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Primordial Gravity Waves and Weak Lensing

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Inflation produces a primordial spectrum of gravity waves in addition to the density perturbations which seed structure formation. We compute the signature of these gravity waves in the large scale shear field. In particular, the shear can be divided into a gradient mode (G or E) and a curl mode (C or B). The former is produced by both density perturbations and gravity waves, while the latter is produced only by gravity waves, so the observations of a non-zero curl mode could be seen as evidence for inflation. We find that the expected signal from inflation is small, peaking on the largest scales at $l(l+1)C_l/2\pi < 10^{-11}$ at $l = 2$ and falling rapidly there after. Even for an all-sky deep survey, this signal would be below noise at all multipoles. Part of the reason for the smallness of the signal is a cancellation on large scales of the standard line-of-sight effect and the effect of "metric shear."

The theory of inflation was proposed over twenty years ago [1]. For the first years after its discovery/invention, cosmologists worked out some of its cosmological implications and particle physics implementations. While this work is still going on, the most exciting development in the last several years has been the confirmation of some of inflation's basic predictions. It now appears that the universe is flat, the most robust prediction of inflation [2]. Further, observations of large scale structure and anisotropies in the cosmic microwave background (CMB) strongly support the idea that small, adiabatic, nearly scale-invariant fluctuations were present in the early universe. Inflation predicts the existence of precisely this class of perturbations [3].

The case for inflation is not airtight though. Inflation explains why the universe appears flat today even if the curvature is not exactly equal to zero. But it is possible that the curvature really is exactly zero, so that inflation is not needed to account for the apparent flatness. Similarly, the adiabatic, scale-free perturbation spectrum predicted by inflation might have been laid down by some other mechanism. Indeed, it is somewhat of an embarrassment for proponents of inflation that this spectrum is called the "Harrison-Zel'dovich-Peebles" spectrum, named after the three eminent cosmologists who first proposed it, long before inflation had been suggested.

Are there any signatures unique to inflation? Just as the primordial density perturbations were produced by quantum fluctuations during inflation, primordial gravity waves were produced by quantum fluctuations of the metric during the inflationary epoch [4]. If we were to detect these gravity waves, the case for inflation would be strengthened considerably. After the detection of anisotropies in the CMB by the COBE satellite [5], there was hope that the gravity waves would eventually be extracted from an accurate measurement of the full anisotropy spectrum [6]. Recent work has shown though that other cosmological parameters, especially late reionization, can mimic the effects of gravity waves in the anisotropy spectrum [7], making detection more difficult.

In 1996, two groups [8, 9] showed that gravity waves produce a polarization pattern in the CMB that cannot be caused by scalar (density) perturbations to first order in the amplitude. The polarization field can be decomposed into two modes. One is technically a *scalar* pattern on the sky, just as a gradient is, and the names given to such modes have variously been "scalar", "G" (for gradient) , or "E" (for electric since an electric field is a gradient of a scalar potential). Another orthogonal set of modes are *pseudoscalar* patterns on the sky, and can be obtained by rotating the polarization at each direction in a *scalar* pattern by 45°. Such modes change sign under parity like a curl does, and names given to them have variously been "pseudoscalar", "C" (for curl), and "B" (for magnetic since a magnetic field is also odd under parity). The polarization pattern produced by a single sinusoidal[1] density perturbation must be rotationally symmetric about the wavenumber $\mathbf{k}$ and also symmetric under reflections in the plane perpendicular to $\mathbf{k}$. Such a pattern cannot contain any C (≡B) modes. In linear theory the polarization pattern produced by density perturbations is just the sum of the pattern from individual sine waves and hence we derive the general rule that scalar perturbations produce no C modes[2]. This symmetry argument does not apply to gravity waves which do not have the same reflection symmetry because each gravity wave is polarized in a particular direction. Gravity waves do produce C mode polarization patterns in the CMB. Thus detection of C modes can provide fairly unambiguous evidence for gravity waves, thereby verifying a unique prediction of inflation. It will take quite a while to reach the sensitivity needed to make this test a

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[1] or pseudo-sinusoidal in the case of an FRW cosmology with non-zero curvature
[2] Nonlinearities can lead to C modes even from scalar inhomogeneities but will be most important on small angular scales. The gravity wave signal discussed here is on large angular scales.

powerful probe of inflation [10]. Along the way, there are many systematic effects which might prove to be spoilers, prominent among them the possibility that foregrounds are polarized.

