APPLICATION OF SCALING PROPERTIES OF THE VLASOV AND FOKKER-PLANCK EQUATIONS TO IMPROVED MACROPARTICLE MODELS

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1 INTRODUCTION

There are two important partial differential equations (PDE's) which describe the evolution of phase space density distributions (denoted ψ) for beams in cyclic accelerators and storage rings. The Vlasov equation

$$\frac{\partial \psi}{\partial t} + \dot{\tau}\frac{\partial \psi}{\partial \varepsilon} + \dot{\varepsilon}\frac{\partial \psi}{\partial \tau} = 0 \tag{1}$$

applies when the forces are derivable from a potential. It is written here just for the longitudinal degree of freedom in conjugate time τ and energy ε variables. The Fokker-Planck equation applies when there are incoherent cooling or diffusive processes to be taken into account:

$$\frac{\partial \psi}{\partial t} = -\frac{\partial}{\partial E}\left[C(E)\psi - D(E)\frac{\partial \psi}{\partial E}\right] \quad . \tag{2}$$

Here $\psi(E;t)$ is the energy distribution of a coasting beam. It is often easier in practical applications to follow the evolution of phase space density by tracking a representative distribution of macroparticles using the equations of motion. There are scaling relations derived from the PDE's which can make dramatic improvements in the efficiency or practical scope of macroparticle models. Scaling rules reported previously are recapitulated below. Their usefulness in speeding up macroparticle model calculations is discussed with most attention to the simpler case of time scaling in the Vlasov equation.

Andre Gerasimov found scaling for the Fokker-Planck equation for longitudinal stochastic cooling permitting macroparticle solutions correct even for macroparticle number six or seven orders of magnitude less than the beam population.[1] These ideas were used by the author in some cooling simulations described briefly at the end of this note. Their success encouraged the search for an appropriate scaling for speeding up large scale longitudinal dynamics modeling for synchrotrons, allowing time steps longer than a beam turn without misrepresenting the dynamics. The result was more powerful than anticipated, being for a broad range of cases the fourth power of the time scaling constant rather than the first power one might expect.

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2 TIME SCALING IN THE VLASOV EQUATION

The objective for scaling the Vlasov equation (eq. 1) is to find variables in which the evolution of the density function is accelerated but not otherwise changed. In the absence of a multiparticle potential it is obvious from inspection of the equations of motion that multiplying the potential and the phase-slip-per-time-step by a constant accelerates the change in the distribution proportionally. This scaling constant is hereafter denoted by λ. Because the Hamiltonian H is the generating function for infinitesimal canonical transformations of coordinates from their value at time t to time $t+\mathrm{d}t$, the Hamiltonian for the system that evolves λ times as fast is simply

$$H'(t') = \lambda H(t) \;\text{ for } t' = \lambda t;\, \varepsilon' = \varepsilon \;\; , \tag{3}$$

from which one can get particle equations of motion for an accelerated tracking. Note that writing the eq. 1 in the scaled system

$$\frac{\partial \psi'}{\partial t'} + \frac{\partial \psi'}{\partial \tau'}\frac{\mathrm{d}\tau'}{\mathrm{d}t'} + \frac{\partial \psi'}{\partial \varepsilon'}\frac{\mathrm{d}\varepsilon'}{\mathrm{d}t'} = 0 \tag{4}$$

and substituting according to eq. 3 leads directly to

$$\lambda\frac{\partial \psi}{\partial t} + \lambda\frac{\partial \psi}{\partial \tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} + \lambda\frac{\partial \psi}{\partial \varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = 0 \;\; . \tag{5}$$

Thus, the invariance of the Vlasov equation under scaling is almost trivial. This derivation of time scaling, which starts from the existence of the multiparticle Hamiltonian, complements the previously reported derivation from the particle equations of motion.[2] Somewhat more subtle is how the scaled time will show up in the potential; any terms relating to frequencies, including rf potentials and frequency domain expressions for impedance, will be scaled.

