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# Betatron Motion with Coupling of Horizontal and Vertical Degrees of Freedom*

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The Courant-Snyder parametrization of one-dimensional linear betatron motion is generalized to two-dimensional coupled linear motion. To represent the 4×4 symplectic transfer matrix the following ten parameters were chosen: four beta-functions, four alpha-functions and two betatron phase advances which have a meaning similar to the Courant-Snyder parametrization. Such a parametrization works equally well for weak and strong coupling and can be useful for analysis of coupled betatron motion in circular accelerators as well as in transfer lines. Similarly, the transfer matrix, the bilinear form describing the phase space ellipsoid and the second order moments are related to the eigen-vectors. Corresponding equations can be useful in interpreting tracking results and experimental data.

## 1. Introduction

In many applications analysis of coupled betatron motion [1] [2] is an important part of the machine design. The development of accelerator technology has stimulated additional interest in the subject in recent years. Initially betatron coupling was an undesired effect and efforts were made to suppress it. However, over recent years betatron coupling has become an intrinsic part of many accelerator proposals. This paper introduces a new parametrizarion of coupled betatron motion, where in addition to four beta-functions and two betatron phases we introduce four alpha-functions. That yields a complete set of ten independent functions to parametrize a 4×4 symplectic transfer matrix. The beta-functions have similar meaning to the Courant-Snyder parametrization, and the definition of alpha-functions coincides with the standard.

## 2. Equations of Motion and Condition of Symplecticity

The two-dimensional linear motion of a particle in a focusing lattice structure can be described by the following matrix equations:

$$\frac{d\hat{\mathbf{x}}}{ds} = \mathbf{UH}\hat{\mathbf{x}} \quad , \tag{2.1}$$

where the Hamiltonian matrix **H** and the unit symplectic matrix **U** are introduced as follows,

$$\mathbf{H} = \begin{bmatrix} K_x^{\,2} + k + \frac{R^2}{4} & 0 & N & -R/2 \\ 0 & 1 & R/2 & 0 \\ N & R/2 & K_y^{\,2} - k + \frac{R^2}{4} & 0 \\ -R/2 & 0 & 0 & 1 \end{bmatrix} \quad , \qquad \mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} . \tag{2.2}$$

Here $x$ and $y$ are the horizontal and vertical particle displacements from the ideal orbit; the derivatives are calculated along the longitudinal coordinate $s$; $K_{x,y} = eB_{y,x}/Pc$; $k = eG/Pc$; $N = eG_s/Pc$; $R = eB_s/Pc$; $B_x$, $B_y$ and $B_s$ are the corresponding components of the magnetic field; $G$ is the normal component of the magnetic field gradient; and $G_s$ is the skew component of the magnetic field gradient.

Introducing transfer matrix from coordinate 0 to $s$, for the canonical variables, $\hat{\mathbf{x}} = \hat{\mathbf{M}}(0,s)\hat{\mathbf{x}}_0$, one can expresses the symplecticity condition for particle motion in the following form

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$$\hat{\mathbf{M}}(0,s)^T \mathbf{U} \hat{\mathbf{M}}(0,s) = \mathbf{U} \quad , \tag{2.3}$$

Consider a circular accelerator with the total transfer matrix $\hat{\mathbf{M}}$. The transfer matrix has four eigen-values, $\lambda_i$, and four corresponding eigen-vectors, $\hat{\mathbf{v}}_i$ ($i = 1, 2, 3, 4$),

$$\hat{\mathbf{M}}\hat{\mathbf{v}}_i = \lambda_i \hat{\mathbf{v}}_i \quad . \tag{2.4}$$

