



## Circular Modes, Beam Adapters and their Applications in Beam Optics

A. Burov\*, S. Nagaitsev,

*Fermi National Accelerator Laboratory, Batavia, IL 60510, USA*

Ya. Derbenev

*Thomas Jefferson National Accelerator Facility, Newport News, VA 23606, USA* †

(June 13, 2001)

In the optics of charged particle beams, circular transverse modes can be introduced; they provide an adequate basis for rotation-invariant transformations. A group of these transformations is shown to be identical to a group of the canonical angular momentum preserving mappings. These mappings and the circular modes are parameterized similar to the Courant-Snyder forms for the conventional uncoupled case. The uncoupled-to-circular and reverse transformers (beam adapters) are introduced in terms of the circular and uncoupled modes; their implementation on the basis of skew quadrupole blocks is described. Various kinds of matching for beams, adapters and solenoids are considered. Applications of the uncoupled-to-circular, circular-to-uncoupled and circular-to-circular transformers are discussed. A range of applications includes round beams at the interaction region of circular colliders, flat beams for linear colliders, relativistic electron cooling and ionization cooling.

PACS Codes: 29.27.-a, 29.27.Eg

## I. INTRODUCTION

Linear beam optics normally employs transformations, which either do not couple vertical and horizontal degrees of freedom, or the coupling is weak. In the canonical 4D phase space,

$$\mathbf{x} \equiv \begin{pmatrix} X \\ Y \end{pmatrix} \equiv \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}, \quad (1)$$

these uncoupled transformations  $\mathcal{P}$  are described by 4D block-diagonal matrices with independent 2D unimodular blocks for the vertical and horizontal sub-spaces. For them, particle trajectories are conventionally described by means of linear, or uncoupled, modes; taken in the Courant-Snyder form [1], the related four basis vectors can be arranged as columns of a block-diagonal matrix:

$$\mathbf{V} = \begin{pmatrix} \frac{\sqrt{\beta_x} \cos(\phi_x)}{-\alpha_x \cos(\phi_x) - \sin(\phi_x)} & \frac{\sqrt{\beta_x} \sin(\phi_x)}{-\alpha_x \sin(\phi_x) + \cos(\phi_x)} & 0 & 0 \\ \frac{0}{\sqrt{\beta_x}} & \frac{0}{\sqrt{\beta_x}} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{\beta_y} \cos(\phi_y)}{-\alpha_y \cos(\phi_y) - \sin(\phi_y)} & \frac{\sqrt{\beta_y} \sin(\phi_y)}{-\alpha_y \sin(\phi_y) + \cos(\phi_y)} \\ 0 & 0 & \frac{0}{\sqrt{\beta_y}} & \frac{0}{\sqrt{\beta_y}} \end{pmatrix}. \quad (2)$$

The uncoupled transformations preserve a structure of the linear-polarized basis (2): the resulting 4 images  $\tilde{\mathbf{V}} = \mathcal{P}\mathbf{V}$  have the same structure as the original vectors, with new values for the Courant-Snyder parameters  $\alpha_{x,y}, \beta_{x,y}$  and the phases  $\phi_{x,y}$ .

Note that the basis vectors (2) are normalized in such a way, which makes the matrix  $\mathbf{V}$  symplectic:

$$\mathbf{V}^T \mathcal{S} \mathbf{V} = \mathcal{S} \quad (3)$$

where

$$\mathcal{S} = \begin{pmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{J} \end{pmatrix}; \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \mathcal{S}^2 = -\mathcal{I} \quad (4)$$

---

\*e-mail: burov@fnal.gov, voice: (630) 840-8852, fax: (630) 840-6311

†On leave from Spin Physics Center, University of Michigan, Ann Arbor, MI 48109-1120

is the symplectic unit matrix,  $\mathcal{I}$  is the  $4 \times 4$  identity matrix and the superscript  $T$  stands for the transposing. A convenience to choose basis vectors in the symplectic form relates to mapping symplecticity: without friction, any (coupled or uncoupled) transformation  $\mathcal{T}$  from the initial to the final state is symplectic: taken in a canonical basis, it satisfies the symplecticity condition (3) (see, e. g. [2], p. 52):

$$\mathcal{T}^T \mathcal{S} \mathcal{T} = \mathcal{S}. \quad (5)$$

The symplectic matrices form a group; so when a symplectic mapping  $\mathcal{T}$  acts on a symplectic basis  $\mathbf{W}$ , the new basis  $\tilde{\mathbf{W}} = \mathcal{T} \mathbf{W}$  is also symplectic. The inverted statement is true as well: any transformation which maps one symplectic basis  $\mathbf{W}$  onto another  $\tilde{\mathbf{W}}$  is symplectic itself.

Any initial phase space vector  $\mathbf{x}$  can be expanded over the basis (2)

$$\mathbf{x} = \mathbf{V} \cdot \mathbf{a} \quad (6)$$

with its amplitudes  $\mathbf{a} = (a_1, a_2, a_3, a_4)^T$ . Due to the basis symplecticity, the amplitudes  $\mathbf{a}$  can be considered as new canonical variables. Uncoupled mappings change basis parameters  $\alpha_{x,y}$ ,  $\beta_{x,y}$ , and  $\phi_{x,y}$ , while leaving the amplitudes  $\mathbf{a}$  constant. The actions and initial phases  $J_{x,y}$ ,  $\chi_{x,y}$  in the 4D phase space  $\mathbf{x}$  can be presented in terms of the amplitudes  $\mathbf{a}$  as

$$\mathbf{a} = (\sqrt{2J_x} \sin \chi_x, \sqrt{2J_x} \cos \chi_x, \sqrt{2J_y} \sin \chi_y, \sqrt{2J_y} \cos \chi_y) \quad (7)$$

where the actions are given by the Courant-Snyder invariants

$$\begin{aligned} J_x &= (a_1^2 + a_2^2)/2 = \gamma_x x^2/2 + \alpha_x x p_x + \beta_x p_x^2/2 \\ J_y &= (a_3^2 + a_4^2)/2 = \gamma_y y^2/2 + \alpha_y y p_y + \beta_y p_y^2/2 \end{aligned} \quad (8)$$

with  $\gamma_{x,y} = (1 + \alpha_{x,y}^2)/\beta_{x,y}$ .

The structure of the linear-polarized basis (2) is preserved only by the uncoupled transformations; for general 4D symplectic transformations the proper basis structure is more complicated; various forms were presented in Refs. [3–5]. The uncoupled transformations are not, however, the only point of interest where the symplectic basis can be presented in a specific reduced form.

Starting from an analogy, there are two conventional descriptions of the light polarization: it can be described in terms of either linear or circular modes. The linear basis is good when the medium is characterized by two fixed orthogonal optical axes in a plane normal to the beam propagation. The circular basis is adequate when the medium is invariant under rotations about the axis of the light propagation.

For charged particle beams, focusing by means of solenoids or round electrostatic lenses and bending by index  $n = 1/2$  dipoles gives a continuous, or local-invariant optics, i. e. such that mapping between any two places is (rotation-) invariant. However, mapping between two specific places can be designed as rotation-invariant even on a base of such non-invariant elements as quadrupoles and constant-field dipoles [6]; the whole mapping in this case can be referred to as block-invariant. Optical schemes with actual local or block invariance are discussed for muon transport [7], circular colliders (see list of references in e. g. [9]) and relativistic electron cooling [6]. An important property of rotation-invariant mappings is that they preserve the canonical angular momentum (CAM); this and inverse statements are proven in the next section. For invariant transformations, the adequate basis is constructed from circular modes; this is as obvious for charged particle beams, as it is for light. A symplectic circular basis, analogous to the Courant-Snyder uncoupled form, is presented in Section III.

After both linear and circular polarized modes are introduced, a problem of their mutual transformation can be considered. When both bases are symplectic, they can be mapped onto each other; thus, this transformation can always be done. Such an idea was originally proposed by one of the authors (Ya. Derbenev) to reduce the beam-beam effects in circular colliders; he found that an uncoupled beam state can be transformed into a round whirled state and back. He then called these linear-circular transformers "beam adapters" [10]. If one of the emittances of the coming uncoupled beam can be neglected, then the outgoing beam would be of definite-sign spirality, CAM dominated state [6], and its transverse motion could be completely cancelled inside a matched solenoid. This effective elimination of transverse temperature of coming flat beam can be essential for relativistic electron cooling [11,12]. A particular realization of the adapting optics has been found in Ref. [13]. One more application of the adapter was proposed in Ref. [14]: to get a flat electron beam for a linear collider from a round beam emitted by a magnetized cathode. In Ref. [15], a general requirement on the magnetized-to-flat mapping was discussed; the properties of the involved quadrupole blocks were formulated in terms of the Courant-Snyder parameters; the emittance preservation for both the canonical emittances was shown. Recently, the magnetized-to-flat beam transformation was demonstrated experimentally at Fermilab [16]. In the paper below, the beam adapters are considered as linear-circular transformers, which gives a straightforward way to present all their features. Finally, various applications of the circular modes and beam adapters are discussed.