Multiparticle tracking using the equations of motion from the scaled Hamiltonian will go faster by the factor λ because the clock runs faster by that factor. However, it is through the scaling of frequencies and a statistical observation that the scaling gets most of its power, another three powers of λ. Scaling any frequency f with the time scaling gives $f' = \lambda f$. Writing the potential in frequency domain, one can see that it takes λ^{-1} as many harmonics of the beam circulation frequency to span the domain; the sampling is, of course, sparser by that same factor. If the potential is smooth enough that the reduced sampling can be accepted, the scaling will not compromise fidelity of the calculated distribution. In fact, for the space charge term, whose frequency domain form is linear, there is *no* loss of fidelity from reduced sampling.[2] Fewer harmonics in the potential mean fewer bins needed in calculating fourier components of the beam current. It has been shown rigorously for the space charge force in time domain[3] and heuristically for smooth potentials in frequency domain[4] that when the bin number is reduced by λ^{-1}, the macroparticle number can be reduced by λ^{-3} for the same statistical accuracy (numerical noise) in the distribution. Therefore, in suitable circumstances λ may be considerably larger than one, and a tracking simulation can be accelerated by λ^4.

Besides sampling deficiency, there is another way in which the fidelity of a scaled macroparticle model can be compromised; *viz.*, the expanded time step can be too large a fraction of a characteristic period in the motion like, especially, the synchrotron period. One must observe a limit on scaling $\lambda\nu_c << 1$, where ν_c is the synchrotron tune or the tune of some parameter variation. Usually a factor ten for each $<$ sign is acceptable.

3 HYBRID MACROPARTICLE MODELS VLASOV + FOKKER-PLANCK

Gerasimov's scaling rules[1] for the Fokker-Planck equation eq. 2 can be expressed in terms of their effect on the gain G_ℓ at each harmonic ℓ of the beam circulation frequency, the cooling power P_ℓ at each harmonic, the beam circulation frequency as a function of energy $\omega(E)$, the scaled time t, and the scaled number of particles N. There are three rules which are called here number scaling, bandwidth scaling, and gain scaling. The scale factors are denoted k_N, k_B, and k_G respectively; $\omega_\circ$ is the beam circulation angular frequency at the central orbit energy $E_\circ$. The rules are expressed in the following table:

Scaled	Scaling Rule		
Result	Number	Bandwidth	Gain
G'_ℓ	$k_N G_\ell$	$G_{k_B\cdot\ell}$	$k_G G_\ell$
P'_ℓ	$k_N P_\ell$	P_ℓ/k_B	$k_G P_\ell$
$\omega' - \omega_\circ$	$\omega - \omega_\circ$	$k_B(\omega - \omega_\circ)$	$k_G(\omega - \omega_\circ)$
t'	t/k_N	$k_B t$	t/k_G
N'	N/k_N	N	N

There are limits on the size of the scale factors which involve number of harmonics and particle statistics; they are akin to those discussed above for time scaling in the Vlasov equation. The reader is referred to ref. [1] for these limits and the relations between tabulated quantities and the cooling coefficient $C(E)$ and diffusion coefficient $D(E)$ of eq. 2. For the macroparticle approach, the tabulated quantities are the ones of interest.

Consider a case where the beam particles are acted on by an rf system, for example, as well as a cooling system. If the Fokker-Planck scaling rules can be used to give an adequate macroparticle model with a time step appropriate for the phase motion at a macroparticle number low enough to be practical, one can build a combined macroparticle model by applying both the map for the equations of motion and the Fokker-Planck operator for the cooling on each iteration. Notice it is also possible to scale the map for larger time step as discussed above. In the case of the hybrid model, however, the macroparticle number will be determined by the Fokker-Planck scaling so that the Schottky noise is correctly scaled. This technique has been used to study rf stacking onto a cooled stack in the Fermilab Recycler storage ring. It was meaningful to simulate minutes of real time of cooling/stacking with fewer than 10^4 macroparticles; computing times were a few hours.

4 CONCLUSIONS

Numerical simulations of cooling processes over minutes or hours of real time are usually carried out using direct solution of the Fokker-Planck equation. However, by using scaling rules derived from that equation, it is possible to use macroparticle representations of the beam distribution. Besides having applications for cooling alone, the macroparticle approach allows combining the cooling process with other dynamical processes which are represented by area-preserving maps. A time-scaling rule derived from the Vlasov equation can be used to adjust the time step of a map-based dynamics calculation to one more suitable for combining with a macroparticle Fokker-Planck calculation. The time scaling for the Vlasov equation is also useful for substantially more rapid calculations when a macroparticle model of a conservative multiparticle system requires a large number of macroparticles to faithfully produce the collective potential or when the model must simulate a long time period.

5 REFERENCES

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