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ACCELERATOR PHYSICS DEPARTMENT

TITLE: Comment on the Exact Evaluation of Sympletic Maps

AUTHOR: Leo Michelotti
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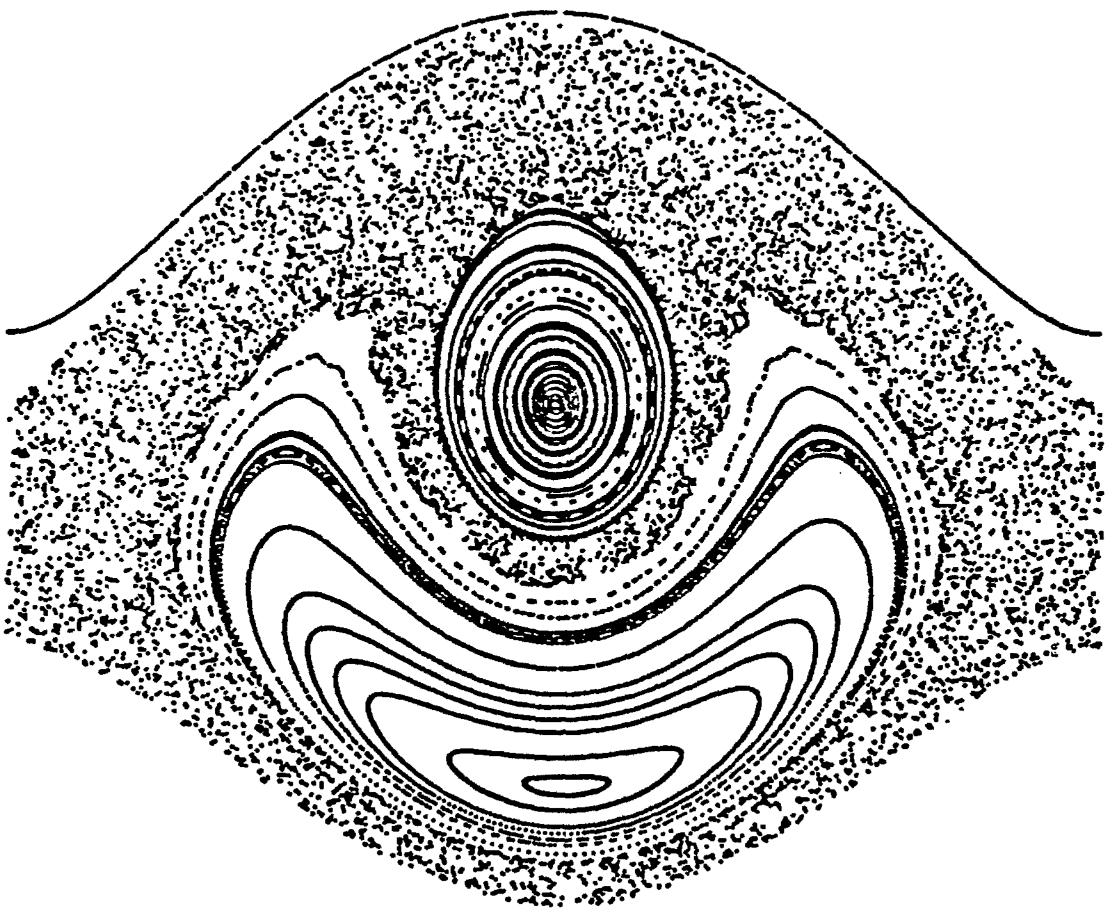

Comment on the exact evaluation of symplectic maps

Leo Michelotti

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Gjaja [1] has recently reported closed form expressions for $\exp[a : x^n p^m :]x$ and $\exp[a : x^n p^m :]p$. We present here a simpler derivation of these results using the fact that the vector field associated with a Hamiltonian H is $X_H = - : H :$. (See, for example, [2].) This means that $\exp[t : x^n p^m :]$ is the time-evolution operator associated with the Hamiltonian, $H = -x^n p^m$, and the desired expressions can be found by solving Hamilton's equations of motion.

