



## Perturbations of a Stringy Black Hole

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We extend the three dimensional stringy black hole of Horne and Horowitz to four dimensions. After a brief discussion of the global properties of the metric, we discuss the stability of the background with respect to small perturbations, following the methods of Gilbert and of Chandrasekhar. The potential for axial perturbations is found to be positive definite.

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## 1. Introduction

The discovery of exact and approximate solutions of perturbative string theory with black hole behavior has garnered much attention recently. In addition to furthering our understanding of the nature of the solutions of the field equations of general relativity and/or the nonperturbative content of string theory, one might even hope that a stringy black hole would have observational and measureable features which might not require a detailed knowledge of how the fundamental string theory devolves into the Standard Model.

In one approach[1][2], one starts with a so-called “string inspired” low energy effective action that would naturally devolve from a fundamental string theory. The key difference from a non-string inspired model is then the presence of the dilaton.

The presence of the dilaton has important effects. Gilbert has recently shown that the four dimensional charged black hole derived in [1] is catastrophically unstable to small perturbations, in sharp contrast to similar backgrounds in general relativity[3,4]. This instability is completely distinct from other effects, such as possible quantum mechanical evaporation. It may be difficult, however, to further investigate the “stringiness” of these solutions, as it is not known what conformal field theory they correspond to, and such solutions generically include an electromagnetic field whose stringy origins are somewhat unclear, while they often do not contain an axion field which is expected on general grounds.

In the second approach, gauged Wess-Zumino-Witten (WZW) models have been found to correspond to string propagation in curved backgrounds, backgrounds with black hole singularities with an appropriate choice of groups and gauged subgroups. Higher dimension analogs of Witten’s two dimensional black hole[5]have been found by Bars and Sfetsos[6,7], Horne and Horowitz [8]and Horava [9]. The availability of such higher dimensional analogs, especially in four dimensions, presents some potentially important oppourtunities. It would be very interesting, for example, to consider a four dimensional black hole of this type and subject it to the same perturbative analysis which the “string-inspired” model has recently undergone. That is precisely what will be attempted here.

First, in Section 2, we will extend the black hole of Horne and Horowitz [8] to four dimensions. The global properties of the metric will be briefly discussed in Section 3. The metric, dilaton, and axion have sufficiently simply forms that the perturbative analysis in Section 4 will not be overly difficult. The equations for the perturbations separate, as in [3], into two sets, referred to as axial and polar. The potential for the axial perturbations will

be explicitly derived. Numerical evidence strongly suggests that this potential is strictly positive, although this is not obvious from the analytic expressions. Further discussion and conclusions will be collected in Section 5.

## 2. A Four Dimensional Black Hole

Let us begin by reviewing the three dimensional black string of Horne and Horowitz[8], which in turn is an extension of Witten's 2-d black hole based on the gauged  $SL(2, R)/U(1)$  WZW model. We begin with a Lorentz metric  $ds^2 = 2d\sigma_+\sigma_-$  on the world sheet  $\Sigma$ . In terms of the group valued function  $g$ , the WZW action is

$$L(g) = \frac{k}{4\pi} \int d^2\sigma \text{Tr}(g^{-1}\partial_+gg^{-1}\partial_-g) - \frac{k}{12\pi}\Gamma(g). \quad (2.1)$$

$\Gamma(g)$  is the Wess-Zumino-Witten term, normally written as an integral over a three manifold  $B$  whose boundary is  $\Sigma$ ,

$$\Gamma(g) = \int_B d^3y \text{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg). \quad (2.2)$$

We now gauge a one dimensional subgroup  $H$  of the symmetry group, with action  $g \rightarrow hgh$ . The symmetry is promoted to a local symmetry by the introduction of a gauge field  $A_i$ , with values in the Lie algebra of  $H$ . The local symmetry acts as

$$\delta g = \epsilon g + g\epsilon, \delta A_i = -\partial_i\epsilon, \quad (2.3)$$

where  $\epsilon$  is an infinitesimal parameter. The action for which this transformation law is a symmetry, the gauged WZW action, is

$$L(g, A) = L(g) + \frac{k}{2\pi} \int d^2\sigma \text{Tr}(A_+\partial_-gg^{-1} + A_-\partial_+g + A_+A_- + A_+gA_-g^{-1}). \quad (2.4)$$

The 2-d black hole of Witten[5] follows by letting  $G = SL(2, R)$ , with  $H$  the subgroup generated by  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , and integrating out the gauge fields, which appear quadratically in the action.