The pattern of shearing of images due to gravitational, or any other kind of, lensing is described by a symmetric traceless tensor field on the sky, just as is the polarization pattern, and like polarization can be decomposed into G and C modes. The gravitational field causing the lensing can be scalar (caused by density inhomogeneities), vector (caused by vorticity), or tensor (gravity waves). The symmetry argument stated above tells us that C mode shear pattern can be produced by gravity waves and not by density inhomogeneities [11]. As with CMB polarization, we expect the dominant shear to be G modes produced by density inhomogeneities, but we can hope to detect gravity waves through C modes in the shear pattern. This raises the question of whether the primordial gravity waves produced by inflation might be detectable by measuring the C mode of shear. Here, we explore this question.

One measures the shear field by correlating the shapes of distant galaxies, in particular the ellipticity of galactic light distribution on the sky [12], and as with polarization we can expect various other effects to mimic C mode shear from gravity waves in such a measurement. We will not address the issue of "foregrounds" in this paper, but as we will show (see also Ref. [13]) the amplitude of shear produced by gravity waves is so small that the expected signal is below even the statistical noise expected in the largest experiment imaginable.

Gravitational lensing is caused by the deflection of light trajectories due to metric perturbations. The metric of a flat Friedman-Robertson-Walker cosmology in synchronous coordinates[3] is given by

$$g_{\mu\nu} = a^2 \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{I}+\mathbf{H} \end{pmatrix} \tag{1}$$

where $a(\eta)$ is the scale factor of the universe; η is the conformal time variable; and the 3×3 matrices $\mathbf{I}$ and $\mathbf{H}$ are respectively the identity matrix and the metric perturbation which is symmetric $\mathbf{H} = \mathbf{H}^{\mathrm{T}}$. The deflection of light is described by the geodesic equation,

$$\ddot{\mathbf{r}} = \frac{1}{2}\left(\dot{\mathbf{r}}\cdot\dot{\mathbf{H}}\cdot\dot{\mathbf{r}}\right)\dot{\mathbf{r}} - (\mathbf{I}+\mathbf{H})^{-1}\cdot\left(\dot{\mathbf{r}}\cdot\frac{d}{d\eta}\mathbf{H} - \frac{1}{2}\nabla(\dot{\mathbf{r}}\cdot\mathbf{H}\cdot\dot{\mathbf{r}})\right) \tag{2}$$

where $\dot{} = \frac{\partial}{\partial\eta}$ and $\frac{d}{d\eta} = \frac{\partial}{\partial\eta} + \dot{\mathbf{r}}\cdot\nabla$. A full solution of this equation (ray-tracing) is unnecessary since we may not only linearize the equation in $\mathbf{H}$, but also linearize the solution in $\mathbf{H}$ by evaluating the gravitational accelerations on an unperturbed trajectory, *i.e.* the Born approximation. A "Born trajectory" arriving at the origin at $\eta = \eta_0$ coming from direction $\hat{\mathbf{n}}$ is

$$\mathbf{r}(\eta) = \left(\hat{\mathbf{n}} - \frac{1}{2}\mathbf{H}_0\cdot\hat{\mathbf{n}}\right)(\eta_0-\eta) - \int_\eta^{\eta_0} d\eta' \times \left[\mathbf{H}\cdot\hat{\mathbf{n}} + \frac{1}{2}(\eta'-\eta)\left((\hat{\mathbf{n}}\cdot\dot{\mathbf{H}}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}} - \nabla(\hat{\mathbf{n}}\cdot\mathbf{H}\cdot\hat{\mathbf{n}})\right)\right]_{(\eta',\mathbf{x}')} \tag{3}$$

where $\mathbf{x}' = (\eta_0-\eta')\hat{\mathbf{n}}$. The initial conditions are chosen such that the trajectory is light-like to the angle between two trajectories $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ intersecting at the origin is $\cos^{-1}\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}'$ (both to 1st order in $\mathbf{H}$).

One may decompose the perturbation of the trajectory into a displacement along the line-of-sight (LOS), *i.e.* a time delay; and a displacement perpendicular to the LOS, $\delta\mathbf{r}_\perp = \mathbf{r} - (\hat{\mathbf{n}}\cdot\mathbf{r})\,\hat{\mathbf{n}}$, which describes all lensing effects. It is convenient to convert from displacement in coordinate distance to displacement in angle on the sky by defining $\vec{\Delta} = \delta_\perp\mathbf{r}/(\eta_0-\eta)$, which can be thought of as a 2-d vector in the tangent space of the direction sphere parameterized by $\hat{\mathbf{n}}$.