We will consider the case of a stable betatron motion, meaning all four eigen-values are confined to a unit circle and none of them is equal to ±1. For any two eigen-vectors the symplecticity condition of Eq. (2.3) yields following set of orthogonality conditions:

$$\begin{aligned} &\hat{\mathbf{v}}_1^{+}\mathbf{U}\hat{\mathbf{v}}_1 = -2i \quad , && \hat{\mathbf{v}}_2^{+}\mathbf{U}\hat{\mathbf{v}}_2 = -2i \quad , \\ &\hat{\mathbf{v}}_1^{T}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad , && \hat{\mathbf{v}}_2^{T}\mathbf{U}\hat{\mathbf{v}}_2 = 0 \quad , \\ &\hat{\mathbf{v}}_2^{T}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad , && \hat{\mathbf{v}}_2^{+}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad . \end{aligned} \tag{2.5}$$

## 3. Relation between Eigen-vectors and Emittance Ellipsoid in 4D Phase Space

The turn-by-turn particle positions and angles (at the beginning of the lattice) can be represented as a linear combination of four independent solutions,

$$\begin{aligned} \hat{\mathbf{x}} &= \mathrm{Re}\left(A_1 e^{-i\psi_1}\hat{\mathbf{v}}_1 + A_2 e^{-i\psi_2}\hat{\mathbf{v}}_2\right) \\ &= A_1\left(\hat{\mathbf{v}}_1{}' \cos\psi_1 + \hat{\mathbf{v}}_1{}'' \sin\psi_1\right) + A_2\left(\hat{\mathbf{v}}_2{}' \cos\psi_2 + \hat{\mathbf{v}}_1{}'' \sin\psi_2\right) \quad , \end{aligned} \tag{3.1}$$

where four real parameters, $A_1$, $A_2$, $\psi_1$ and $\psi_2$ , represent the betatron amplitudes and phases. The amplitudes remain constant in the course of betatron motion, while the phases change after each turn.

Introducing the following real symplectic matrix

$$\hat{\mathbf{V}} = \left[\hat{\mathbf{v}}_1{}', -\hat{\mathbf{v}}_1{}'', \hat{\mathbf{v}}_2{}', -\hat{\mathbf{v}}_2{}''\right] \quad \text{, where} \quad \hat{\mathbf{V}}^{-1} = -\mathbf{U}\hat{\mathbf{V}}^T\mathbf{U} \quad , \tag{3.2}$$

one can rewrite Eq. (2.1) in the compact form

$$\hat{\mathbf{x}} = \hat{\mathbf{V}}\mathbf{A}\boldsymbol{\xi}_A \quad , \tag{3.3}$$

where the amplitude matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\xi}_A = \begin{bmatrix} \cos\psi_1 \\ -\sin\psi_1 \\ \cos\psi_2 \\ -\sin\psi_2 \end{bmatrix} \quad . \tag{3.4}$$

Let us consider an ensemble of particles, whose motion (at the beginning of lattice) is contained in a 4D ellipsoid. A 3D surface of this ellipsoid is determined by particles with extreme betatron amplitudes. For any of these particles, Eq. (3.3) describes the 2D-subspace of single-particle motion, which is a subspace of the 3D surface of the ellipsoid, described by the bilinear form

$$\hat{\mathbf{x}}^T\hat{\boldsymbol{\Xi}}\hat{\mathbf{x}} = 1 \quad . \tag{3.5}$$

Applying Eqs. (3.1) – (3.5), one can express the bilinear form, $\hat{\boldsymbol{\Xi}}$, in terms of its diagonal form as follows :

$$\hat{\boldsymbol{\Xi}} = \mathbf{U}\hat{\mathbf{V}}\hat{\boldsymbol{\Xi}}'\hat{\mathbf{V}}^T\mathbf{U}^T \quad \text{, where } \hat{\boldsymbol{\Xi}}' = \mathbf{A}^{-1}\mathbf{A}^{-1} \text{ is a diagonal matrix} \tag{3.6}$$

