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Derivation of Prismatic Coupling Coefficients in Magnetostatic Code-GFUN

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DERIVATION OF PRISMATIC COUPLING COEFFICIENTS IN MAGNETOSTATIC CODE–GFUN

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Abstract

A new derivation of the analytical representation of coupling coefficients for prismatic elements in the magnetostatic code GFUN3D[1] is given. Although the numerical results agree with the existing code the explicit derivation given below does not to my knowledge exist in the open literature. In particular the derivation makes use of new algebraic equivalences that simplify the presentation and rationalize the sign changes needed in the subroutine TERMS.

I. Introduction.

It has been noted that in the magnetostatic code GFUN3D[1] the AMPFAC(actual excitation current divided by ideal iron excitation current for a given central field) is larger than expected. Many refinements have been added to the code such as the use of preconditioned matrices. Although the use of these matrices has removed internal looping of the magnetization vector the AMPFAC is always too high. It is possible that the limitation placed on the number of elements by the local computing environment is responsible for the difficulty. However, since one must live with the local environment, other conceivable sources of the difficulty were explored. In particular, since the use of tetrahedral elements gave a lower AMPFAC than the use of prismatic elements, it was thought desirable to derive the coupling coefficients for the latter elements independently.

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II. General.

In Fig. 1 use h for prism axis and let $\bar{r}$ designate the vector from P(field point) to Q(source point) on any of the plane faces of the prism shown here as triangular for simplicity.

Thus:

$$\bar{r} = \bar{i}(x_q - x_p) + \bar{j}(y_q - y_p) + \bar{k}(h_q - h_p) \tag{1}$$

with

$$r = |\bar{r}| \, . \tag{2}$$

And

$$\bar{n}da = \begin{cases} (d\bar{s} \times \bar{k})dh & (a,b,c..) \\ \bar{k}dxdy & (e) \\ -\bar{k}dxdy & (f) \end{cases} \quad . \tag{3}$$

In GFUN[1] the magnetization is taken to be constant throughout any one element. For this element the contribution to the field at P is given by

$$\Delta\bar{H}(P) = \bar{\bar{\omega}} \cdot \bar{M}(Q) \, , \tag{4}$$

where $\bar{\bar{\omega}}$ is the solid angle tensor or dyadic

$$\bar{\bar{\omega}} = \oint_s \bar{n}_q \nabla_p \left(\frac{1}{r_{pq}}\right) da_q \, . \tag{5}$$

The point Q is on the surface S of the element and P is taken to be the centroid of any element if one is forming the set of equations from which $\bar{M}$ is to be found. Or, having found $\bar{M}$, P is taken to be any point at which the field is desired. The vector $\bar{n}_q$ is the outward normal to the surface at the point Q . Contraction of the dyadic gives the solid angle

$$\omega_p = \oint_s \bar{n}_q \cdot \nabla_p \left(\frac{1}{r_{pq}}\right) da_q \, , \tag{6}$$

which is equal to 4π if P is inside the closed surface or zero if P is outside.

On executing the gradient the solid angle tensor in dyadic notation becomes

$$\bar{\bar{\omega}} = \oint_s \frac{\bar{n}\,\bar{r}}{r^3}\,da\,, \tag{7}$$

where $\bar{n}$ is the outward normal at Q, $\bar{r}$ is the vector from P to Q and r is $|\bar{r}|$.

III. Contribution to $\bar{\bar{\omega}}$ from Sides (a,b,c..) of the Prism

$$\Delta\bar{\bar{\omega}} = \iint \frac{(d\bar{s}\times\bar{k})\;\bar{r}}{r^3}\,dh = \int dh\left[\oint_c \frac{(d\bar{s}\times\bar{k})\;\bar{r}}{r^3}\right]. \tag{8}$$