The extension of Horne and Horowitz[8] follows by adding a free boson  $x_1$  to the action, that is, letting  $G = SL(2, R) \times R$ , and by simultaneously tensoring in translations in  $x_1$  to the subgroup  $H$ . The extension we will consider here is to add another free boson  $x_2$ , and mod out by translations in both  $x_1$  and  $x_2$ . Parametrizing  $SL(2, R)$  as

$$g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix} \quad (2.5)$$

we have the ungauged action

$$\begin{aligned} L(g) = & -\frac{k}{4\pi} \int d^2\sigma (\partial_+u\partial_-v + \partial_-u\partial_+v + \partial_+a\partial_-b + \partial_-a\partial_+b) \\ & + \frac{k}{2\pi} \int d^2\sigma \log u (\partial_+a\partial_-b - \partial_-a\partial_+b) + \frac{1}{\pi} \int d^2\sigma \Sigma_i \partial_+x_i \partial_-x_i. \end{aligned} \quad (2.6)$$

The gauge transformations are

$$\delta a = 2\epsilon a, \delta b = -2\epsilon b, \delta u = \delta v = 0, \delta x_i = 2\epsilon c_i, \delta A_i = -\partial_i \epsilon, \quad (2.7)$$

where the  $c_i$  are constants. The gauged action is then

$$\begin{aligned} L(g, A) = L(g) &+ \frac{k}{2\pi} \int d^2\sigma A_+ (b\partial_- a - a\partial_- b - u\partial_- v + v\partial_- u + \frac{4c_i}{k} \partial_- x_i) \\ &+ \frac{k}{2\pi} \int d^2\sigma A_- (b\partial_+ a - a\partial_+ b + u\partial_+ v - v\partial_+ u + \frac{4c_i}{k} \partial_+ x_i) \\ &+ \frac{k}{2\pi} \int d^2\sigma 4A_+ A_- (1 + \frac{2c^2}{k} - uv), \end{aligned} \quad (2.8)$$

where a sum on  $i = 1, 2$  is assumed, and  $c^2 = c_1^2 + c_2^2$ . We now fix the gauge by setting  $a = \pm b$ , depending on the sign of  $1 - uv$ , and integrate out the gauge fields, thus obtaining

$$L = L_1 + L_2 + L_3, \quad (2.9)$$

$$L_1 = -\frac{k}{8\pi} \int d^2\sigma \frac{\lambda_t (v^2 \partial_+ u \partial_- u + u^2 \partial_+ v \partial_- v) + (2 - 2uv + 2\lambda_t - \lambda_t uv) (\partial_+ u \partial_- v + \partial_- u \partial_+ v)}{(1 - uv)(1 + \lambda_t - uv)} \quad (2.10)$$

$$L_2 = \frac{1}{\pi} \int d^2\sigma \frac{1 + \lambda_2 - uv}{1 + \lambda_t - uv} \partial_+ x_1 \partial_- x_1 + (1 \leftrightarrow 2) - \frac{(\lambda_1 \lambda_2)^{1/2}}{1 - \lambda_t - uv} (\partial_+ x_1 \partial_- x_2 + \partial_- x_1 \partial_+ x_2), \quad (2.11)$$

$$L_3 = \frac{1}{2\pi} \int d^2\sigma \frac{c_i}{1 + \lambda_t - uv} (v \partial_+ u \partial_- x_i - v \partial_- u \partial_+ x_i - u \partial_+ v \partial_- x_i + u \partial_- v \partial_+ x_i), \quad (2.12)$$

where  $\lambda_i = 2c_i^2/k$ ,  $\lambda_t = \lambda_1 + \lambda_2$ .

