_A} - \frac{L_{BC}}{v_{B0} + \Delta v_B}$$

$$= \underbrace{L_{BC}\left(\frac{1}{v_{A0}} - \frac{1}{v_{B0}}\right)}_{\equiv\ \delta t_{BC0}} - \frac{L_{BC}}{v_{A0}}\left(\frac{\Delta v}{v}\right)_A + \underbrace{\frac{L_{BC}}{v_{B0}}}_{\equiv\ \tau_B}\left(\frac{\Delta v}{v}\right)_B$$

Then

$$\delta t_C \equiv \delta t_{BC} - \delta t_{BC0} = -\frac{L_{BC}}{v_{A0}}\left(\frac{\Delta v}{v}\right)_A + \tau_B\left(\frac{\Delta v}{v}\right)_B \qquad (2)$$

The basic equations (1) and (2) are

$$\boxed{\begin{cases} \delta t_B = -\tau_A\left(\frac{\Delta v}{v}\right)_A + \frac{1}{\omega_0}\Delta\phi_A - \frac{1}{\omega_0}\Delta\phi_B \\ \delta t_C = -\frac{L_{BC}}{v_{A0}}\left(\frac{\Delta v}{v}\right)_A + \tau_B\left(\frac{\Delta v}{v}\right)_B \end{cases}} \qquad (3)$$

where $\Delta\phi_B$ and $\left(\frac{\Delta v}{v}\right)_B$ are to be related to $\Delta\phi_A$ and $\left(\frac{\Delta v}{v}\right)_A$ by beam dynamics. Eq. (3) then gives $\Delta\phi_A$ and $\left(\frac{\Delta v}{v}\right)_A$ in terms of measured δt_B and δt_C.

(B) Adiabatic Beam Dynamics

$$\begin{cases} \frac{d}{dz}(\Delta\phi) = -\frac{2\pi}{\beta\lambda}\left(\frac{\Delta v}{v}\right) \equiv -f\left(\frac{\Delta v}{v}\right) \\ \frac{d}{dz}\left(\frac{\Delta v}{v}\right) = \frac{eE_0}{mc^2}\frac{1}{\beta^2\gamma^3}\left[\sin(\phi_s+\Delta\phi) - \sin\phi_s\right] \\ \qquad\qquad \equiv g\left[\sin(\phi_s+\Delta\phi) - \sin\phi_s\right] \end{cases} \qquad (4)$$

$$\begin{cases} f = \dfrac{1}{\beta \lambda\!\!\!^{-}} \\ g = \dfrac{eE_0 \beta \lambda\!\!\!^{-}}{mc^2} \dfrac{1}{(\beta\gamma)^3 \lambda\!\!\!^{-}} \end{cases} \qquad \lambda\!\!\!^{-} \equiv \frac{\lambda}{2\pi}$$

$$H = -\frac{f}{2}\left(\frac{\Delta v}{v}\right)^2 + g\left[\cos(\phi_s + \Delta\phi) + \Delta\phi \sin\phi_s\right]$$

Unstable fixed point at $\left(\Delta\phi = \pi - 2\phi_s,\ \frac{\Delta v}{v} = 0\right)$. The separatrix passes through the UFP, hence has H value $H_u = g\left[\cos(\pi - \phi_s) + (\pi - 2\phi_s)\sin\phi_s\right]$ and the equation $H = H_u$ or

$$-\frac{f}{2}\left(\frac{\Delta v}{v}\right)^2 + g\left[\cos(\phi_s + \Delta\phi) + \Delta\phi \sin\phi_s\right] = g\left[\cos(\pi - \phi_s) + (\pi - 2\phi_s)\sin\phi_s\right]$$

or

$$\cos(\phi_s + \Delta\phi) + \Delta\phi \sin\phi_s = \frac{f}{2g}\left(\frac{\Delta v}{v}\right)^2 - \cos\phi_s + (\pi - 2\phi_s)\sin\phi_s \quad (5)$$

For small $\Delta\phi$ if one keeps only linear terms in Eq. (4) one gets

$$\begin{cases} \dfrac{d}{dz}(\Delta\phi) = -f\left(\dfrac{\Delta v}{v}\right) \\ \dfrac{d}{dz}\left(\dfrac{\Delta v}{v}\right) = g\cos\phi_s(\Delta\phi) \end{cases} \quad (6)$$

$$H = -\frac{f}{2}\left(\frac{\Delta v}{v}\right)^2 - \frac{g}{2}\cos\phi_s(\Delta\phi)^2 \qquad \text{(ellipse)}$$