From this the tensor components are

$$\Delta\omega_{xx} = \int dh \oint_c \frac{1}{r^3}[\bar{i}\cdot(d\bar{s}\times\bar{k})][\bar{r}\cdot\bar{i}] = \int dh \oint_c \frac{x}{r^3}\,dy\,, \tag{9}$$

$$\Delta\omega_{xy} = \int dh \oint_c \frac{1}{r^3}[\bar{i}\cdot(d\bar{s}\times\bar{k})][\bar{r}\cdot\bar{j}] = \int dh \oint_c \frac{y}{r^3}\,dy\,, \tag{10}$$

$$\Delta\omega_{xz} = \int dh \oint_c \frac{1}{r^3}[\bar{i}\cdot(d\bar{s}\times\bar{k})][\bar{r}\cdot\bar{k}] = \int h\,dh \oint_c \frac{1}{r^3}\,dy\,, \tag{11}$$

$$\Delta\omega_{yx} = \int dh \oint_c \frac{1}{r^3}[\bar{j}\cdot(d\bar{s}\times\bar{k})][\bar{r}\cdot\bar{i}] = -\int dh \oint_c \frac{x}{r^3}\,dx\,, \tag{12}$$

$$\Delta\omega_{yy} = \int dh \oint_c \frac{1}{r^3}[\bar{j}\cdot(d\bar{s}\times\bar{k})][\bar{r}\cdot\bar{j}] = -\int dh \oint_c \frac{y}{r^3}\,dx\,, \tag{13}$$

$$\Delta\omega_{yz} = \int dh \oint_c \frac{1}{r^3}[\bar{j}\cdot(d\bar{s}\times\bar{k})][\bar{r}\cdot\bar{k}] = -\int h\,dh \oint_c \frac{1}{r^3}\,dx\,, \tag{14}$$

$$\Delta\omega_{zx} = \int dh \oint_c \frac{1}{r^3}[\bar{k}\cdot(d\bar{s}\times\bar{k})][\bar{r}\cdot\bar{i}] = 0\,, \tag{15}$$

$$\Delta\omega_{zy} = \int dh \oint_c \frac{1}{r^3}[\bar{k}\cdot(d\bar{s}\times\bar{k})][\bar{r}\cdot\bar{j}] = 0\,, \tag{16}$$

$$\Delta\omega_{zz} = \int dh \oint_c \frac{1}{r^3}[\bar{k}\cdot(d\bar{s}\times\bar{k})][\bar{r}\cdot\bar{k}] = 0\,, \tag{17}$$

where

$$x = x_q - x_p,\;\; y = y_q - y_p,\;\; h = h_q - h_p,\;\; r = \sqrt{x^2 + y^2 + z^2}\,. \tag{18}$$

Individual Side Expressions:

$$\Delta\omega_{xx} = \int dh \oint_c \frac{x}{r^3}\,dy = \Sigma_{a,b,c..}\Delta\omega_{xx}(\text{side})\,, \tag{19}$$

$$\Delta\omega_{xx}(\text{side}) = \int dh \int \frac{x}{r^3}\,dy = \int x\,dy \int \frac{\partial}{\partial h}\left[\frac{1}{x^2+y^2}\,\frac{h}{r}\right] dh\,. \tag{20}$$

For side a let

$$y = C + Sx \,, \tag{21}$$

where q, the transverse part of the Q(variable) point relative to the P(fixed) point, ranges from (x_1,y_1) to (x_2,y_2). For side b, characterized by a different C and S, (x,y) will range from (x_2,y_2) to (x_3,y_3); etc. Formulas will be given only for side a . The others may be obtained by induction. Thus

$$\Delta\omega_{xx}(\text{side}) = hS\int_{q_1}^{q_2} \frac{x}{x_r^2+y^2}\,\frac{1}{r}\,dx\,\Bigg|_{h=h_1}^{h=h_2} . \tag{22}$$

Let

$$R = r^2 = C^2 + h^2 + 2CSx + (1+S^2)x^2 , \tag{23}$$

And following G&R[2] 2.252 also let

$$t = x + \frac{CS}{1+S^2} \,. \tag{24}$$

Then Eq.(22) becomes

$$\Delta\omega_{xx}(\text{side}) = \frac{hS}{\sqrt{(1+S^2)^3}} \int \frac{1}{\frac{C^2}{(1+S^2)^2} + t^2} \times \frac{t - \frac{CS}{1+S^2}}{\sqrt{\frac{C^2+(1+S^2)h^2}{(1+S^2)^2} + t^2}}\,dt\,\Bigg|_{h=h_1}^{h=h_2} . \tag{25}$$