## II. ROTATION-INVARIANT TRANSFORMATIONS

Group of rotations in the transverse plane through angles  $\theta$  can be presented by matrices

$$\mathcal{R}(\theta) = \begin{pmatrix} \text{cl} & \text{sl} \\ -\text{sl} & \text{cl} \end{pmatrix} \quad (9)$$

with  $\text{c} = \cos \theta$ ,  $\text{s} = \sin \theta$  and  $\text{l}$  as the  $2 \times 2$  identity matrix. Rotation invariance of a transformation  $\mathcal{T}$  means that it commutes with the rotations:

$$\mathcal{R} \cdot \mathcal{T} - \mathcal{T} \cdot \mathcal{R} = 0. \quad (10)$$

This condition is equivalent to its particular case of an infinitesimal rotation by an angle  $d\theta$  when

$$\mathcal{R} = \mathcal{I} + \mathcal{G} \cdot d\theta; \quad \mathcal{G} = \begin{pmatrix} 0 & \text{l} \\ -\text{l} & 0 \end{pmatrix}; \quad \mathcal{G}^2 = -\mathcal{I} \quad (11)$$

where  $\mathcal{I}$  and  $\text{l}$  are  $4 \times 4$  and  $2 \times 2$  identity matrices correspondingly. Then, the invariance condition reduces to a commutation of the mapping  $\mathcal{T}$  with the infinitesimal operator  $\mathcal{G}$

$$\mathcal{G} \cdot \mathcal{T} - \mathcal{T} \cdot \mathcal{G} = 0. \quad (12)$$

It can be shown now that symplectic invariant transformations  $\mathcal{T}$  preserve the canonical angular momentum (CAM)

$$M \equiv xp_y - yp_x \equiv \frac{1}{2} \mathbf{x}^T \cdot \mathcal{L} \cdot \mathbf{x} \quad (13)$$

where

$$\mathcal{L} = \begin{pmatrix} 0 & \text{J} \\ -\text{J} & 0 \end{pmatrix}; \quad \text{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \mathcal{L}^2 = \mathcal{I}. \quad (14)$$

Note that the CAM matrix  $\mathcal{L}$  is rotation-invariant:

$$\mathcal{G} \cdot \mathcal{L} - \mathcal{L} \cdot \mathcal{G} = 0. \quad (15)$$

In terms of its matrix  $\mathcal{L}$ , CAM preservation at the mapping  $\mathcal{T}$  can be expressed as

$$\mathcal{T}^T \mathcal{L} \mathcal{T} = \mathcal{L}. \quad (16)$$

To prove that this is true when conditions (5, 12) are provided, it is convenient to use the relation between the infinitesimal operator  $\mathcal{G}$ , the symplectic unit matrix  $\mathcal{S}$ , and the CAM matrix  $\mathcal{L}$ :

$$\mathcal{S} \cdot \mathcal{L} = \mathcal{L} \cdot \mathcal{S} = -\mathcal{G}. \quad (17)$$

which is straightforward to prove. It means that the matrices  $\mathcal{S}$ ,  $\mathcal{L}$  and  $\mathcal{G}$  form an algebra: any their product returns one of them. From (15, 17) the symplecticity matrix can be presented as

$$\mathcal{S} = -\mathcal{L} \cdot \mathcal{G}. \quad (18)$$

Being substituted in the symplecticity condition (5), after the commutation (12), it leads to the CAM preservation (16). Thus, the invariant transformations preserve the CAM.

In fact, the more general statement called "Generalized Busch's Theorem" was proven in Ref. [6]. Namely, it was shown that if a symplectic transformation preserves the rotation symmetry of any specific laminar beam, but, generally speaking, does not preserve the symmetry of other beams, an absolute value of the CAM of any particle of this specific beam is preserved.

One more generalization follows from the described algebra of the matrices  $\mathcal{S}$ ,  $\mathcal{G}$  and  $\mathcal{L}$ . Namely, if a symplectic mapping is invariant under a continuous one-parametric group of transformations with an infinitesimal operator  $\mathcal{G}$  (not necessarily rotations), it preserves a quadratic form associated with a matrix  $\mathcal{L} = \mathcal{S} \cdot (\mathcal{G} - \mathcal{G}^T)/2$ .

Having shown that the mapping invariance leads to the CAM preservation, a reverse statement can be proven as well: if a symplectic mapping preserves the CAM of any initial state, it is rotationally invariant. Indeed, with the matrix  $\mathcal{T}^T$  expressed from the symplecticity condition (5) and substituted in the CAM preservation (16), it leads to

what can be seen as the invariance property (12) when Eq. (17) is used. Thus, mapping invariance gives rise to CAM preservation and vice versa, so these properties are absolutely equivalent.

A general form of the CAM-preserving matrices was found by E. Pozdeev [17] and E. Perevedentsev [18]; in the rest of this section we are following E. Perevedentsev. The invariance condition (12) applied to the mapping presented in a block  $2 \times 2$  form

$$\mathcal{T} = \begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{pmatrix} \quad (19)$$

immediately yields  $T_{xx} = T_{yy}$  and  $T_{xy} = -T_{yx}$  or

$$\mathcal{T} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}. \quad (20)$$

The symplecticity condition (5) applied to a matrix of such a form (20) results in

$$A^T \cdot J \cdot A + B^T \cdot J \cdot B = J \quad (21)$$

$$A^T \cdot J \cdot B - B^T \cdot J \cdot A = 0 \quad (22)$$

For arbitrary matrix A, it is true that  $A^T \cdot J \cdot A = |A|J$ ; thus, Eq. (21) gives

$$|A| + |B| = 1. \quad (23)$$

The condition (22) presented as

$$J \cdot B \cdot A^{-1} = (B \cdot A^{-1})^T \cdot J$$

yields

$$B = A \cdot \text{const}. \quad (24)$$

It follows from (23, 24) that the matrices A and B can be presented as

$$A = T \cdot \cos \theta, \quad B = T \cdot \sin \theta \quad (25)$$

where T is an arbitrary  $2 \times 2$  matrix with  $|T| = 1$  and  $\theta$  is an arbitrary parameter. Thus, it leads to a conclusion that  $4 \times 4$  matrices of a form

$$\mathcal{T} = \begin{pmatrix} T \cdot \cos \theta & T \cdot \sin \theta \\ -T \cdot \sin \theta & T \cdot \cos \theta \end{pmatrix} \equiv \mathcal{R}(\theta) \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \quad (26)$$

present a group of symplectic rotation-invariant mappings identical to the CAM-preserving group of transformations.

One more interesting transformation is mirror reflection:

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (27)$$

This symplectic transformation does not commute with rotations, so it is not rotation-invariant and cannot be implemented by rotation-invariant optics. Combined with the rotation invariant group, it leads to such mappings as

$$\mathcal{T}_- = \mathcal{T}\mathcal{M} = \mathcal{R}(\theta) \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix} \quad (28)$$

which invert a sign of the CAM, preserving the CAM absolute value and beam rotation symmetry. In Ref. [6] a generalized Busch's theorem was proven: if a rotation-invariant laminar beam is linearly transformed into a rotation-invariant beam again, then the absolute value of the CAM is preserved for any particle of this beam. Transformations  $\mathcal{T}$  (26) and  $\mathcal{T}_-$  (28) together form a group of CAM-value preserving mappings. Thus, according to this theorem, they cover all the linear symplectic transformations, which preserve beam rotation symmetry for any initial round beam state. Reflection-like transformations  $\mathcal{T}_-$  can be implemented by means of two adapters, discussed in sections IV and V.

Parameterization of the  $2 \times 2$  unimodular matrix  $\mathbf{T}$  can be taken in the conventional Courant-Snyder form, in terms of its input  $\alpha_1, \beta_1$  and output  $\alpha_2, \beta_2$  parameters and a phase advance  $\mu$ : (see e. g. [2]):

$$\mathbf{T} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos \mu + \alpha_1 \sin \mu) & \sqrt{\beta_1 \beta_2} \sin \mu \\ -\frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin \mu + \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1 \beta_2}} \cos \mu & \sqrt{\frac{\beta_1}{\beta_2}}(\cos \mu - \alpha_2 \sin \mu) \end{pmatrix}, \quad (29)$$

where the subscript 1 of the Courant-Snyder parameters relates to an initial state, and 2 to a final state.

### III. CIRCULAR BASIS

The uncoupled transformations preserve the structure of the linear-polarized basis (2), changing only its (Courant-Snyder) parameters. In this sense, the linear-polarized modes (2) form an eigen-basis of the uncoupled transformations. An expansion of a particle trajectory  $\mathbf{x}(s)$  over this variable basis has constant coefficients, leading to the Courant-Snyder invariants.

In this section, the eigen-basis of the rotation-invariant transformations is constructed. First, some heuristic ideas are used for the construction of the matrix, comprised of four basis vectors. Then, free parameters of this form are taken to make this form symplectic. Finally, it is shown how the remaining free parameters are changed under the rotation-invariant transformations (26).