As in the three dimensional case, this action can be greatly simplified by the substitution

$$u = e^{\sqrt{2}t/\sqrt{k(1+\lambda_t)}} \sqrt{\hat{r} - (1 + \lambda_t)}, v = -e^{-\sqrt{2}t/\sqrt{k(1+\lambda_t)}} \sqrt{\hat{r} - (1 + \lambda_t)}. \quad (2.13)$$

The action is then

$$L = \frac{1}{\pi} \int d^2\sigma (g_{\mu\nu} \partial_- x^\mu \partial_+ x^\nu + B_{\mu\nu} (\partial_- x^\mu \partial_+ x^\nu - \partial_+ x^\mu \partial_- x^\nu)), \quad (2.14)$$

where the metric  $g_{\mu\nu}$  corresponds to the line element

$$ds^2 = -(1 - \frac{1 + \lambda_t}{\hat{r}}) dt^2 + (1 - \frac{\lambda_i}{\hat{r}}) dx_i^2 + \frac{k d\hat{r}^2}{8\hat{r}^2} [(1 - \frac{1 + \lambda_t}{\hat{r}})(1 - \frac{\lambda_t}{\hat{r}})]^{-1} - (dx^1 dx^2 + dx^2 dx^1) \frac{\sqrt{\lambda_1 \lambda_2}}{\hat{r}}, \quad (2.15)$$

and where the antisymmetric tensor  $B_{\mu\nu}$  is given by

$$B_{tx_i} = \sqrt{\frac{\lambda_i}{1 + \lambda_t}} \left(1 - \frac{1 + \lambda_t}{\hat{r}}\right). \quad (2.16)$$

Henceforth, we will only consider the case where  $\lambda_1 = \lambda_2 = \lambda/2$ , in which case, we can diagonalize the metric by introducing  $x = \frac{1}{2}(x_1 + x_2)$  and  $y = \frac{1}{2}(x_1 - x_2)$ , we find the exact same metric and antisymmetric tensor as Horne and Horowitz[8], with  $y$  a flat coordinate, i.e.,

$$ds^2 = -\left(1 - \frac{1 + \lambda}{\hat{r}}\right)dt^2 + \left(1 - \frac{\lambda}{\hat{r}}\right)dx^2 + dy^2 + \left(1 - \frac{1 + \lambda}{\hat{r}}\right)^{-1} \left(1 - \frac{\lambda}{\hat{r}}\right)^{-1} \frac{k d\hat{r}^2}{8\hat{r}^2}, \quad (2.17)$$

$$B_{tx} = \sqrt{\frac{\lambda}{1 + \lambda}} \left(1 - \frac{1 + \lambda}{\hat{r}}\right). \quad (2.18)$$

The dilaton can be calculated by considering the determinant induced by our choice of gauge, as in [6], but clearly this case will follow the three dimensional case [8]. There, it was shown that demanding that the fields be an extremum of the low energy effective action[10]

$$S = \int e^{\Phi} \left( R + (\nabla\Phi)^2 - \frac{1}{12} H^2 + \frac{8}{k} \right), \quad (2.19)$$

(where the  $\frac{8}{k}$  cosmological constant term corresponds to the usual  $D - 26$  [5]) requires

$$\Phi = \ln(\hat{r}) + a. \quad (2.20)$$

Here  $a$  is an arbitrary constant, which as in previous cases, will be related to the mass of the black hole.

Since the fields are so closely related to what is found in the three dimensional case, many but not all of the results still hold in four dimensions. For example, the scalar curvature is

$$R = \frac{4(2\hat{r} + r\lambda\hat{r} - 7\lambda - 7\lambda^2)}{k\hat{r}^2}, \quad (2.21)$$

indicating that only  $\hat{r} = 0$  is a true singularity, even though the metric components are ill-defined at  $\hat{r} = \lambda, 1 + \lambda$  as well. As an aside, the coordinates  $u, v, x, y$  are ill-defined at  $uv = 1$  ( $\hat{r} = \lambda$ ), where our gauge fixing procedure breaks down. It is at this point, and presumably in some as yet uncertain region around this point, where Eguchi has argued that the theory is described by a topological field theory [11]. The coordinate singularity at  $\hat{r} = 1 + \lambda$  seems to have no such interpretation.

At this point, we would like to express the fields in terms of the mass and axionic charge of the black hole, rather than the arbitrary parameters  $\lambda$  and  $a$ . Here, we omit the details, which follow almost immediately from the discussion in [8], and which yields

$$Q = e^a \sqrt{\frac{2\lambda(1+\lambda)}{k}} \quad (2.22)$$

for the axionic charge per unit area, and

$$M = \sqrt{\frac{2}{k}}(1+\lambda)e^a \quad (2.23)$$

for the mass per unit area. Introducing the rescaled coordinate  $r$  such that

$$\hat{r} = r e^{-a} \sqrt{\frac{k}{2}}, \quad (2.24)$$

the final form of the fields is

$$ds^2 = -\left(1 - \frac{M}{r}\right)dt^2 + \left(1 - \frac{Q^2}{Mr}\right)dx^2 + dy^2 + \left(1 - \frac{M}{r}\right)^{-1}\left(1 - \frac{Q^2}{Mr}\right)^{-1}\frac{kdr^2}{8r^2}, \quad (2.25)$$

$$H_{rtx} = Q/r^2, \quad (2.26)$$

$$\Phi = \ln(r) + \frac{1}{2}\ln\frac{k}{2}. \quad (2.27)$$

### 3. Global Structure

The presence of two coordinate singularities reminds one of the Reissner-Nordström metric