Following [11] the displacement pattern on the direction sphere described by $\vec{\Delta}$ can alternately be described in terms of the convergence, $\kappa = -\frac{1}{2}\Delta^a{}_{:a}$, which is the covariant divergence of $\vec{\Delta}$; and the rotation $\omega = \frac{1}{2}(\Delta_a\epsilon^{ab})_{:b}$, which is the is the covariant divergence of $\vec{\Delta}$ rotated by $90°$. The G and C modes are given by κ and ω, respectively. In constrast to the G modes, where the gravity wave contribution will be dwarfed by the density perturbation, the C modes will have no contribution from density perturbations on large scales, so we are interested in ω and not κ. Rexpressing the 2-d relation $\omega = \frac{1}{2}(\Delta_a\epsilon^{ab})_{:b}$ in terms of the 3-d trajectory we find[4]

$$\begin{aligned}
\omega(\hat{\mathbf{n}},\eta) &= -\frac{1}{2}\frac{1}{\eta_0-\eta}\hat{\mathbf{n}}\cdot(\nabla_{\hat{\mathbf{n}}}\times\mathbf{r}(\hat{\mathbf{n}},\eta)) \\
&= \frac{1}{2}\int_\eta^{\eta_0} d\eta'\,[\hat{\mathbf{n}}\cdot(\nabla\times\mathbf{H})\cdot\hat{\mathbf{n}}]_{(\eta',\hat{\mathbf{n}}(\eta_0-\eta'))}
\end{aligned} \tag{4}$$

using the Born solution. Henceforth one can consider η as being a measure of the distance to the background galaxies whose shear we measure.

The contribution of the gravity waves (tensor modes) to $\mathbf{H}$ is transverse and traceless, *i.e.* $\mathrm{Tr}\,\mathbf{H} = 0$, $\nabla\cdot\mathbf{H} = 0$, and one can Fourier decompose the tensor constribution as

$$\mathbf{H}^{(\mathrm{T})}(\mathbf{x},\eta) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{k}\,e^{i\mathbf{k}\cdot\mathbf{x}}\sum_{i=1}^{2}\tilde{\mathbf{H}}^{(\mathrm{T})}_{(i)}(\mathbf{k},\eta) \tag{5}$$

where

$$\tilde{\mathbf{H}}^{(\mathrm{T})}_{(i)}(\mathbf{k},\eta) = T_{(\mathrm{T})}(k,\eta)\,\tilde{H}_{(i)}(\mathbf{k})\mathbf{e}_{(i)}(\mathbf{k})\ , \tag{6}$$

[3] Linearized tensor (gravity wave) perturbations always lead to synchronous coordinates, but other coordinate choices (gauges) are available for vector (vorticity) and scalar (density) perturbations.

[4] One can see from this equation that a scalar (density) perturbation does not contribute to ω and hence to C modes.

$k = |\mathbf{k}|$, we define the transfer function such that $T_{(\mathrm{T})}(k,0)=1$, and i sums over the two (linear) polarization states defined by the polarization tensors, $\mathbf{e}_{(i)}$, which obey $\mathrm{Tr}\,\mathbf{e}_{(i)} = \mathrm{Im}\,\mathbf{e}_{(i)} = 0$, $\mathbf{e}_{(i)} = \mathbf{e}^{\mathrm{T}}_{(i)}$, $\mathbf{k}\cdot\mathbf{e}_{(i)}(\mathbf{k}) = \mathbf{0}$, $\mathrm{Tr}\left(\mathbf{e}_{(i)}(\mathbf{k})\cdot\mathbf{e}_{(j)}(\mathbf{k})\right) = 2\delta_{ij}$. Assuming isotropy and no preferred handedness we may define a power spectrum:

$$\left\langle \tilde{H}_{(i)}(\mathbf{k})\tilde{H}^{*}_{(j)}(\mathbf{k}')\right\rangle = (2\pi)^3 P_{(\mathrm{T})}(k)\,\delta_{ij}\,\delta^{(3)}(\mathbf{k}-\mathbf{k}')\ . \quad (7)$$

In inflationary models with Hubble parameter H_{I} during inflation

$$P_{(\mathrm{T})}(k) = \frac{8\pi}{(2\pi)^3}\left(\frac{H_{\mathrm{I}}}{M_{\mathrm{Planck}}}\right)^2 k^{-3}\ . \quad (8)$$

which gives equal metric perturbation on all scales. Presently CMB observations limit [14] $H_{\mathrm{I}} \leq 2\times 10^{14}$GeV.