Let:

$$\Delta\omega_{xx}(\text{side}) = (\, I_1 + I_2 \,)\,\Bigg|_{x=x_1}^{x=x_2}\Bigg|_{h=h_1}^{h=h_2} , \tag{26}$$

where I_1 is the integral associated with $CS/(1+S^2)$ in the numerator and I_2 is the integral associated with t in the numerator. For I_1 G&R[2] 2.252 suggests the transformation

$$v = \frac{t}{\sqrt{\frac{C^2+(1+S^2)h^2}{(1+S^2)^2} + t^2}} \,. \tag{27}$$

If a parameter r_c is defined as

$$r_c = \sqrt{x^2 + y^2 - \frac{C^2}{1+S^2}} \,, \tag{28}$$

Appendix A indicates that

$$v = \frac{r_c}{r}\,\mathrm{sign}\left[\, x + \frac{CS}{1+S^2} \right] \equiv \frac{r_c}{r}\,\mathrm{sgnb}\ . \tag{29}$$

Application of Eq(27) to the I_1 part of Eq.(25) yields

$$I_1 = -\ \frac{CS^2}{\sqrt{(1+S^2)^3}}\,\frac{1}{h}\int \frac{1}{\frac{C^2}{(1+S^2)h^2} + v^2}\,dv\ . \tag{30}$$

Then utilizing G&R[2] 2.124 and Eq.(29)

$$I_1 = -\,\mathrm{sgnb}\,\frac{S^2}{1+S^2}\,\frac{C\,|h|}{h\,|C|}\,\tan^{-1}\frac{|h|\sqrt{1+S^2}\ r_c}{|C|\ r}\,. \tag{31}$$

Since only the principle value of the multivalued function is desired, the arctangent function has the sign of its argument, thus permitting the absolute value signs around h and C to be removed.

Hence

$$I_1 = -\,\mathrm{sgnb}\,\frac{S^2}{1+S^2}\tan^{-1}\frac{h\sqrt{1+S^2}\ r_c}{C r}\,. \tag{32}$$

For I_2, the t–term in Eq.(25), an expression in Appendix A indicates that

$$\frac{r^2}{1+S^2} = \frac{C^2 + (1+S^2)h^2}{(1+S^2)^2} + t^2 \tag{33}$$

would be a useful transformation.

Thus

$$I_2 = \frac{1}{2}\,\frac{S}{1+S^2}\int\left[\frac{1}{r-h} - \frac{1}{r+h}\right]dr = -\,\frac{1}{2}\,\frac{S}{1+S^2}\ln\left[\frac{r+h}{r-h}\right]. \tag{34}$$

However $[r+h/(r-h)] = [(r+h)^2/(r^2-h^2)]$. But, since r^2-h^2 does not contain h, Eq.(26) indicates that it may be dropped.

Hence $$I_2 = -\,\frac{S}{1+S^2}\ln(r+h)\,. \tag{35}$$

Summarizing:

$$\Delta\omega_{xx}(\mathrm{side}) = \left\{-\,\mathrm{sgnb}\,\frac{S^2}{1+S^2}\tan^{-1}\frac{h\sqrt{1+S^2}\ r_c}{C r} - \frac{S}{1+S^2}\ln(r+h)\right\}\Bigg|_{q=q_1}^{q=q_2}\Bigg|_{h=h_1}^{h=h_2}. \tag{36}$$

Next

$$\Delta\omega_{xy} = \int dh \oint_c \frac{y}{r^3}\, dy = \Sigma_{a,b,c..} \Delta\omega_{xy}(\text{side}) \,, \tag{37}$$

$$\begin{aligned}\Delta\omega_{xy}(\text{side}) &= \int dh \int \frac{y}{r^3}\, dy \\ &= \int y dy \int \frac{\partial}{\partial h}\Big[\frac{1}{x^2+y^2}\frac{h}{r}\Big]\, dh\,.\end{aligned} \tag{38}$$

Utilizing Eq.(21)

$$\Delta\omega_{xy}(\text{side}) = hS\int_{q_1}^{q_2} \frac{C + Sx}{x^2+ y^2}\frac{1}{r}\, dx \,\Big|_{h=h_1}^{h=h_2}. \tag{39}$$

And following G&R[2] 2.252 use Eq.(24) to give

$$\Delta\omega_{xy}(\text{side}) = \frac{hS}{\sqrt{(1+S^2)^3}}\int \frac{1}{\frac{C^2}{(1+S^2)^2} + t^2} \times \frac{St - \frac{C}{1+S^2}}{\sqrt{\frac{C^2+(1+S^2)h^2}{(1+S^2)^2} + t^2}}\, dt \,\Bigg|_{h=h_1}^{h=h_2}. \tag{40}$$