(c) Linear ΔT relations

From Eqs (6) we can write

$$\begin{pmatrix} \Delta\phi \\ \frac{\Delta v}{v} \end{pmatrix}_B = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \begin{pmatrix} \Delta\phi \\ \frac{\Delta v}{v} \end{pmatrix}_A \qquad (7)$$

where

$$a = \cos \Omega L_{AB} \qquad \Omega \equiv \sqrt{f g \cos\phi_s}$$

$$b = -\frac{f}{\Omega} \sin \Omega L_{AB} \qquad = \frac{1}{\lambda\!\!\!^{-}} \sqrt{\frac{e E_0 \lambda\!\!\!^{-}}{m c^2} \frac{\cos\phi_s}{(\beta\gamma)^3}} \cos\phi_s$$

$$c = \frac{\Omega}{f} \sin \Omega L_{AB} \qquad \left(\lambda\!\!\!^{-} \equiv \frac{\lambda}{2\pi} \right)$$

Eq. (3) then becomes

$$\left\{ \begin{aligned} \delta t_B &= \frac{1-a}{\omega_0} \Delta\phi_A - \left(\tau_A + \frac{b}{\omega_0}\right)\left(\frac{\Delta v}{v}\right)_A \\ &= -\tau_A \left(\frac{\Delta v}{v}\right)_A + \frac{1-a}{\omega_0}\left[\Delta\phi_A - \frac{b}{1-a}\left(\frac{\Delta v}{v}\right)_A\right] \\ \delta t_C &= c\,\tau_B \Delta\phi_A + \left(a\tau_B - \frac{L_{BC}}{v_{A0}}\right)\left(\frac{\Delta v}{v}\right)_A \\ &= -\tau_B\left(1 + \frac{v_{B0}}{v_{A0}}\right)\left(\frac{\Delta v}{v}\right)_A + c\,\tau_B\left[\Delta\phi_A - \frac{b}{1-a}\left(\frac{\Delta v}{v}\right)_A\right] \end{aligned} \right. \qquad (8)$$

This shows that with given $\left(\frac{\Delta v}{v}\right)_A$ all $\Delta\phi_A$-curves with different E_0 will cross at the point

$$\left\{ \begin{aligned} \delta t_B &= -\tau_A \left(\frac{\Delta v}{v}\right)_A \\ \delta t_C &= -\tau_B\left(1 + \frac{v_{B0}}{v_{A0}}\right)\left(\frac{\Delta v}{v}\right)_A \end{aligned} \right. \qquad (9)$$

at

$$\Delta\phi_A = \frac{b}{1-a}\left(\frac{\Delta v}{v}\right)_A \tag{10}$$

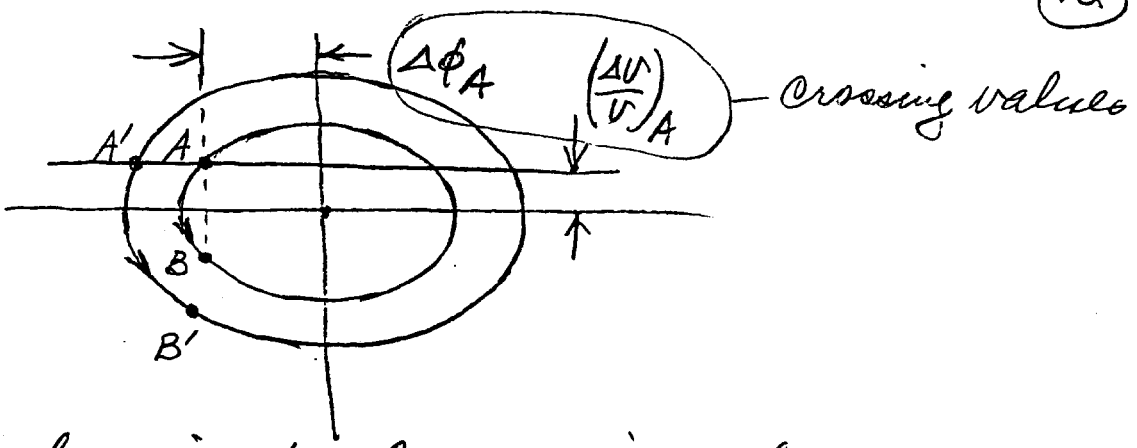
At that point

$$\begin{cases} \Delta\phi_B = \left(\frac{ab}{1-a}+b\right)\left(\frac{\Delta v}{v}\right)_A = \frac{b}{1-a}\left(\frac{\Delta v}{v}\right)_A = \Delta\phi_A \\ \left(\frac{\Delta v}{v}\right)_B = \left(\frac{bc}{1-a}+a\right)\left(\frac{\Delta v}{v}\right)_A = \frac{a-1}{1-a}\left(\frac{\Delta v}{v}\right)_A = -\left(\frac{\Delta v}{v}\right)_A \end{cases} \tag{11}$$