An integral required to compute ω is

$$\begin{aligned}\Omega_{(i)}(\mathbf{k},\hat{\mathbf{n}},\eta) &= -\frac{i}{2}k\sin^2\theta\,\sin 2\phi \\ &\times\int_{\eta}^{\eta_0} d\eta' e^{ik\,\cos\theta\,(\eta_0-\eta')}T(k,\eta) \quad (9)\end{aligned}$$

where the two angles, θ and ϕ, are defined by $\hat{\mathbf{n}}\cdot\mathbf{k} = k\cos\theta$ and $\hat{\mathbf{n}}\cdot\left(\mathbf{k}\times\mathbf{e}_{(i)}(\mathbf{k})\right)\cdot\hat{\mathbf{n}} = k\sin^2\theta\sin 2\phi$. Eq.s (4,5,6) becomes

$$\omega(\hat{\mathbf{n}},\eta) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{k}\sum_{i=1}^{2}\tilde{H}_{(i)}(\mathbf{k})\,\Omega_{(i)}(\mathbf{k},\hat{\mathbf{n}},\eta) \quad (10)$$

If we define

$$\tilde{\omega}_{(l,m)}(\eta) = \int d^2\hat{\mathbf{n}}\,Y^{*}_{(l,m)}(\hat{\mathbf{n}})\omega(\hat{\mathbf{n}},\eta) \quad (11)$$

then

$$\begin{aligned}C_l^{\otimes}(\eta) &= \frac{1}{2l+1}\sum_{m=-l}^{l}\left\langle|\tilde{\omega}_{(l,m)}(\eta)|^2\right\rangle \\ &= 2\int d^3\mathbf{k}\,P_{(\mathrm{T})}(k)\,\left|T_l^{\otimes}(k,\eta)\right|^2 \quad (12)\end{aligned}$$

where, $w = k(\eta_0-\eta)$ and $w_0 = k\eta_0$, then

$$\begin{aligned}T_l^{\otimes}(k,\eta) &= \sqrt{\frac{1}{2l+1}\sum_{m=-l}^{l}\left|\int d^2\hat{\mathbf{n}}Y^{*}_{(l,m)}(\hat{\mathbf{n}})\Omega_{(i)}(\mathbf{k},\hat{\mathbf{n}},\eta)\right|^2} \\ &= \sqrt{\frac{\pi}{2}\frac{(l+2)!}{(l-2)!}}\int_0^w dw'\,T(k,\frac{w_0-w'}{k})\,\frac{j_l(w')}{w'^2}\ . \quad (13)\end{aligned}$$

Like $C_l^{\otimes}$, but unlike $\tilde{\omega}_{(l,m)}$, $T_l^{\otimes}$ is invariant to a rotation of coordinates, and is computed most easily when $\mathbf{k}$ is in the direction of the coordinate "North Pole".

The quantity ω gives the rotation of the apparent position angle (PA) wrt the coordinate grid due to the bending of light along the LOS. This is related to the apparent shear of the coordinate grid, γ_{ab} by the relation on the sphere $(\nabla^2+2)\,\omega = -(\gamma^{ab}\epsilon_b{}^c)_{:ac}$. For this γ_{ab} to be indicative of the shear measured by looking at galaxy PA's one requires these PA's not to be preferentially aligned wrt to the coordinate grid. However since $\mathbf{H}$ at the source (the galaxy) is non-zero and non-isotropic, this would not be true if galaxy orientation were isotropically distributed in physical space.

Assuming physical isotropy, we must add a "metric shear" caused by the shearing of the coordinates wrt physical space, *i.e.* $\Delta\gamma_{ab}$, which is just the traceless tranverse projection of $-\frac{1}{2}\mathbf{H}$. Metric shear does not cause rotation of the images, but to compare to the above we compute $\Delta\omega$, which is the rotation which would be caused by the bending of light required to produce $\Delta\gamma_{ab}$:

$$\begin{aligned}(\nabla^2+2)\,\Delta\omega &= -(\Delta\gamma^{ab}\epsilon_b{}^c)_{:ac} \\ &= \frac{(\eta_0-\eta)^2}{2}\,\hat{\mathbf{n}}\cdot\left(\left(\nabla_{\perp}-\frac{4\,\hat{\mathbf{n}}}{\eta_0-\eta}\right)\cdot(\nabla\times\mathbf{H})^{\mathrm{T}}\right)\Bigg|_{(\eta,(\eta_0-\eta)\hat{\mathbf{n}})} \quad (14)\end{aligned}$$

where $\nabla_{\perp} \equiv \nabla - \hat{\mathbf{n}}\,\hat{\mathbf{n}}\cdot\nabla$. Correcting for the metric shear leads us to correct the transfer function by adding

$$\begin{aligned}\Delta T_l^{\otimes}(k,\eta) &= \frac{1}{(l+2)(l-1)}\sqrt{\frac{\pi}{2}\frac{(l+2)!}{(l-2)!}} \\ &\times\left(\frac{l-1}{w}\,j_l(w) - j_{l-1}(w)\right)\,T(k,\frac{w_0-w}{k}) \quad (15)\end{aligned}$$

to eq (13). Note that none of these metric corrections are small especially at large redshift and for the longest wavelengths the two terms nearly cancel.