Rotation-invariant transformations preserve the CAM. The simplest vector with a non-zero CAM can be given as

$$\mathbf{u}_0 = (b, 0, 0, p_t)^T \quad (30)$$

with an arbitrary offset  $b$  and the tangential momentum  $p_t$ . This vector gives rise to a pair of rotation-equivalent orthogonal vectors by turning through angles  $\phi_+$  and  $\phi_+ - \pi/2$  resulting in

$$\begin{aligned} \mathbf{u}_1 &= (b \cos \phi_+, -p_t \sin \phi_+, b \sin \phi_+, p_t \cos \phi_+)^T \\ \mathbf{u}_2 &= (b \sin \phi_+, p_t \cos \phi_+, -b \cos \phi_+, p_t \sin \phi_+)^T \end{aligned} \quad (31)$$

where  $\phi_+$  is an arbitrary parameter. Then, the CAM of the original vector  $\mathbf{u}_0$  can be negated by changing the sign  $b \rightarrow -b$  and an additional pair of orthogonal vectors can be constructed from the vector

$$\tilde{\mathbf{u}}_0 = (-b, 0, 0, p_t)^T \quad (32)$$

by rotations through an arbitrary angle  $-\phi_-$  and  $-\phi_- + \pi/2$ :

$$\begin{aligned} \mathbf{u}_3 &= (-b \cos \phi_-, p_t \sin \phi_-, b \sin \phi_-, p_t \cos \phi_-)^T \\ \mathbf{u}_4 &= (-b \sin \phi_-, -p_t \cos \phi_-, -b \cos \phi_-, p_t \sin \phi_-)^T \end{aligned} \quad (33)$$

A structure of the four vectors (31, 33) is preserved by rotations  $\mathcal{R}$ , but it is not general enough to be preserved by the rotation-invariant transformations (26). A reason is that these vectors contain only the tangential momentum  $p_t$  having zero its normal (radial) component  $p_n = (xp_x + yp_y)/\sqrt{x^2 + y^2}$ , which is not general enough. After the addition of the normal momentum  $p_n$ , the matrix of the vectors  $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$  changes as follows:

$$\mathbf{U} = \begin{pmatrix} b \cos \phi_+ & b \sin \phi_+ & -b \cos \phi_- & -b \sin \phi_- \\ -p_t \sin \phi_+ + p_n \cos \phi_+ & p_t \cos \phi_+ + p_n \sin \phi_+ & p_t \sin \phi_- - p_n \cos \phi_- & -p_t \cos \phi_- + p_n \sin \phi_- \\ b \sin \phi_+ & -b \cos \phi_+ & b \sin \phi_- & -b \cos \phi_- \\ p_t \cos \phi_+ + p_n \sin \phi_+ & p_t \sin \phi_+ - p_n \cos \phi_+ & p_t \cos \phi_- + p_n \sin \phi_- & p_t \sin \phi_- - p_n \cos \phi_- \end{pmatrix}. \quad (34)$$

To be a valid basis for the rotation-invariant transformations, it is necessary for the set  $\mathbf{U}$  to be symplectic. It is straightforward to see that the symplecticity condition (3) is satisfied for the matrix  $\mathbf{U}$  if the tangential momentum  $p_t$  is in a specific relation with the offset:  $p_t = 1/(2b)$ . This enables the CAM to have certain values for the basis vectors:  $M = 1/2$  for the first pair  $\mathbf{u}_1$  and  $\mathbf{u}_2$ , and  $M = -1/2$  for the second pair  $\mathbf{u}_3$  and  $\mathbf{u}_4$ .

After the symplecticity of the set of vectors (34) is fixed, the final remaining point is to find out how it is changed under the invariant transformations (26). Instead of the offset  $b$  and the normal momentum  $p_n$ , new parameters  $\beta$  and  $\alpha$  are more convenient to use:

$$b = \sqrt{\beta/2}; p_n = -\alpha/\sqrt{2\beta}. \quad (35)$$

The matrix  $\mathbf{U}$  then becomes the following function of its parameters  $\alpha, \beta, \phi_+, \phi_-$ :

$$\mathbf{U} \equiv \mathbf{U}(\alpha, \beta, \phi_+, \phi_-) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{\beta} \cos \phi_+}{-\sin \phi_+ - \alpha \cos \phi_+} & \frac{\sqrt{\beta} \sin \phi_+}{\cos \phi_+ - \alpha \sin \phi_+} & \frac{-\sqrt{\beta} \cos \phi_-}{\sin \phi_- + \alpha \cos \phi_-} & \frac{-\sqrt{\beta} \sin \phi_-}{-\cos \phi_- + \alpha \sin \phi_-} \\ \frac{\sqrt{\beta} \sin \phi_+}{\sqrt{\beta}} & \frac{-\sqrt{\beta} \cos \phi_+}{\sqrt{\beta}} & \frac{\sqrt{\beta} \sin \phi_-}{\sqrt{\beta}} & \frac{-\sqrt{\beta} \cos \phi_-}{\sqrt{\beta}} \\ \frac{\cos \phi_+ - \alpha \sin \phi_+}{\sqrt{\beta}} & \frac{\sin \phi_+ + \alpha \cos \phi_+}{\sqrt{\beta}} & \frac{\cos \phi_- - \alpha \sin \phi_-}{\sqrt{\beta}} & \frac{\sin \phi_- + \alpha \cos \phi_-}{\sqrt{\beta}} \end{pmatrix}. \quad (36)$$

The same notations  $\beta, \alpha$  as for the linear-polarized basis (2) are used here on purpose. The fact is that under the rotation-invariant transformations (26) the circular set (36) is transformed similarly to how the linear basis is transformed under the uncoupled mappings.

The invariant transformation  $\mathcal{T}$  (26) parameterized by block  $\mathbf{T}$  (29) can be applied to the set of circular vectors  $\mathbf{U}$  (36). Without any loss of generality, the input Courant-Snyder parameters of the mapping can be matched with the vectors:  $\alpha_1 = \alpha, \beta_1 = \beta$ . After that, the output vector is found as

$$\tilde{\mathbf{U}} \equiv \mathcal{T} \cdot \mathbf{U}(\alpha, \beta, \phi_+, \phi_-) = \mathbf{U}(\alpha_2, \beta_2, \phi_+ + \mu - \theta, \phi_- + \mu + \theta). \quad (37)$$

This result completes the basis construction for the rotation-invariant mappings. It shows that the structure of the symplectic set of vectors  $\mathbf{U}$  (36) is preserved under these transformations; thus, this set forms the eigen-basis of the rotation-invariant mappings. After this mapping, the vectors expand (change their  $\beta$  parameter), acquire some normal momentum (change their  $\alpha$  parameter), and turn (change their phases  $\phi_+$  and  $\phi_-$ ). Note that vectors of the same spirality (sign of the CAM) are turned by the same angles. Circular modes for beam optics were considered in Ref. [10] for the description of round beams in the IP region of circular colliders. In a slightly different form than (36), the circular modes were derived in Ref. [5] as a particular case of the general 4D symplectic eigen-vectors presented in that paper.

Having defined the circular Courant-Snyder parameters according to Eq. (37), any phase space vector  $\mathbf{x}$  can be expanded over this rotating basis:

$$\mathbf{x} = \mathbf{U} \cdot \mathbf{a}. \quad (38)$$

In this presentation, the parameters of the circular basis are changed after the transformation, while the 4D vector of amplitudes  $\mathbf{a} = (a_1, a_2, a_3, a_4)^T$  remains constant. Similar to the uncoupled basis (2), relative values of the same-pair amplitudes (same-spirality for the circular basis) relate to the phases of the excited modes, while the sums of the same-pair amplitude squared give the corresponding actions. These actions, or Courant-Snyder invariants of the circular modes, can be expressed in terms of the particle coordinates  $(x, p_x, y, p_y)$ . Canonically conjugated actions  $J_{\pm}$  and phases  $\chi_{\pm}$  for positive and negative spirality modes are given by the same canonical transformation as for the uncoupled modes (7):

$$\mathbf{a} = (\sqrt{2J_+} \sin \chi_+, \sqrt{2J_+} \cos \chi_+, \sqrt{2J_-} \sin \chi_-, \sqrt{2J_-} \cos \chi_-) \quad (39)$$

Taking the amplitudes from their definition (38), the actions can be expressed in terms of 2D vectors of the offset and transverse momentum  $\vec{r} = (x, y), \vec{p} = (p_x, p_y)$ :

$$J_{\pm} = \gamma \vec{r}^2 / 4 + \alpha \vec{r} \vec{p} / 2 + \beta \vec{p}^2 / 4 \pm M / 2 \quad (40)$$

where  $\gamma \equiv (1 + \alpha^2) / \beta$  and  $M = xp_y - yp_x$  is the CAM. Note a similarity of this expression to the corresponding formula in the uncoupled case (8).