The slopes of the $\Delta\phi_A$-curves (straight lines) are

$$m = \frac{d(\delta t_C)/d(\Delta\phi_A)}{d(\delta t_B)/d(\Delta\phi_A)} = \omega_0\tau_B\frac{c}{1-a} = \frac{\omega_0 L_{BC}}{v_{B0}}\frac{\Omega}{f}\frac{\sin\Omega L_{AB}}{1-\cos\Omega L_{AB}}$$

$$= \Omega L_{BC}\cot\frac{\Omega L_{AB}}{2} = m(E_0) \tag{12}$$



E_0 is also given by the crossing values

$$\left[\frac{\Delta\phi_A}{\left(\frac{\Delta v}{v}\right)_A}\right]_{\text{crossing value}} = \frac{b}{1-a} = -\frac{f}{\Omega}\frac{\sin\Omega L_{AB}}{1-\cos\Omega L_{AB}}$$

$$= -\frac{1}{\Omega\lambda}\cot\frac{\Omega L_{AB}}{2} \tag{13}$$

The straight-line joining all the crossing points passes thru origin at $\left(\frac{\Delta U}{U}\right)_A$ and has the slope

$$\frac{d(\delta t_C)}{d(\delta t_B)} = \frac{\tau_B}{\tau_A}\left(1+\frac{U_{B0}}{U_{A0}}\right) = \frac{\tau_B}{\tau_A} + \frac{L_{BC}}{L_{AB}} \qquad (14)$$

(D) Non-linear ΔT relations

We can not write the solutions of Eqs. (4) explicitly as Eq. (7) but many of the features are still valid.

(1) With given $\left(\frac{\Delta U}{U}\right)_A$ all $\Delta\phi_A$-curves with different E_0 values cross at the point when

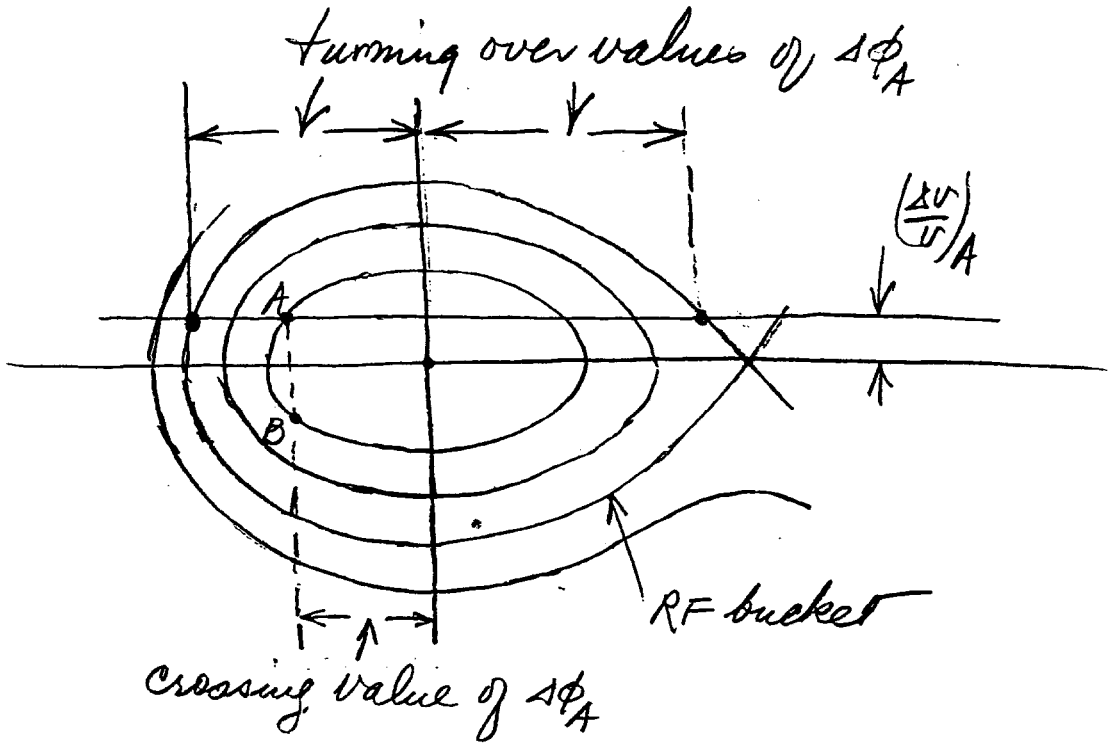
$$\Delta\phi_B = \Delta\phi_A \quad \text{and} \quad \left(\frac{\Delta U}{U}\right)_B = -\left(\frac{\Delta U}{U}\right)_A$$