Starting from the Hamiltonian, $H = -x^n p^m$, the vector field components for x and p are written,

$$\begin{aligned} \dot{x} &= -m x^n p^{m-1} = mH/p \qquad (1) \\ \dot{p} &= n x^{n-1} p^m = -nH/x \ . \qquad (2) \end{aligned}$$

From these, and the fact that H is autonomous, we obtain immediately that xp is a linear function of time.

$$\begin{aligned} \frac{d}{dt}(xp) &= \dot{x}p + x\dot{p} = (m-n)H \\ xp &= x_0 p_0 + (m-n)Ht \end{aligned}$$

We use this in conjunction with Eq.(1) to obtain the solution for $x(t)$.

$$\begin{aligned} \dot{x} = mH/p &= mHx/xp \\ &= mHx/(x_0 p_0 + (m-n)Ht) \\ \frac{dx}{x} &= \frac{mH\,dt}{x_0 p_0 + (m-n)Ht} \end{aligned}$$

The case $m = n$ is simplest:

$$m = n \ \Rightarrow \ x = x_0 \exp[mHt/x_0 p_0] = x_0 \exp[-m x_0^{m-1} p_0^{m-1} t] \ .$$

If $m \neq n$, then the integral is only slightly more complicated.

$$x = C\,(x_0 p_0 + (m-n)Ht)^{\frac{m}{m-n}}$$

Here, C is a constant, which will very quickly be replaced by a function of x_0 and p_0. We substitute $H = -x_0^n p_0^m$ and perform a little factorization to get the final result.

$$\begin{aligned} x &= C\,(x_0 p_0 + (n-m) x_0^n p_0^m t)^{\frac{m}{m-n}} \\ &= C\,(x_0 p_0)^{\frac{m}{m-n}}\,(1 + (n-m) x_0^{n-1} p_0^{m-1} t)^{\frac{m}{m-n}} \\ &= x_0\,(1 + (n-m) x_0^{n-1} p_0^{m-1} t)^{\frac{m}{m-n}} \end{aligned} \tag{3}$$

A completely analagous calculation, starting from Eq.(2), provides the corresponding result for p.

$$p = \begin{cases} p_0 \exp[m x_0^{m-1} p_0^{m-1} t]\ , & n = m \\ p_0\,(1 + (n-m) x_0^{n-1} p_0^{m-1} t)^{\frac{n}{n-m}}\ , & n \neq m \end{cases}$$

These results correspond to Equations 10 through 14 of Gjaja [1].

An even simpler development gives the same result when $m \neq n$, for we then can use the invariance of H itself to find the solutions.

$$\begin{aligned} H \;=\; -x^n p^m &= -x^{n-m}(xp)^m \\ &= -x^{n-m}(x_0 p_0 + (n-m) x_0^n p_0^m t)^m \\ x^{n-m} &= x_0^n p_0^m\,(x_0 p_0 + (n-m) x_0^n p_0^m t)^{-m} \\ &= x_0^{n-m}\,(1 + (n-m) x_0^{n-1} p_0^{m-1} t)^{-m} \end{aligned}$$

Taking the $(n-m)^{\text{th}}$ root reproduces Eq.(3).

We leave as a little exercise for the reader to determine what happens in the vicinity of $t \approx ((m-n) x_0^{n-1} p_0^{m-1})^{-1}$, and correspondingly to find the error in the solution as presented.

References

[1] I. Gjaja. Exact evaluation of arbitrary symplectic maps. Preprint from University of Maryland. College Park, MD., January 1992.

[2] Leo Michelotti. From vector fields to normal forms. Presented at the Fifth Advanced ICFA Beam Dynamics Workshop. Corpus Christi, Texas. October 3-8, 1991.