Preservation of the circular actions  $J_{\pm}$  under the invariant mappings means that both their sum and difference are preserved as well:

$$J_+ - J_- = M = \text{const}; \quad J_+ + J_- = \gamma \vec{r}^2 / 2 + \alpha \vec{r} \vec{p} + \beta \vec{p}^2 / 2 = \text{const}. \quad (41)$$

Inverse expressions are found as

$$\begin{aligned}
\vec{r}^2 &= \beta (J_+ + J_- + 2\sqrt{J_+ J_-} \cos \psi) \\
\vec{p}^2 &= \beta^{-1} \left( (J_+ + J_-)(1 + \alpha^2) + 2\sqrt{J_+ J_-}(-1 + \alpha^2) \cos \psi + 4\sqrt{J_+ J_-} \alpha \sin \psi \right) \\
\vec{r}\vec{p} &= -\alpha(J_+ + J_-) - 2\sqrt{J_+ J_-} \alpha \cos \psi - 2\sqrt{J_+ J_-} \sin \psi
\end{aligned} \tag{42}$$

where  $\psi = \phi_+ + \chi_+ + \phi_- + \chi_-$ . When only one of the two circular modes is excited (either  $J_+$  or  $J_-$  is zero), then

$$\vec{r}^2 = \beta J, \quad \vec{p}^2 = \gamma J, \quad \vec{r}\vec{p} = -\alpha J, \quad M = \pm J. \tag{43}$$

Due to the basis symplecticity, the amplitudes  $\mathbf{a}$  can be considered as new canonical coordinates, where  $a_1$  is conjugated with  $a_2$  and  $a_3$  with  $a_4$ . One more useful canonical transformation is given by the circular basis  $\mathbf{U}(\alpha, \beta, \phi_+, \phi_-)$  taken for some fixed values of the phases  $\phi_+, \phi_-$ , say,  $\phi_+ = 0, \phi_- = 0$ . Let

$$\mathbf{U}_0(\alpha, \beta) \equiv \mathbf{U}(\alpha, \beta, 0, 0) \tag{44}$$

be such a fixed-phase basis; then new canonical coordinates  $\tilde{\mathbf{a}} \equiv (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4)$  can be introduced by a symplectic transformation  $\mathbf{U}_0$  as

$$\mathbf{x} = \mathbf{U}_0 \cdot \tilde{\mathbf{a}}. \tag{45}$$

These new coordinates

$$\tilde{\mathbf{a}} = \mathbf{U}_0^{-1} \cdot \mathbf{x} = \mathbf{U}(\alpha, \beta, 0, 0)^{-1} \cdot \mathbf{U}(\alpha, \beta, \phi_+, \phi_-) \cdot \tilde{\mathbf{a}}$$

are nothing else but the constant amplitudes,  $\mathbf{a}$ , rotated by the phases  $\phi_+, -\phi_-$ :

$$\begin{aligned}
\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} &= \begin{pmatrix} \cos \phi_+ & \sin \phi_+ \\ -\sin \phi_+ & \cos \phi_+ \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \sqrt{2J_+} \begin{pmatrix} \sin(\phi_+ + \chi_+) \\ \cos(\phi_+ + \chi_+) \end{pmatrix}, \\
\begin{pmatrix} \tilde{a}_3 \\ \tilde{a}_4 \end{pmatrix} &= \begin{pmatrix} \cos \phi_- & \sin \phi_- \\ -\sin \phi_- & \cos \phi_- \end{pmatrix} \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = \sqrt{2J_-} \begin{pmatrix} \sin(\phi_- + \chi_-) \\ \cos(\phi_- + \chi_-) \end{pmatrix}.
\end{aligned} \tag{46}$$

#### IV. ADAPTERS

Both uncoupled  $\mathbf{V}$  (2) and circular  $\mathbf{U}$  (36) basic sets are symplectic; therefore, they can be mapped on each other. Symplectic transformations

$$\mathcal{C} = \mathbf{U} \cdot \mathbf{V}^{-1} \quad \text{and} \quad \tilde{\mathcal{C}} = \mathbf{V} \cdot \mathbf{U}^{-1} \tag{47}$$

map the uncoupled basis  $\mathbf{V}$  on the circular basis  $\mathbf{U}$ , and back, respectively. Note that the uncoupled-to-circular transformation  $\mathcal{C}$  maps the horizontal and vertical phase spaces on the modes of opposite spiralities. The initial state of a particle  $\mathbf{x}$  expanded over the uncoupled basis as  $\mathbf{x} = \mathbf{V} \cdot \mathbf{a}$  is characterized by a vector of the amplitudes  $\mathbf{a} = (a_1, a_2, a_3, a_4)^T$ . Then, the uncoupled-to-circular transformation  $\mathcal{C}$  converts the initial state  $\mathbf{x}$  into a new one

$$\tilde{\mathbf{x}} = \mathcal{C} \cdot \mathbf{x} = \mathbf{U} \cdot \mathbf{V}^{-1} \cdot \mathbf{V} \cdot \mathbf{a} = \mathbf{U} \cdot \mathbf{a}$$

with the same amplitudes of expansion over the circular basis; the same statement is true for the opposite transformation  $\tilde{\mathcal{C}}$ . As a consequence, the corresponding uncoupled and circular Courant-Snyder invariants are equal:

$$J_x = J_+; \quad J_y = J_-. \tag{48}$$

Note that every invariant assumes here its own Courant-Snyder parameters:  $J_{x,y}$  are calculated with  $\alpha_{x,y}, \beta_{x,y}$  of the uncoupled basis  $\mathbf{V}$  (2), while  $J_{\pm}$  assumes  $\alpha, \beta$  of the circular basis  $\mathbf{U}$  (36).

Principal ideas of the uncoupled-to-circular  $\mathcal{C}$  or reverse  $\tilde{\mathcal{C}}$  mappings were originally proposed in Ref. [10] for round beam schemes in circular colliders and, later, for electron cooling [11]. Optical devices realizing such transformations were named by the author as *beam adapters*, which underlines their shaping role for the beam phase portrait. Adaptive transformations are illustrated schematically by Fig. 1.

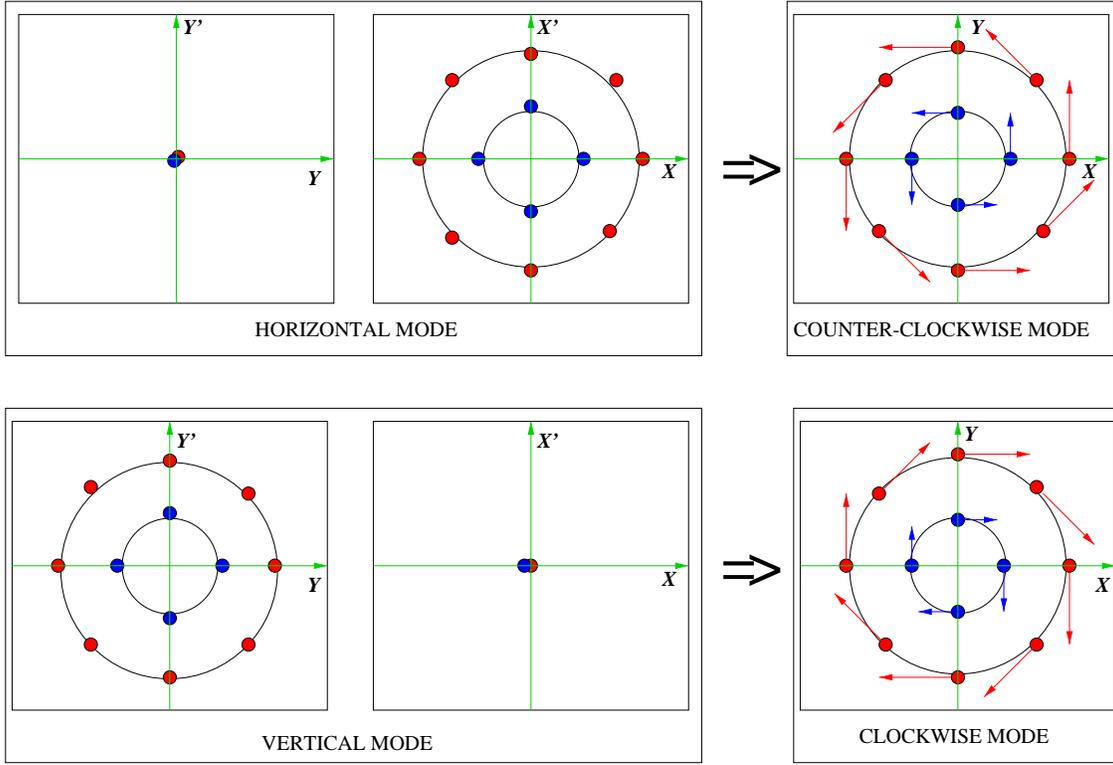


FIG. 1. Schematic illustration of the uncoupled-to-circular beam adapter: horizontally and vertically polarized modes are transformed into circular modes of opposite spiralities. Blue and red dots represent particles with smaller or larger actions. Arrows on the circular mode portraits show particle momenta, proportional to the offsets. For simplicity, all the phase portraits are depicted as circles; generally, tilted ellipses are mapped onto each other. Direction of external arrows  $\Rightarrow$  specify the direction of transformation. Reverse direction of both upper and lower arrows ( $\Leftarrow$ ) would correspond to the reverse, circular-to-uncoupled adapter.

For circular colliders, round beams in the interaction region can significantly increase the beam-beam limit of the luminosity [8,9,19]. It can be shown that a proper adapter transforms an incoming uncoupled beam into a rotation-invariant outgoing beam, and the rotation invariance would be guaranteed not only at the interaction point, but in the whole space around it, bounded by the nearest up- and downstream quadrupoles. Indeed, homogeneous distributions over the horizontal and vertical phases for the incoming uncoupled beam turn into a homogeneous distribution over the circular phases in the outgoing beam if the uncoupled-to-circular mapping  $\mathcal{C} = \mathbf{U} \cdot \mathbf{V}^{-1}$  (47) is matched with the beam, i. e. the Courant-Snyder parameters  $\alpha_{x,y}, \beta_{x,y}$  of the mapping are equal to those of the beam. Thus, any matched adapter transforms uncoupled phase-homogeneous beams into rotation-invariant beams. After the interaction region, the round beam can be turned back to a new uncoupled state by means of the reverse transformation. Note that the matched uncoupled-to circular adapter  $\mathcal{C}$  makes outgoing beams round for any ratio of the vertical to horizontal emittances and any machine tunes, contrary to the schemes like those proposed in Ref. [8] and implemented at CESR [19]. Note also that the revolution matrix at the interaction point makes only a transverse turn of the circular basis; this matrix is obviously rotation-invariant.