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2d\Omega^2. \quad (3.1)$$

The singularity at  $r = r_+ = M + \sqrt{M^2 - Q^2}$  is an event horizon, while that at  $r = r_- = M - \sqrt{M^2 - Q^2}$  is known as the inner horizon and is believed to be unstable with respect to time dependant perturbations[13]. The spacetime is timelike geodesically complete. Let us now compare these results to those for the metric in (2.25).

#### A. $Q < M$

As in the three dimensional case [8], for the metric in (2.25), the singularity at  $r = M$  is an event horizon similar to that at  $r = r_+$  in (3.1), while the singularity at  $r = Q^2/M$  is an inner horizon similar to that at  $r = r_-$  in (3.1), with the only difference being that the Killing vector  $\partial/\partial t$  (timelike at spatial infinity) becomes spacelike from the outer horizon

all the way to the singularity at  $r = 0$ , unlike the Reissner-Nordström case, where the Killing vector is spacelike only between the two horizons.

There is a difference, however, between the three and four dimensional cases when we consider geodesics, since there is an additional conserved quantity associated with the new coordinate  $y$ . Let  $\xi^\mu$  be tangent to an affinely parametrized geodesic, and let  $E = -\xi \cdot \frac{\partial}{\partial t}$ ,  $P = \xi \cdot \frac{\partial}{\partial x}$ ,  $R = \xi \cdot \frac{\partial}{\partial y}$  denote the conserved quantities. Then the geodesics must satisfy

$$\frac{k\dot{r}^2}{8r^2} = E^2 - P^2 - R^2 + \frac{1}{r}(P^2 M - \frac{E^2 Q^2}{M} + R^2(M + \frac{Q^2}{M})) - \frac{Q^2}{r^2} R^2 - \alpha(1 - M/r)(1 - Q^2/Mr), \quad (3.2)$$

where the dot denotes a derivative with respect to the affine parameter, and  $\alpha$  is 0 for null geodesics and  $-1$  for timelike geodesics. In either case, if the right hand side of (3.2) is positive for large  $r$  then it will continue to be positive within the horizons, and therefore geodesics which begin at large  $r$  cross both horizons. Also in both cases, the geodesic equation (3.2) is eventually dominated by a  $-1/r^2$  term, so that neither null nor timelike geodesics reach the singularity (in the three dimension case [8], this was only true for timelike geodesics). Thus the spacetime is timelike and lightlike geodesically complete. The exact form of the geodesics will not be needed.

The rest of the global structure, including the Penrose diagram, is essentially identical to the three dimensional case. For example, the Hawking temperature can be found by analytically continuing  $t = i\tau$ . The horizon is a regular point only if  $\tau$  has period  $\pi M \sqrt{2k/(M^2 - Q^2)}$ , corresponding to a temperature

$$T = \frac{1}{\pi M} \sqrt{\frac{M^2 - Q^2}{2k}}. \quad (3.3)$$

The temperature vanishes at  $Q = M$ , suggesting is the black hole will eventually settle at  $Q = M$ , assuming that charge cannot be radiated away.

## B. $Q = M$

For the extremal case of  $Q = M$ , (2.25) reads

$$ds^2 = (1 - M/r)(-dt^2 + dx^2) + dy^2 + (1 - M/r)^{-2} k dr^2 / 8r^2. \quad (3.4)$$

This is similar to the extremal Reissner-Nordström metric, but in fact, in is inappropriate to consider values of  $r < M$ , as geodesics no longer go through the horizon. The geodesic equation for this case reads

$$\frac{k\dot{r}^2}{8r^2} = E^2 - P^2 - R^2 + \frac{1}{r}(P^2 M - E^2 M + 2R^2 M) - \frac{M^2 R^2}{r^2} + \alpha(1 - M/r)^2. \quad (3.5)$$