Therefore, a symplectic transform $\hat{\mathbf{V}}$ reduces matrix $\hat{\boldsymbol{\Xi}}$ to its diagonal form. Then, in the new coordinate frame the 3D ellipsoid enclosing the total 4D phase-space of the beam can be described by the following equation:

$$\hat{\Xi}' = \hat{\mathbf{V}}^T \hat{\Xi} \hat{\mathbf{V}} \quad . \tag{3.7}$$

It is natural to define the beam emittance as a product of the ellipsoid axes:

$$\varepsilon_{4D} = \frac{1}{\sqrt{\det(\hat{\Xi}')}} = \frac{(A_1 A_2)^2}{\left|\det(\hat{\mathbf{V}})\right|} = (A_1 A_2)^2 \quad . \tag{3.8}$$

Thus, the squares of amplitudes $A_1$ and $A_2$ can be considered as 2D emittances $\varepsilon_1$ and $\varepsilon_2$ corresponding to the eigen-vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$. They coincide with the horizontal and vertical emittances of the uncoupled motion so that $\varepsilon_1 \varepsilon_2 = \varepsilon_{4D}$.

Similarly to the one-dimensional case the particle ellipsoid shape, described by matrix $\hat{\Xi}$, determines the beam emittances $\varepsilon_1$ and $\varepsilon_2$, and the eigen-vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$. In this case the beam emittances are reciprocal to the roots of the following characteristic equation,

$$\det\left(\hat{\Xi} - i\lambda \mathbf{U}\right) = 0 \quad . \tag{3.9}$$

Furthermore, knowing the beam emittances and consequently $\hat{\Xi}'$, one can find the eigen-vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ by solving the following equation

$$\left(\hat{\Xi} - \frac{i}{\varepsilon_l}\mathbf{U}\right)\hat{\mathbf{v}}_l = 0 \quad , \qquad \text{where } l = 1, 2 \tag{3.10}$$

One can easily prove [3] that matrix $\hat{\Xi}$ is the inverse of the second moments of the beam distribution matrix .

## 4. Beta-functions for Coupled Motion

Employing previously introduced notation, one can parametrize the eigen-vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ in the following form:

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} \sqrt{\beta_{1x}} \\ -\dfrac{i(1-u)+\alpha_{1x}}{\sqrt{\beta_{1x}}} \\ \sqrt{\beta_{1y}}\, e^{i\nu_1} \\ -\dfrac{iu+\alpha_{1y}}{\sqrt{\beta_{1y}}} e^{i\nu_1} \end{bmatrix} \qquad \hat{\mathbf{v}}_2 = \begin{bmatrix} \sqrt{\beta_{2x}}\, e^{i\nu_2} \\ -\dfrac{iu+\alpha_{2x}}{\sqrt{\beta_{2x})}} e^{i\nu_2} \\ \sqrt{\beta_{2y}} \\ -\dfrac{i(1-u)+\alpha_{2y}}{\sqrt{\beta_{2y}}} \end{bmatrix} \quad . \tag{4.1}$$

Here $\beta_{1x}$, $\beta_{1y}$, $\beta_{2x}$ and $\beta_{2y}$ are the beta-functions; the alphas: $\alpha_{1x}$, $\alpha_{1y}$, $\alpha_{2x}$ and $\alpha_{2y}$ are negative half-derivatives of the betas at regions with zero longitudinal magnetic field and two phases: $\nu_1$, $\nu_2$ constitute ten independent parameters – the generalized Twiss functions.

The presented parametrization [3] has been proven useful for both analytic and numerical analysis of coupled betatron motion in circular machines and transfer lines. Knowing the eigen-vectors, one can easily obtain the generalized Twiss functions. The formalism also allows one to perform the inverse operation of finding eigen-vectors from the generalized Twiss functions.

## References


[1] D.A. Edwards and L.C. Teng, *IEEE Trans. Nucl. Sci.* **20**, 3, pp. 885-889 (1973)

[2] F. Willeke and G. Ripken, Methods of Beam Optics, *Proceedings of US Particle Accelerator School (1987 and 1988), AIP Conf. Proc. 184,* New York 1989

[3] V.A. Lebedev and S.A. Bogacz, J-Lab internal publication, TN-00-022, May 2000