Adapters can also be effectively used for purposes of the relativistic electron cooling, transforming a naturally flat and hot electron beam in a cooling storage ring into a cold elliptical or round beam inside the matched cooling solenoid [11]; the resulting dramatical reduction of the electron temperature in the cooling section can be crucial for the cooling process. Indeed, when  $J_y = 0$ , only a positive circular mode is excited after mapping  $\mathcal{C}$ , making the canonical angular momentum (CAM) a function of the beam offset:  $M = r^2/\beta$ , according to Eq. (43). Immersing this beam inside the solenoid with the field

$$B = 2c/(e\beta) \quad (49)$$

turns the transverse motion to zero ; in this matched solenoid, electrons travel strictly along the magnetic field, having zero Larmor radii.

A pair of matched adapters can provide a reflection-like mapping  $\mathcal{T}_-$ , Eq. (28). Indeed, if the first adapter transforms initial circular modes into uncoupled modes, say,  $+\Rightarrow x, -\Rightarrow y$ , the second can make these uncoupled

modes circular again, but with switched spiralities,  $x \Rightarrow -, y \Rightarrow +$ ; thus, eventually the circular modes are transformed as  $+ \Rightarrow -, - \Rightarrow +$ , which means that the sign of the CAM is negated.

## V. IMPLEMENTATION OF ADAPTERS

Being symplectic, the adaptive transformations  $\mathcal{C}, \tilde{\mathcal{C}}$  are doable. Being linear, they can be realized of quadrupoles. To provide coupling, some quadrupoles must be skew. The question is, "How can it be done?" Principle ideas were proposed in Refs. [10,11] and then in more detail in [13]. It was found that particular adaptive transformations can be provided by a skew quadrupole triplet.

The transformation  $\mathcal{C} = \mathbf{U} \cdot \mathbf{V}^{-1}$  is constructed from given circular and uncoupled bases. Let the circular basis  $\mathbf{U}$  be taken for a waist point, where  $\alpha = 0$ , with the phases  $\phi_+ = -\phi_- = -\pi/4$ , while the uncoupled basis is taken with  $\alpha_x = \alpha_y = \alpha_0$ ,  $\beta_x = \beta_y = \beta_0$  and  $\phi_x = \phi_y = \phi_0$ . It is straightforward to show that in this particular case, the adaptive transformation  $\mathcal{C}$  reduces to an uncoupled transformation in a frame rotated by  $\pi/4$ . This can be expressed as

$$\mathcal{C} = \mathcal{R}(\pi/4) \langle \mathbf{M}, \mathbf{N} \rangle \mathcal{R}(-\pi/4) \quad (50)$$

where  $\langle \mathbf{M}, \mathbf{N} \rangle$  stands for a block-diagonal  $4 \times 4$  matrix with  $\mathbf{M}$  and  $\mathbf{N}$  as its  $2 \times 2$  diagonal blocks:

$$\mathbf{M} = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos \phi_0 - \alpha_0 \sin \phi_0) & -\sqrt{\beta\beta_0} \sin \phi_0 \\ \frac{\alpha_0 \cos \phi_0 + \sin \phi_0}{\sqrt{\beta\beta_0}} & \sqrt{\frac{\beta_0}{\beta}} \cos \phi_0 \end{pmatrix} \quad (51)$$

and

$$\mathbf{N} = \begin{pmatrix} -\sqrt{\frac{\beta}{\beta_0}}(\alpha_0 \cos \phi_0 + \sin \phi_0) & -\sqrt{\beta\beta_0} \cos \phi_0 \\ \frac{\cos \phi_0 - \alpha_0 \sin \phi_0}{\sqrt{\beta\beta_0}} & -\sqrt{\frac{\beta_0}{\beta}} \sin \phi_0 \end{pmatrix}. \quad (52)$$

For this transformation, the phases  $\phi_x, \phi_y$  of the initial uncoupled phase space vector and those  $\phi_+, \phi_-$  of the final circular vector are related as  $\phi_+ = \phi_x - \phi_0 - \pi/4$ ,  $\phi_- = -\phi_y + \phi_0 + \pi/4$ . Note that the blocks  $\mathbf{M}$  and  $\mathbf{N}$  look almost like the standard Courant-Snyder form, Eq. (29). Obviously, these  $2 \times 2$  matrices are characterized by identical sets of the Courant-Snyder parameters: in terms of Eq. (29),  $\alpha_1 = \alpha_0$ ,  $\alpha_2 = 0$  and  $\beta_1 = \beta_0$ ,  $\beta_2 = \beta$  for both of them, with the phase advances shifted by  $\pi/2$ , namely  $\mu_{\mathbf{M}} = -\phi_0$ ,  $\mu_{\mathbf{N}} = -\pi/2 - \phi_0$  for  $\mathbf{M}$  and  $\mathbf{N}$  blocks respectively. This relation can also be formulated as

$$\mathbf{N} = \mathbf{F} \cdot \mathbf{M} \quad \mathbf{F} = \begin{pmatrix} 0 & -\beta \\ 1/\beta & 0 \end{pmatrix}. \quad (53)$$

In other words, this particular adapter can be realized as a sequence of skew quadrupoles, with the condition (53) between the horizontal and vertical matrices in the natural (unrotated) frame of the quadrupoles. This condition on the unimodular  $2 \times 2$  matrices is equivalent to  $2 \cdot 2 - 1 = 3$  independent conditions on their elements; thus, a skew triplet of quadrupoles with variable gradients can do the job. If the circular  $\beta$  parameter is not fixed, only 2 conditions remain, so 2 variable quadrupoles are sufficient.

It becomes clearer now how adapting transformation  $\mathcal{C}$  for the arbitrary given uncoupled  $\alpha_{x,y}, \beta_{x,y}$  and circular  $\alpha, \beta$  Courant-Snyder parameters can be realized. First of all, the initial uncoupled basis can be mapped onto another uncoupled basis with identical Courant-Snyder parameters,  $\alpha_x = \alpha_y, \beta_x = \beta_y$ , which could be done by means of 2 quadrupoles with variable field gradients. Then, the described specific adapter can be applied to this second basis, mapping it onto an unspecified circular basis, which would require 2 more quadrupoles. Finally, a transformation of this unspecified circular basis onto the given circular basis can be provided by 2 quadrupoles upstream from the specific adapter. Thus, 2+2+2=6 quadrupoles with variable strength can provide the mapping of a given uncoupled beam state onto a given circular state.

For some purposes, it could be useful to have a laminar vortex state not round. This goal can be reached by applying the uncoupled-to-circular adapter (50) to an initially flat beam which Courant-Snyder parameters differ from the uncoupled parameters of the adapter. In this case, the outgoing beam would have a cross-section as a tilted ellipse, which tilt and aspect ratio would depend on the beam and adapter parameters.

## VI. CIRCULAR EIGENMODES FOR A SOLENOID

The circular modes described in section III present an adequate basis for any rotation-invariant transformations. The choice of the initial Courant-Snyder parameters can be made taking into account the properties of the incoming beam or some ideas related to the convenience or physical sense of the description. In some cases, this choice can be made by the optics itself. In this section, specific circular modes for a solenoid are discussed. Inside an extended solenoid, the modes can be defined in such a way that, while the beam travels along the field, their Courant-Snyder parameters remain constant, and only the phases run. Being rotation-invariant, solenoidal transformation  $\mathcal{T}_s$  from the entrance to an arbitrary coordinate  $z$  inside the solenoid can be presented as Eq. (26):

$$\mathcal{T}_s = \mathcal{R}(-\theta_s/2) \cdot \langle \mathbb{T}_s, \mathbb{T}_s \rangle \quad (54)$$

with

$$\mathbb{T}_s = \begin{pmatrix} \cos(\theta_s/2) & \beta_s \sin(\theta_s/2) \\ -\beta_s^{-1} \sin(\theta_s/2) & \cos(\theta_s/2) \end{pmatrix}. \quad (55)$$

Here  $\theta_s = eBz/(p_0c) \equiv z/\rho$  is the cyclotron phase advance inside the field  $B$  for a particle with the longitudinal momentum  $p_0$ . The parameter

$$\beta_s = 2c/(eB) \quad (56)$$

can be referred to as the Larmor  $\beta$ -function. From here, it follows that the Courant-Snyder parameters of the circular basis with  $\beta = \beta_s$  and  $\alpha = 0$  are preserved inside the solenoid: the first pair of the basis vectors turns by an angle  $\Delta\phi_+ = \theta_s/2 + \theta_s/2 = \theta_s$  and the second pair by  $\Delta\phi_- = -\theta_s/2 + \theta_s/2 = 0$ , i. e. remains unturned.

It is straightforward to see that the canonical variables  $\tilde{\mathbf{a}}$  (45) associated with these circular modes describe the kinetic momenta

$$k_y = p_y + x/\beta_s \quad k_x = p_x - y/\beta_s \quad (57)$$

and coordinates of the Larmor center

$$d_x = x/2 - \beta_s p_y/2 \quad d_y = y/2 + \beta_s p_x/2; \quad (58)$$

namely,

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} = \sqrt{\frac{\beta_s}{2}} \begin{pmatrix} k_y \\ k_x \end{pmatrix}, \quad \begin{pmatrix} \tilde{a}_3 \\ \tilde{a}_4 \end{pmatrix} = -\sqrt{\frac{2}{\beta_s}} \begin{pmatrix} d_x \\ d_y \end{pmatrix}. \quad (59)$$

These special canonical coordinates in the solenoidal field were proposed in Ref. [20]; they are considered in Ref. [6] in more details as cyclotron and drift canonical variables.