Physical isotropy need not be the correct approximation as galaxies will oscillate in phase with the gravity waves just as a Weber bar does, and also because the initial galaxy shapes may have residual correlation with the gravity wave due to small shear they exert on the galaxy when it is formed. Just how the galaxy shapes react to these forces depends on the details of galaxy dynamics and formation. Here we simply ignore such effects. Both the metric and induced shear take us away from the "bending of light" in flat space picture for the origin of shear, and while they are negligible in the case of scalar perturbations, they are not for the very small amplitude of tensor shear.

In the observationally indicated flat FRW cosmology the tensor metric, $\mathbf{H}$, obeys the wave equation $\ddot{\mathbf{H}}-\nabla^2\mathbf{H}+2\frac{\dot{a}}{a}\mathbf{H}=0$. With zero cosmological constant $T_{(\mathrm{T})}(k,\eta) = 3j_1(k\eta)/(k\eta)$, and this is also very for accurate for the observationally indicated Λ, given by $\Omega_\Lambda \approx 0.7$.

Figure 1 shows the spectrum of the shear produced by inflation. This depends on the redshift of the background galaxies; here we have chosen a very optimistic source redshift of $z=3$ for all background galaxies. The gravity wave shear $l(l+1)C_l^{\otimes}$ decreases with increasing l, in contrast to density perturbations which increases with l. The dashed curve in Figure 1 shows the results using only Eq. (13), i.e. ignoring the metric shear. We see that the metric shear correction partially cancels the signal on

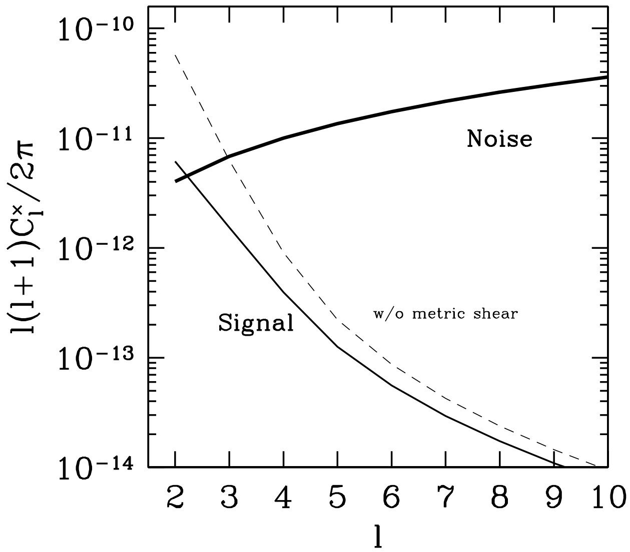

FIG. 1: Expected signal in an all-sky survey in the curl mode in a model with $\Omega_m = 0.3, \Omega_\Lambda = 0.7$ with (solid) and without (dashed) the metric shear term of Eq. (15). The Hubble rate during inflation which determines the amplitude of the gravity wave power spectrum has been set to 2×10^{14} GeV. The noise estimate here assumes an all-sky survey with 1.5×10^{10} galaxies and the variance of the intrinsic shear equal to 0.1. Background galaxies are all assumed to be at fixed redshift 3.

large scales. So a very small, virtually undetectable signal becomes completely undetectable due to the metric shear. Also shown in Figure 1 is the shape noise:

$$\Delta C_l^{\otimes}(\eta) = \sqrt{\frac{2}{(2l+1)f_{\rm sky}}}\frac{\langle\gamma^2\rangle}{N_{\rm gal}}, \qquad (16)$$

where $\langle\gamma^2\rangle$ is the intrinsic rms ellipticity of the galaxies ($N_{\rm gal}$ in all) and $f_{\rm sky}$ is the fraction of sky covered by the survey. Figure 1 shows the noise for an all-sky survey with 100 galaxies per square arcminute, or 1.5×10^{10} galaxies in total, roughly the density anticipated by the proposed SNAP and LSST missions, though they are not all-sky.

It would be wonderful if inflation-produced gravity waves produced a C mode of cosmic shear that could be detected by observing ellipticities of distant galaxies. Alas, the signal is too small to be detected, and the best hope of observing C modes remains in the polarization of the cosmic microwave background.

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