Near the (single) horizon at  $r = M$ , the last term is negligible, while the remainder changes sign. Therefore, geodesics do not penetrate the horizon and the similarity with the Reissner-Nordström case ends. As in [8], it is appropriate therefore to introduce a new coordinate  $\bar{r}^2 = r - M$ , in terms of which the metric becomes

$$ds^2 = \frac{\bar{r}^2}{\bar{r}^2 + m} (-dt^2 + dx^2) + \frac{k d\bar{r}^2}{2\bar{r}^2} + dy^2. \quad (3.6)$$

In the new coordinate system,  $\bar{r} = 0$  is the horizon, and geodesics do cross the horizon. However, the horizon does not surround a singularity, but rather separates two identical asymptotically flat regions.

### C. $Q > M$

A situation similar to that for  $Q = M$  occurs here. Namely, the metric has a change of sign at  $r = Q^2/M$ , but this can be removed by an appropriate choice of coordinates, as suggested by the geodesics equation. Setting  $\bar{r} = r - Q^2/M$ , one finds [8],

$$ds^2 = -\frac{Q^2 - M^2 + M\bar{r}^2}{Q^2 + M\bar{r}^2} dt^2 + \frac{M\bar{r}^2}{Q^2 + M\bar{r}^2} dx^2 + dy^2 + \frac{Mk}{2(Q^2 - M^2 + M\bar{r}^2)} d\bar{r}^2. \quad (3.7)$$

The metric has no horizons or curvature singularities. The conical singularity at  $\bar{r} = 0$  can be removed by requiring  $x$  to be periodic, which is equivalent to changing the spacetime structure at infinity from  $R^4$  to  $R^3 \times S^1$ . As shown in [8], this can also be derived from our original gauged WZW action, by gauging the two translations together with the subgroup of  $SL(2, R)$  generated by  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .

## 4. Perturbation Analysis

We now proceed to consider perturbations to the dilaton, metric, and antisymmetric tensor, following the general procedure of Gilbert [3], which follows along the lines of earlier work of Chandrasekhar [12]. The equations of motion derived from the action (2.19) are

$$\nabla_\lambda H^{\lambda\mu\nu} + \nabla_\lambda \Phi H^{\lambda\mu\nu} = 0, \quad (4.1)$$

$$-\frac{1}{6} H^2 + \nabla^2 \Phi + (\nabla \Phi)^2 - \frac{8}{k} = 0, \quad (4.2)$$

$$R_{\mu\nu} = \nabla_\mu \nabla_\nu \Phi + g_{\mu\nu} \left( \frac{1}{2} \nabla^2 \Phi + \frac{1}{2} (\nabla \Phi)^2 - \frac{4}{k} - \frac{H^2}{12} \right). \quad (4.3)$$

The analysis of [3]and [12]utilizes the fact that the unperturbed metric has some symmetry, that is the components are independant of (at least) one coordinate. Denote the components of the unperturbed contravariant form of the metric as

$$g_{\mu\nu} = e^{2f_\mu} \delta_{\mu\nu}, \quad (4.4)$$

( $\mu, \nu = 0, 1, 2, 3$ ). In the present case, we will order our coordinates as  $(x^0, x^1, x^2, x^3) = (t, x, r, y)$ , thus keeping our notation as close as possible to [3]and [12]. In this case, the perturbed metric can be chosen as

$$g^{\mu\nu} = \begin{pmatrix} -e^{-2f_0} & -\chi_0 e^{-2f_0} & 0 & 0 \\ -\chi_0 e^{-2f_0} & g^{11} & \chi_2 e^{-2f_2} & \chi_3 e^{-2f_3} \\ 0 & \chi_2 e^{-2f_2} & e^{-2f_2} & 0 \\ 0 & \chi_3 e^{-2f_3} & 0 & e^{-2f_3} \end{pmatrix} \quad (4.5)$$

for the contragradient form, and

$$g_{\mu\nu} = \begin{pmatrix} (e^{2f_1} \chi_0^2 - e^{2f_0}) & -e^{2f_1} \chi_1 & e^{2f_1} \chi_2 \chi_0 & e^{2f_1} \chi_0 \chi_3 \\ -e^{2f_1} \chi_0 & e^{2f_1} & -e^{2f_1} \chi_2 & -e^{2f_1} \chi_3 \\ e^{2f_1} \chi_2 \chi_0 & -e^{2f_1} \chi_2 & (e^{2f_1} \chi_2^2 + e^{2f_2}) & e^{2f_1} \chi_2 \chi_3 \\ e^{2f_1} \chi_0 \chi_3 & -e^{2f_1} \chi_3 & e^{2f_1} \chi_2 \chi_3 & (e^{2f_1} \chi_3^2 + e^{2f_3}) \end{pmatrix}, \quad (4.6)$$