Let the uncoupled-to-circular adaptive transformation  $\mathcal{C}$  be matched with an adjacent downstream solenoid, i. e.  $\alpha = 0, \beta = \beta_s$ . In this case, the horizontal degree of freedom of the incoming uncoupled beam transforms into the cyclotron mode inside the solenoid, while the vertical one transforms into the drift mode. Due to symplecticity, the corresponding emittances are equal:

$$\begin{aligned} \varepsilon_x^2 &\equiv \langle x^2 \rangle \langle p_x^2 \rangle - \langle xp_x \rangle^2 = \varepsilon_c^2 \equiv \langle \tilde{a}_1^2 \rangle \langle \tilde{a}_2^2 \rangle - \langle \tilde{a}_1 \tilde{a}_2 \rangle^2 = (4/\beta^2) (\langle k_x^2 \rangle \langle k_y^2 \rangle - \langle k_x k_y \rangle^2) \\ \varepsilon_y^2 &\equiv \langle y^2 \rangle \langle p_y^2 \rangle - \langle yp_y \rangle^2 = \varepsilon_d^2 \equiv \langle \tilde{a}_3^2 \rangle \langle \tilde{a}_4^2 \rangle - \langle \tilde{a}_3 \tilde{a}_4 \rangle^2 = (\beta^2/4) (\langle d_x^2 \rangle \langle d_y^2 \rangle - \langle d_x d_y \rangle^2) \end{aligned} \quad (60)$$

with the brackets  $\langle \dots \rangle$  standing for an ensemble averaging. For a particular case of the round beam inside the solenoid, when  $\langle d_x^2 \rangle = \langle d_y^2 \rangle \equiv d^2$ ,  $\langle d_x d_y \rangle = 0$  and similar momentum relations, it yields

$$\varepsilon_x = \beta k^2/2, \quad \varepsilon_y = 2d^2/\beta. \quad (61)$$

Note that the solenoid with an opposite field switches mapping: the horizontal degree of freedom is mapped onto the drift mode and the vertical plane is mapped onto the cyclotron mode.

Similar relations take place for the reverse, circular-to-uncoupled transformations  $\tilde{\mathcal{C}}$ .

## VII. LOCAL ROTATION INVARIANCE

In a case when the rotation invariance is local (continuous), the circular Courant-Snyder parameters and phases satisfy certain differential equations, similar to the uncoupled case. The derivatives of the  $\beta$ -function and phases can be found similar to that in Ref. [5]. For any circular basis vector, the slope  $x'(s) = dx(s)/ds$  can be expressed by means of the kinetic momentum:  $x' = k_x/p_0 = (p_x - y/\beta_s)/p_0$ , with a consequent substitution of the canonic momentum  $p_x$  and the coordinate  $y$  in terms of the Courant-Snyder parameters of this basis vector. On the other hand, the slope can be found by a direct derivation of the offset  $x(s)$  expressed through the  $\beta$ -function and phase. Equating these two expressions for the same value leads to the following relations for the circular modes:

$$\frac{d\beta}{ds} = -\frac{2\alpha}{p_0}, \quad \frac{d\phi_{\pm}}{ds} = \frac{1}{p_0} \left( \frac{1}{\beta} \pm \frac{1}{\beta_s} \right). \quad (62)$$

A size of an axisymmetric laminar beam  $r_m(s)$  satisfies the envelope equation (see e. g. [34], Eq. (4.79)):

$$r_m'' + \frac{\gamma_0' r_m'}{\beta_0^2 \gamma_0} + \frac{\gamma_0'' r_m}{2\beta_0^2 \gamma_0} + \frac{r_m}{\beta_s^2 p_0^2} - \frac{M_m^2}{p_0^2} \frac{1}{r_m^3} - \frac{K}{r_m} = 0. \quad (63)$$

Here  $\beta_0$  and  $\gamma_0$  are the relativistic factors,  $p_0 = mc\beta_0\gamma_0$  is the total (longitudinal) momentum,  $M_m$  is the CAM of the boundary particle with the offset  $r_m$  and  $K = \frac{2Ie}{mc^3\beta_0^3\gamma_0^3}$  is the so-called generalized perveance, which takes into account the space charge. The term  $\propto \gamma_0'$  gives the adiabatic damping during acceleration, and the term  $\propto \gamma_0''$  relates to the electrostatic focusing. The envelope equation gives a simplest way to obtain the second-order equation for the circular  $\beta$ -function. Indeed, the laminar beam is a beam where only one of the two circular modes is excited; thus, according to Eq. (43),  $r_m = \sqrt{\beta|M_m|}$ , which leads to an equation for the circular  $\beta$ -function:

$$\beta'' - \frac{\beta'^2}{2\beta} + \frac{(\gamma_0'\beta)'}{\beta_0^2 \gamma_0} + \frac{2\beta}{p_0^2} \left( \frac{1}{\beta_s^2} - \frac{1}{\beta^2} \right) - \frac{2K}{|M_m|} = 0. \quad (64)$$

## VIII. DIAGONALIZATION OF BEAM MATRIX

Beam distributions are conventionally described by means of the so-called  $\Sigma$ -matrix, or the matrix of second moments,  $\Sigma \equiv \langle \mathbf{x} \otimes \mathbf{x} \rangle$ , with the sign  $\otimes$  standing for the outer product and  $\mathbf{x}$  is the 4D phase-space vector (1); in other words,  $\Sigma_{i,j} = \langle \mathbf{x}_i \mathbf{x}_j \rangle$  (see e. g. Ref. [2], p. 56). If  $\mathcal{M}$  is an arbitrary  $4 \times 4$  transfer matrix, then the resulting new  $\Sigma$ -matrix is determined by  $\mathcal{M}\Sigma\mathcal{M}^T$ . Unimodular transformations preserve a determinant of the  $\Sigma$ -matrix. This relates to all the symplectic transformations, but not only: a transfer from the kinetic to the canonical momenta is not symplectic, but its determinant is also a unit; thus, the  $\Sigma$ -matrix determinant is the same in the kinetic and canonical bases. The square root of this determinant is the beam emittance in the 4D phase space. The uncoupled state is described by the block-diagonal  $\Sigma$ -matrix in the original Cartesian coordinates (1); its 4D emittance is a product of the 2D emittances. Normally the phase distributions are homogeneous, in this case the  $\Sigma$ -matrix is diagonal in the matched uncoupled basis (the transfer matrix in this case  $\mathcal{M} = \mathbf{V}^{-1}$ ):

$$\Sigma = \text{Diag}(\varepsilon_x, \varepsilon_x, \varepsilon_y, \varepsilon_y), \quad (65)$$

where  $\text{Diag}(\dots)$  is a diagonal matrix with elements listed as the arguments. Suppose this uncoupled beam is transformed into a round beam by the uncoupled-circular adapter. In the matched circular basis, this new vortex state has the same diagonal beam matrix (65). However, this vortex state represents an arbitrary round beam; thus, it can be concluded that the  $\Sigma$ -matrix of any round beam distribution is diagonalized in a proper circular basis. Assuming that the matrix of a round beam is given in the original Cartesian coordinates  $\mathbf{x}$ , its two pairs of circular eigen-vectors and two canonical emittances can be found; this is a problem treated in this section.

First of all, the  $\Sigma$ -matrix of a round beam can be expressed in rotation-invariant terms. Substitution of

$$\begin{aligned} x &= r \cos \theta, \quad y = r \sin \theta, \\ p_x &= p_n \cos \theta - p_t \sin \theta \\ p_y &= p_n \sin \theta + p_t \cos \theta \\ p_x^2 + p_y^2 &= p_n^2 + p_t^2 \equiv p^2 \end{aligned} \quad (66)$$

and averaging over the angle  $\theta$  leads to the following  $2 \times 2$  block form of the  $4 \times 4$   $\Sigma$ -matrix:

$$\Sigma = \frac{1}{2} \begin{pmatrix} \Sigma & \langle rp_t \rangle J \\ -\langle rp_t \rangle J & \Sigma \end{pmatrix}; \quad \Sigma = \begin{pmatrix} \langle r^2 \rangle & \langle rp_n \rangle \\ \langle rp_n \rangle & \langle p^2 \rangle \end{pmatrix}. \quad (67)$$

Here, the  $2 \times 2$  matrix  $J$  is determined in Eq. (14); the normal and tangential canonical momenta  $p_n, p_t$  are independent of the angle  $\theta$  due to the beam symmetry.

It is straightforward to check that this beam matrix is diagonalized by the circular basis (36) with

$$\beta = \frac{\langle r^2 \rangle}{\sqrt{\langle r^2 \rangle \langle p^2 \rangle - \langle rp_n \rangle^2}}, \quad \alpha = -\frac{\langle rp_n \rangle}{\sqrt{\langle r^2 \rangle \langle p^2 \rangle - \langle rp_n \rangle^2}} \quad (68)$$

and arbitrary phases  $\phi_+, \phi_-$ . In this basis, the beam matrix (67) is presented as

$$\Sigma = \text{Diag}(\varepsilon_1, \varepsilon_1, \varepsilon_2, \varepsilon_2) \quad (69)$$

with the emittances

$$2\varepsilon_{1,2} = \pm \langle rp_t \rangle + \sqrt{\langle r^2 \rangle \langle p^2 \rangle - \langle rp_n \rangle^2} \geq 0. \quad (70)$$

Note that these partial emittances are preserved by any symplectic transformation: when the beam matrix for a new state is diagonalized, it will have the same form as (69) with the same eigenvalues  $\varepsilon_{1,2}$  as the initial state.