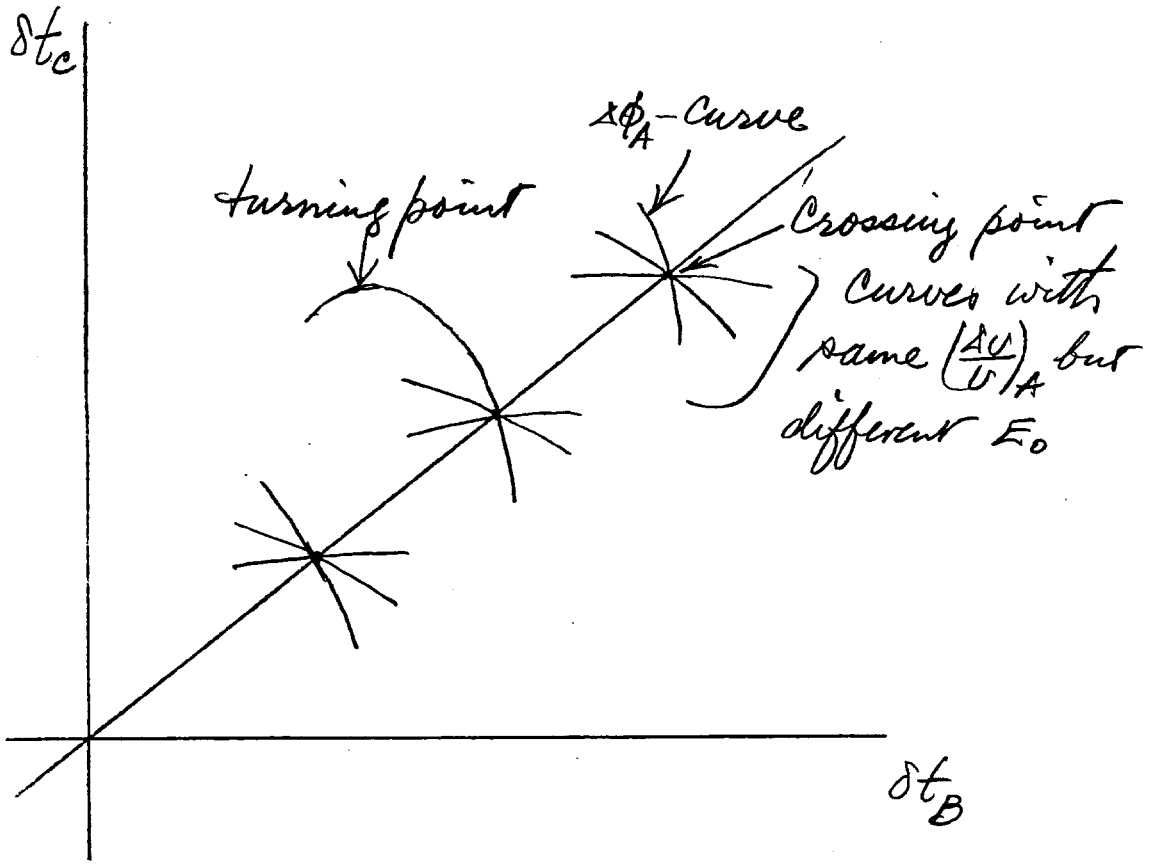
Eq. (3) then gives, at the crossing point

$$\begin{cases} \delta t_B = -\tau_A\left(\frac{\Delta U}{U}\right)_A \\ \delta t_C = -\tau_B\left(1+\frac{U_{B0}}{U_{A0}}\right)\left(\frac{\Delta U}{U}\right)_A \end{cases}$$

same as Eq. (9). The $\Delta\phi_A$-curves are now curved lines and they turn over when $\Delta\phi_A$ goes out of the stable RF bucket (given by Eq. (5)).

The line joining all the crossing points is still a straight line passing through the origin & having the slope given by Eq. (14).


turning over values of $s\phi_A$
$\left(\frac{\Delta U}{U}\right)_A$
A
B
RF bucket
crossing value of $s\phi_A$


δt_C
$s\phi_A$-curve
turning point
Crossing point
Curves with same $\left(\frac{\Delta U}{U}\right)_A$ but different E_0
δt_B