for the covariant form, where  $g^{11} = \chi_3^2 e^{-2f_3} + \chi_2^2 e^{-2f_2} + e^{-2f_1} - \chi_0^2 e^{-2f_0}$ , and where  $\chi_0, \chi_2$  and  $\chi_3$  (as well as all the perturbations of  $f_i$  and the other fields) are functions of  $t, r$  and  $y$  only. Note that (4.6)corresponds to a squared line element of

$$ds^2 = -e^{2f_0} dt^2 + e^{2f_1} (dx - \chi_0 dt - \chi_2 dr - \chi_3 dy)^2 + e^{2f_2} dr^2 + e^{2f_3} dy^2, \quad (4.7)$$

from which one can immediately determine the veirbein  $e_\mu^a$ ,

$$e_\mu^a = e^{f_a}, \quad A = \mu, \quad (4.8)$$

$$e_\mu^1 = -\chi_\mu e^{f_1}, \quad \mu = 0, 2, 3. \quad (4.9)$$

As discussed in [3]and [12], the above form of the perturbed metric has the effect of dividing the perturbations into two classes, known as polar and axial. Polar perturbations are those which leave the sign of the metric unchanged upon a reversal of sign, whereas axial perturbations are those for which one must accompany such a reversal with the change  $x \rightarrow -x$  to keep the sign of the metric invariant. In the present case, the variations of the  $f_i$  are polar, and the  $\chi_i$  are axial perturbations. We expect that, as in previous cases

in general relativity, that the equations for the two types of perturbations will separate. In this paper we will only analyze the axial equations, leaving the polar perturbations for future work.

We begin with the Einstein equation (4.3). It has been shown [12] that for the axial perturbations, we only need to consider the {12} and {13} components of (4.3). Let us denote the right hand side of (4.3) as  $T_{\mu\nu}$ , and let us use the vierbein in (4.8) and (4.9) to go to the orthonormal frame, where Chandrasekhar has already worked out the components of the Ricci tensor, which are

$$R_{12} = \frac{1}{2} e^{-2f_1 - f_0 - f_3} [(e^{3f_1 + f_0 - f_2 - f_3} \chi_{23})_{,3} - (e^{3f_1 - f_0 + f_3 - f_2} \chi_{20})_{,0}], \quad (4.10)$$

and

$$R_{13} = \frac{1}{2} e^{-2f_1 - f_0 - f_2} [(e^{3f_1 + f_0 - f_3 - f_2} \chi_{32})_{,2} - (e^{3f_1 - f_0 + f_2 - f_3} \chi_{30})_{,0}], \quad (4.11)$$

where  $\chi_{ij} = \chi_{i,j} - \chi_{j,i}$ . Since  $\chi_{ij}$  are already assumed to be first order of smallness, when we linearize (4.3), we can take  $f_i$  as their unperturbed values given by (2.25). We now only need the form of  $T_{ab}$ , again in the orthonormal frame. In this frame, all the terms proportional to  $g_{\mu\nu}$  in (4.3) will now be proportional to  $\eta_{ab}$  and therefore will vanish because we are only considering the cases of  $a = 1, b = 2, 3$ . The remaining term in (4.3),  $\nabla_\mu \nabla_\nu \Phi$ , must be carefully evaluated, including the contributions of the vierbein. The result, having eliminated terms quadratic in small quantities, is

$$\delta T_{12} = 0, \quad (4.12)$$

$$\delta T_{13} = -\frac{1}{2r} e^{f_1 - f_3 - 2f_2} \chi_{32}. \quad (4.13)$$

The important thing to notice is that (4.13) does not contain any term proportional to  $\delta\Phi$ . It can also be shown (see below) that neither the dilaton nor antisymmetric tensor equation (with indices raised) (4.3) and (4.2) contain terms involving the  $\chi_{ij}$ .