The total 4D emittance is a product of these partial emittances:

$$4\epsilon \equiv 4\varepsilon_1\varepsilon_2 = \langle r^2 \rangle \langle p_n^2 \rangle + \langle r^2 \rangle \langle p_t^2 \rangle - \langle rp_n \rangle^2 - \langle rp_t \rangle^2. \quad (71)$$

All these results can be expressed in terms of the kinetic momenta  $k_{x,y}$ , related to canonic ones by Eqs. (57). In terms of the normal and tangential components, this can be presented as

$$p_n = k_n, \quad p_t = k_t - r/\beta_s, \quad (72)$$

which leads to

$$2\varepsilon_{1,2} = \pm(\langle rk_t \rangle - \langle r^2 \rangle / \beta_s) + \sqrt{\langle r^2 \rangle \langle k^2 \rangle - \langle rk_n \rangle^2 - 2\langle r^2 \rangle \langle rk_t \rangle / \beta_s + \langle r^2 \rangle^2 / \beta_s^2}. \quad (73)$$

and

$$4\epsilon = \langle r^2 \rangle \langle k_n^2 \rangle + \langle r^2 \rangle \langle k_t^2 \rangle - \langle rk_n \rangle^2 - \langle rk_t \rangle^2; \quad (74)$$

the last result was previously found in Ref. [21]. Note that presentations of the 4D emittance in terms of the canonical and kinetic momenta are absolutely identical: a transfer from one to another is equivalent to rotation imposed on the beam as a whole, which does not change the total emittance (71).

The basis which makes the beam matrix diagonal can be considered as eigen-vectors of the given beam distribution, while the partial emittances can be looked as eigen-values. The deduced circular eigen-vectors for a round beam give a solution to a problem of its transformation into an uncoupled beam, when the  $\Sigma$ -matrix of the round beam is known. Indeed, a circular-to-uncoupled adapter with the circular parameters (68) would make this job.

As an example, a beam born at the magnetized cathode can be considered. At the round cathode,  $\langle r^2 \rangle \equiv 2\sigma_c^2$ ,  $\langle k^2 \rangle \equiv 2mT_c$ ,  $\langle rk_n \rangle = \langle rk_t \rangle = 0$ , where  $\sigma_c$  is its r.m.s. size and  $T_c$  is the temperature; for a homogeneous circle of radius  $a_c$ , the r.m.s. size  $\sigma_c = a_c/2$ . The circular Courant-Snyder parameters for the eigen-vectors (68) come out as

$$\alpha = 0, \quad 1/\beta^2 = 1/\beta_s^2 + k_T^2/\sigma_c^2, \quad (75)$$

and the emittances [5]

$$\varepsilon_{1,2} = \sigma_c^2/\beta_s \left( \sqrt{1 + \beta_s^2 k_T^2/\sigma_c^2} \mp 1 \right), \quad \epsilon \equiv \varepsilon_1\varepsilon_2 = \sigma_c^2 m T_c. \quad (76)$$

If the beam is strongly magnetized,  $\beta_s \ll \sigma_c k_T$ , then

$$\varepsilon_1 = \beta_s k_T^2/2; \quad \varepsilon_2 = 2\sigma_c^2/\beta_s. \quad (77)$$

A problem of eigen-vectors for arbitrary (non-round, coupled) beam distribution has been recently solved by V. A. Lebedev and S. A. Bogacz [5]. It has been found that the beam matrix can be diagonalized, and the two emittances  $\varepsilon_{1,2}$  are given by the positive roots of a characteristic equation for  $\varepsilon$ :

$$\det(\Sigma^{-1} - (i/\varepsilon)\mathcal{S}) = 0,$$

where  $i = \sqrt{-1}$ .

## IX. POSSIBLE APPLICATIONS

In this section, a possible use of the circular mode formalism and beam adapters is discussed.

### A. Round beams for circular colliders

For circular colliders, it should be beneficial to have round beams in the interaction point (IP); a list of references can be found in Ref. [9]. The main reason is that the rotation symmetry of a kick from the round opposite beam accompanied by the revolution matrix invariance leads to angular momentum preservation. This makes the transverse motion equivalent to one-dimensional. Resulting elimination of the betatron resonances is of crucial importance since they are believed to cause the beam lifetime degradation. Optical realization of the round colliding beams has been proposed in Ref. [8], and a similar scheme has been implemented at CESR [19]. For all these cases, the identity of the horizontal and vertical emittances and tunes is required. Another approach to get the beams round, the Möbius accelerator [22], based on beam rotator optics [23], is studied experimentally at CESR [24]. This scheme also leads to emittance identity and effective tune degeneration: the resulting normal tunes are inevitably separated by  $1/2$ . Use of the matched adapter at the IP opens a way that is free from all these limitations. The matched uncoupled-to-circular and reverse adapters make the beams round only in the space between these adapters. This "beam rounder" does not change the uncoupled beta-functions and emittances in the outer part of the storage ring, which would allow to use it as a transparent insert at existing circular colliders. Generally speaking, inserting this device would change the tunes, which can be restored by another local insert (phase trombone). The adapter is absolutely indifferent to such global parameters as tunes. Two tunes of the storage ring with the local beam rounder are independent variables, both available for the working point optimization. The colliding beams are round for any emittance ratio, and the revolution matrix for any point between the two adapters is rotation-invariant. All this guarantees the angular momentum preservation at the beam-beam collisions. Note that the beams would be round not only in the IP itself, but at the whole interval including IP and bounded by the nearest upstream and downstream quadrupoles. A solenoidal magnetic field in the interaction region is not important for the CAM preservation; thus, the adapter can be used either with or without the solenoid inside.

### B. Flat electron beams for linear colliders

The magnetized-to-flat transformation was suggested to be used for preparation of flat electron beams for linear colliders [14], as an alternative to flat beams obtained in damping rings. This method also allows to form electron beams with optimum density distribution in the beam plane to obtain maximum luminosity of a collider. The magnetized-to-flat transformation maps the cathode shape onto, say, a horizontal phase space of the outgoing flat beam. Changing the cathode shape, the surface density distribution of the flat beam can be arbitrarily modified, so any distribution function can be prepared. This optimization would be different for an e-e+ collider and e-e- collider, since the positron beam is shaped in a damping ring.

### C. Relativistic electron cooling

Several beam optics advancements can play a critical role in the development of the relativistic electron cooling projects for hadron beams [6,12,25].

At Fermilab, a project is developed for electron cooling of antiprotons in the Recycler storage ring at 8.9 GeV/c [26]. To provide beam focusing, the cooling section has to be immersed in the solenoidal field. To avoid beam excitation in the cooling section, the cathode, where the beam is born, has to be properly magnetized, providing the same magnetic flux through the beam as in the cooling section. It is important that all the rest of the transport line be free of extended solenoids [6]. At DESY, a possibility is studied for an RF linac-based electron cooling of 20 GeV protons in PETRA [27]. For both of these projects, the electron beam is CAM-dominated; similar optical problems have to be solved, the same methods can be used.

For high-energy electron cooling, with the energy per nucleon  $\geq 100$  GeV, the electron beam can be circulating in a storage ring. The effects of intra-beam scattering for this beam are minimized, if it is flat for the most part of the ring, as it is naturally for an uncoupled lattice. A calm and round beam in the magnetized cooling section can be provided by means of the adapting optics. Schemes of such a kind were proposed for Tevatron [29] and RHIC [28] at full energy.

Recent success in the realization of the energy recovery principle in superconducting electron linacs [30] opens a very promising perspective of linac-based high energy electron cooling. Currently, there are two proposals of this type under development: cooling of heavy ion and proton beams in RHIC [31], and ion cooling in an electron-ion collider [32]; both are based on principles of electron beam transport with a discontinuous solenoid [6]. In view of a high value of electron (average) current required for efficient electron cooling, the incorporation of an electron recirculator ring with the electron linac seems to be an important advancement for future electron cooling devices [33]. Today, this possibility is realized conceptually as a ring with circular modes matched with a solenoid of the cooling section [12,25,35]. In order to extend the lifetime of a high quality beam against intrabeam scattering and (or) quantum radiation, the ring lattice can be complemented by adapters to keep the beam flat in arcs ( similar to the above mentioned electron storage ring case, although the wigglers are not needed here).

There is an interesting possibility to compensate the optical coupling, introduced by the cooling solenoid to the hadron beam. It can be done by changing the sign of the electron CAM in the middle of the cooling section, where the solenoid is disrupted for a special short part of the trajectory, as discussed in section IV. Assuming this CAM flip is provided, the beam enters the second solenoid, where the magnetic field is reversed, so the beam remains calm there as well as it is calm at the first solenoid. This CAM-flip transformation can be provided by two adapters: the first one transforms the CAM-dominated beam into a flat beam, and the second transforms this flat beam into a whirled beam again with an opposite sign of the CAM. As a result of this trick, an average value of the magnetic field in the cooling section is eliminated, which can be beneficial for the cooled particles. This CAM-flip requires 5 quadrupoles: adjacent quadrupoles of two skew triplets can be merged.

The transport of the magnetized electron beam from the electron sources to the cooling section at high energies would also make efficient the electron cooling of high energy positron beams [25]. Due to small positron mass and magnetization, this process is very intense [36,37]; employing sweeping and rate-redistribution dispersive techniques [12,38] could additionally intensify it. The circulating positron beam can be cooled down by a linear electron beam to the emittance of a much lower value than that of the electron beam, obtained from a magnetized source. Finally, the cooled positron beams can be used, in their turn, for a fast and deep cooling of circulating electron beams.

#### **D. Low energy hadron cooler rings with circular modes**

The equilibrium emittances of a beam under cooling can be limited by Coulomb repulsion when cooling intense low-energy beams. The above mentioned concept of round beams in a recirculator ring with circular modes matched with a solenoid of the cooling section prompts a possible way to reduce the space charge effect on the 4D phase space emittance [12]. The principal optical feature of such a ring is that the drift and cyclotron components of the hadron particle motion in the solenoid are not mixed by the outside optical channel. Then, the cyclotron component (related to the hadron beam temperature in the solenoid) will experience a deep cooling, not limited by the space charge. The drift component (i.e. beam size) can be cooled to an equilibrium limited by the space charge, using the dispersive cooling.

#### **E. Ionization cooling**

A central problem for muon colliders and neutrino factories is the effective ionization cooling of muons. When the muons are transported inside an extended solenoid, only their cyclotron mode, related to the Larmor rotation, can be cooled, while the drift emittance, related to positions of the Larmor centers, is preserved. To make the cooling process comprehensive, a transport scheme with an alternating sign of the magnetic field was proposed. An optimized scheme based on the use of long solenoids was suggested in Ref. [39]. The central idea of this proposal is a cross-mapping of the drift and cyclotron modes for the sequential reversed solenoids. Due to the rotation invariance of this symplectic transformation, it could be done by round lenses (short solenoids) or invariant blocks [6]. The canonical angular momentum is preserved by this optics, while cooling makes the CAM value systematically decrease.

#### **F. Electron and ion beams for applied use**

High quality relativistic electron beams obtained from magnetized sources can be used for the effective generation of hard radiation, coherent or incoherent. The electron beam in the generation section can be returned (re-injected) into a solenoid (a strong one), to eliminate beam rotation. Optionally, the beam can be turned to the flat one, to obtain maximum electron concentration, if necessary. The flat electron and ion beams also might be of interest for

technological applications. Flat ion beams with low 4D emittance can be obtained by means of electron cooling in ion rings with circular modes, as described in section IX D. After cooling, a round ion beam can be transformed into a flat one by using beam adapter in the regime of circulation or after ejection from the ring. Optionally, if the use of a very cold ion beam would be compatible with the magnetic field, the beam can be re-injected into the solenoid of the user section, keeping it in a tranquil round state.

## X. SUMMARY

In the optics of charged particle beams, circular transverse modes can be introduced; they might be considered as analogous to the circular modes in the optics of light. These modes provide an adequate basis when the transformations are rotation-invariant. A group of the invariant transformations is shown to be identical to a group of transformations preserving the canonical angular momentum; its matrices are described. The rotation-invariant mappings and circular modes can be parameterized in a way which makes them similar to the Courant-Snyder parameterization in the conventional uncoupled case. The constructed symplectic basis of circular modes make almost obvious an idea of the beam adapters, which are optical transformers of the uncoupled to circular modes and back. The adapters can be implemented on a base of a skew quadrupole block; mapping of a given uncoupled basis onto a given circular basis requires 6 quadrupoles with variable field gradients. Inside a solenoid, there is a particular choice for the circular modes, when one of them describes the cyclotron rotation and another - coordinates of the Larmor center. In case of a beam born at the magnetized cathode, another special choice of the circular modes allows to present a matrix of the beam second moments (the so-called  $\Sigma$ -matrix) in a diagonal form. A proper downstream adapter can transform this beam into an uncoupled, or an  $X - Y$ -uncorrelated state, in which horizontal and vertical emittances are equal to the corresponding circular emittances of the beam at the cathode. Such transformations can be used for flat beams preparation in linear colliders. Beam adapters can also be used for preparation of round beams in the interaction region of the circular colliders. Requiring only local matching and being insensible to the machine tunes, the beam adapters can be added without any change to the main part of the lattice. Providing round beams and a rotation-invariant revolution matrix, such inserts guarantee the angular momentum preservation, which is believed to be crucial for a significant increase of luminosity. Relativistic electron cooling of heavy particles and ionization cooling of muons present other fields of research where use of the circular modes can be quite relevant.

- [1] E. D. Courant, H. S. Snyder, *Ann. Phys.* **3**, p.1 (1958)
- [2] A. Chao and M. Tigner, "Handbook of Accelerator Physics and Engineering", World Scientific, pp.49-52 (1998).
- [3] D. A. Edwards and L. C. Teng, *IEEE Trans. Nucl. Sci.* **20**, 3, p. 885 (1973).
- [4] I. Borchardt, E. Karantzoulis, H. Mais, G. Ripken, DESY 87-161.
- [5] V. A. Lebedev and S. A. Bogacz, e-print JLAB-ACC-99-19.
- [6] A. Burov, Ya. Derbenev, S. Nagaitsev and A. Shemyakin, *Phys. Rev. ST -Accel. Beams*, Vol. **3**, 094002 (2000).
- [7] Status of Muon Collider R & D and Future Plans, BNL-65623, FERMILAB-PUB-98/179, LBNL-41935 (1999).
- [8] V. V. Danilov et al., in Proc. *EPAC'96*, Barcelona, 1996, p. 1149.
- [9] V. Danilov and V. Shiltsev, FNAL, FERMILAB-FN-655 (1997).
- [10] Ya. Derbenev, University of Michigan, UM HE 93-20 (1993).
- [11] Ya. S. Derbenev, University of Michigan, UM HE 98-04 (1998).
- [12] Ya. Derbenev, *NIM-A* **441**, p. 223 (2000).
- [13] A. Burov and V. Danilov, FNAL, FERMILAB-TM-2043 (1998).
- [14] R. Brinkmann, Ya. Derbenev and K. Flttmann, DESY, TESLA 99-09 (1999); also in Proc. *EPAC'2000*, Vienna, 2000.
- [15] A. Burov and S. Nagaitsev, FNAL, FERMILAB-TM-2114 (2000).
- [16] D. Edwards et al., Proc. Int. Conf. Linac 20 00, Monterey, California (2000).
- [17] E. Pozdeev, BINP, private communication (1995).
- [18] E. Perevedentsev, BINP, private communication (1995).
- [19] E. Young et al., Proc. 1997 Part. Acc. Conf., p. 1542
- [20] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Mass., 1980)
- [21] S. Nagaitsev and A. Shemyakin, "Emittance of Round Beams", Fermilab-TM-2107, 2000.
- [22] R. Talman, *Phys. Rev. Lett.* **74**, p. 1590 (1995).
- [23] S. K. Kowalski and H. Enge, in Proc. Int. Conf. on Magnet Tech., Brookhaven, 1972, p. 181.
- [24] S. Henderson et al. in Proc. Part. Acc. Conf., New York, 1999, p. 410.

- [25] Ya. Derbenev, Prospects of High Energy Electron Cooling, in Proc. of EPAC 2000, Vienna, 2000
- [26] S. Nagaitsev et al., NIM-A 441, p. 241 (2000).
- [27] P. Wesolowski et al., in Proc. EPAC'2000, Vienna, 2000.
- [28] A. Burov, V. Danilov, P. Colestock and Ya. Derbenev, NIM-A 441, p. 271 (2000).
- [29] A. Burov, P. Colestock, V. Danilov, Ya. Derbenev and S. Y. Lee, Fermilab internal note (1998).
- [30] G.R. Neil et al., Kilowatt lasing in a Free electron laser, Proc. EPAC 2000, Vienna, 2000
- [31] I. Ben-Zvi et al, in Proc. of "Beam Cooling and Related Topics" workshop, Bad Honnef, Germany, May 13-18, 2001.
- [32] I. Ben-Zvi, in Proc of Second Workshop on Physics with a Polarized-Electron Light-ion Collider (EPIC), September 14-16, 2000, MIT, Cambridge, MA USA.
- [33] G. I. Budker and A. N. Skrinsky, Sov. Phys. Usp. 21 (1978) 277
- [34] M. Reiser, "Theory and Design of Charged Particle Beams", J. Wiley & Sons, Inc., 1994.
- [35] Yu. Martirosyan et al., Conceptual design of recirculator ring for electron cooling at PETRA, Proc. EPAC 2000, Vienna, 2000.
- [36] A. Artamonov, Ya. Derbenev and E.Saldin, Part. Acc. 23, 79 (1988).
- [37] A.Artamonov, Ya.Derbenev, NIM A284, 248 (1989).
- [38] Ya. Derbenev and A. Skrinsky, Sov. Phys. Rev. 1, 165 (1981).
- [39] Ya. Derbenev, Notices on Muon Cooler for Neutrino Factory, January 17-21; Ya. Derbenev, Conceptual Studies on Ionization Cooling of Muon Beam, MUCOOL Notes 2000.