We can now proceed to equate (4.10) with (4.12) and (4.11) with (4.13). First, substitute  $f_3 = 0$ ,  $\frac{k}{8r^4} e^{-2f_2} = e^{2f_0 + 2f_1}$ , and assume that all perturbations vary with time as  $e^{i\omega t}$ . Then the {12} equation can be written as

$$0 = e^{2f_0} \chi_{23,3} - \chi_{20,0}, \quad (4.14)$$

while the {13} equation is

$$\frac{8}{k} e^{2f_0 + 2f_1} \chi_{23} = -2r e^{-2f_1} \left( \frac{4}{k} r e^{4f_1 + 2f_0} \chi_{23} \right)_{,2} - \chi_{30,0}. \quad (4.15)$$

Now differentiate (4.14) with respect to  $x_3$ , and subtract this from the derivative of (4.15) with respect to  $x_2$ . Using the fact that  $\chi_{20,03} - \chi_{30,02} = \chi_{23,00} = -\omega^2 \chi_{23}$ , and defining  $\alpha = r e^{4f_1 + 2f_0} \chi_{23}$ , we have

$$\frac{\partial}{\partial r} \left[ \frac{8}{k} e^{-2f_1} \alpha \right] = -\frac{\partial}{\partial r} \left[ \frac{8r}{k} e^{-2f_1} \frac{\partial \alpha}{\partial r} \right] - \frac{1}{r} e^{-4f_1} \alpha_{,33} - \frac{\omega^2}{r} e^{-4f_1 - 2f_0} \alpha. \quad (4.16)$$

Clearly we can separate variables, and let  $\alpha_{,33} = -\kappa^2 \alpha$ , where  $\kappa$  is some constant, which should be taken to be real so that the solutions are well behaved at  $y \rightarrow \pm\infty$ .

The remaining procedure is now clear. First, multiply (4.16) by  $r e^{4f_1 + 2f_0}$  so that the last term is just  $\omega^2 \alpha$ . Then make a change of variables  $r \rightarrow r^*$ , so that the coefficient of  $\frac{d^2 \alpha}{dr^{*2}}$  is unity. This requires

$$\frac{dr^*}{dr} = \frac{e^{-f_0 - f_1}}{r} \sqrt{\frac{8}{k}}. \quad (4.17)$$

Making this substitution, we have

$$\alpha'' + \alpha' \left( \sqrt{\frac{8}{k}} e^{f_0 + f_1} - f_0' - 3f_1' \right) + \alpha \left( -2e^{f_0 + f_1} \sqrt{\frac{8}{k}} f_1' - \kappa^2 e^{2f_0} + \omega^2 \right), \quad (4.18)$$

where the primes denote derivatives with respect to  $r^*$ .

Denoting by  $X_1(r^*)$  the coefficient of  $\alpha'$ , we now eliminate the this term by multiplying (4.18) by the integrating factor

$$A(r^*) = e^{-\frac{1}{2} \int X_1(x) dx}. \quad (4.19)$$

Substituting  $\alpha(r^*) = A(r^*)g(r^*)$ , and using the fact that  $A'(r^*) = -(1/2)X_1(r^*)A(r^*)$ , etc., we have

$$g'' + \omega^2 g = V(r^*)g, \quad (4.20)$$

where

$$V(r^*) = \kappa^2 e^{2f_0} + \frac{1}{4} X_1^2 + \frac{1}{2} X_1' + 2 \sqrt{\frac{8}{k}} e^{f_0 + f_1} f_1'. \quad (4.21)$$

This potential can easily be evaluated as a function of  $r$  (implicitly a function of  $r^*$  through (4.17)) by using the explicit expressions for the  $f_i$  and equation (4.17). It is not manifestly positive because of the  $X_1'$  term. To simplify matters slightly, let us assume that  $\kappa$  vanishes, as it only contributes positively to the potential (for  $r > M$ ), and thus only enhances stability. Secondly, all the remaining terms are easily seen to be proportional to  $1/k$ , so  $k$  can be chosen at our convenience, say  $k = 8$ . Furthermore, note that (2.25) indicates that  $k$  has units of  $(\text{length})^2$ , so that the remaining factor in the

potential is a dimensionless function of  $r/M$  and  $Q/M$ . The explicit expression for  $X_1$  is then

$$X_1(r) = \sqrt{\left(1 - \frac{M}{r}\right)\left(1 - \frac{Q^2}{Mr}\right)\left(1 - \frac{M}{2r}\left(1 - \frac{M}{r}\right)^{-1} - \frac{3Q^2}{2Mr}\left(1 - \frac{Q^2}{Mr}\right)^{-1}\right)}. \quad (4.22)$$

While the derivative of (4.22) has many terms, the only region (outside the outer horizon) in which the potential can turn negative is near  $r = M$ . But the leading contribution to  $X_1'$  at  $r = M$  goes as

$$X_1'(r) \rightarrow \frac{M^2}{4r^2} \sqrt{1 - \frac{Q^2}{Mr}} \left(1 - \frac{M}{r}\right)^{-1}. \quad (4.23)$$

Since the coefficient is positive, we expect the potential to be positive near  $r = M$ , and therefore the potential should be positive for all  $r$ . For example, for  $Q = M$ , after much cancellation, and setting  $r = M/x$ , the potential is simply

$$V(x) = \frac{1}{4}(1 + 4x - x^2). \quad (4.24)$$

Since  $0 < x < 1$ , the potential is positive definite. For more general values of  $Q/M$ , some typical results are shown in Table 1.

**Table 1:** The axial potential.

$r/M$	$Q = .2M$	$Q = .4M$	$Q = .6M$	$Q = .8M$
2	.348	.365	.402	.484
3	.288	.31	.353	.426
4	.272	.292	.33	.39
5	.265	.283	.315	.365
6	.261	.277	.305	.348
7	.259	.273	.298	.335
8	.257	.27	.292	.325
9	.256	.268	.288	.317
10	.256	.266	.284	.311

Is there further information which can be easily gleaned from the coupled equations (4.2)-(4.3)? A short calculation shows that the  $(t, y)$ ,  $(r, y)$  and  $(r, t)$  components of (4.2) require that  $\delta H^{trv} = \text{constant}$  (recall that by assumption, all perturbations are independent of  $x$ ). The numerous remaining equations all involve polar quantities such as  $\delta f_i$  and  $\delta \Phi$  which will be difficult to disentangle as in the axial case. Further investigation of these coupled equations is currently under way.

## 5. Discussion

The numerical result that the axial potential is positive definite, and that therefore the black hole is stable to axial perturbations, is interesting in how it differs from the “string-inspired” case [3], yet incomplete. One would like a physical explanation for this stability, and of course, one still must check whether the solution is stable with respect to the polar perturbations. Furthermore, we would like to know whether this stability is generic for the ever increasing list of stringy black holes. If it is not, some criteria for stability may shed some light on the underlying conformal field theories. In fact, there is evidence that it is the gauge fields, present in the string inspired models but not in the present one, that guarantees instability, and that this result is independent of  $\alpha'$  corrections or the presence of an axion field [15]. The fact that the axion field did not enter into the equations for the axial perturbations in the present model tends to support this evidence.

There are many other issues that can be addressed now that four dimensional stringy black hole solutions are becoming available. Possibly the most important of these would be to compute scattering amplitudes, as begun in [16]. The fact that these models come from WZW models should enable such calculations, and some preliminary results in the two dimensional case are already becoming available [17]. Such calculations would open the way for discussions of Hawking radiation and other “hair” which a black hole may have, as well as the role of string effects such as winding states.

Another interesting question which was mentioned briefly in Section 2 is the recent result of Eguchi [11] that the horizon in the two dimensional black hole is described by a topological field theory. The first question such a result raises is what region *around* the horizon is described by such a theory. As topological field theories are by construction independent of the metric, it is not clear how any length scale which determines this region can enter the calculation. Assuming the generality of this result, however, we see that in the four dimensional case, the outer horizon, which is stable to perturbations, is described by a topological field theory, whereas the inner horizon, which we expect to be unstable by the analogy with the Reissner-Nordström case, is not. It is tempting indeed to conjecture that this correlation is in fact general.

Yet another avenue for further investigation concerns string loop contributions. In a very recent paper, Ellis et.al. [18] argue that the  $c = 1$  matrix model represents the evaporation endpoint of the two dimensional black hole due to the absence in the matrix model of imaginary parts in higher genus amplitudes. The arguments do not depend on

the number of background dimensions, and therefore appear to be general. They further argue that the static nature of the known black hole solutions will be destroyed at higher genus. If this is true, detailed calculations of the evaporation process and a comparison to conventional black holes would be important to pursue.

In summary, it would seem that there are many interrelated issues involving string theory and black holes, of which only a few have even begun to be addressed. Hopefully, further analysis of these toy models will shed significant light on the important underlying physics.

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