Hence

$$\Delta_{xy}(\text{side}) = \Big(-\frac{1}{S}I_1 + S\, I_2\Big)\Big|_{q=q_1}^{q=q_2}\Big|_{h=h_1}^{h=h_2}, \tag{41}$$

Or

$$\Delta\omega_{xy}(\text{side}) = \Big[\text{sgnb}\,\frac{S}{1+S^2}\tan^{-1}\frac{h\sqrt{1+S^2}\; r_c}{Cr} - \frac{S^2}{1+S^2}\ln(r+h)\Big]\Big|_{q=q_1}^{q=q_2}\Big|_{h=h_1}^{h=h_2}. \tag{42}$$

Next

$$\Delta\omega_{xz} = \int hdh \oint_c \frac{1}{r^3}\, dy = \Sigma_{a,b,c..}\Delta\omega_{xz}(\text{side}) \,, \tag{43}$$

$$\Delta\omega_{xz}(\text{side}) = \int dy \int \frac{h}{r^3}\, dh = -\int dy \int \frac{\partial}{\partial h}\Big[\frac{1}{r}\Big]\, dh\,. \tag{44}$$

Utilizing Eq.(21)

$$\Delta\omega_{xz}(\text{side}) = -S\int_{q_1}^{q_2} \frac{1}{r}\, dx \,\Big|_{h=h_1}^{h=h_2}. \tag{45}$$

Then, from Eq.(23), and G&R[2] 2.261

$$\Delta\omega_{xz}(\text{side}) = -\frac{S}{\sqrt{1+S^2}}\ln\left[2\sqrt{1+S^2}\,r + 2(1+S^2)\,x + 2CS\right], \tag{46}$$

taken between the above limits. Replace $(1+S^2)\,x + CS$ by expression found in Appendix A and suppress the factor $2\sqrt{1+S^2}$ in the argument since it is independent of x and h to give

$$\Delta\omega_{xz}(\text{side}) = (\ I_3\ )\ \Big|_{q=q_1}^{q=q_2}\Big|_{h=h_1}^{h=h_2}, \tag{47}$$

where

$$I_3 = -\frac{S}{\sqrt{1+S^2}}\ln(\ r + \text{sgnb}\ r_c\ )\ . \tag{48}$$

Or

$$\Delta\omega_{xz}(\text{side}) = -\frac{S}{\sqrt{1+S^2}}\ln(\ r + \text{sgnb}\ r_c\ )\ \Big|_{q=q_1}^{q=q_2}\Big|_{h=h_1}^{h=h_2}. \tag{49}$$

Next

Since $(xdx + ydy)/r^3$ is integrable, subtracting Eq.(12) from Eq.(10) gives

$$\Delta\omega_{yx}(\text{side}) = \Delta\omega_{xy}(\text{side}) \tag{50}$$

Next

$$\Delta\omega_{yy} = -\int dh \oint_c \frac{y}{r^3}\,dx = \Sigma_{a,b,c..}\,\Delta\omega_{yy}(\text{side})\ , \tag{51}$$

$$\begin{aligned}\Delta\omega_{yy}(\text{side}) &= -\int dh \int \frac{y}{r^3}\,dx \\ &= -\int\ ydx \int \frac{\partial}{\partial h}\left[\frac{1}{x^2+y^2}\,\frac{h}{r}\right]dh\ .\end{aligned} \tag{52}$$

Utilizing Eq.(21)

$$\Delta\omega_{yy}(\text{side}) = -\,h\int_{q_1}^{q_2}\frac{C + \ Sx}{x^2+\ y^2}\,\frac{1}{r}\,dx\ \Big|_{h=h_1}^{h=h_2}. \tag{53}$$

Hence

$$\Delta_{yy}(\text{side}) = \left(\frac{1}{S^2} I_1 - I_2 \right) \Big|_{q=q_1}^{q=q_2} \Big|_{h=h_1}^{h=h_2}, \tag{54}$$

Or, comparing with Eq.(41),

$$\Delta\omega_{yy}(\text{side}) = \Big[- \text{sgnb} \frac{1}{1+S^2} \tan^{-1} \frac{h\sqrt{1+S^2}\ r_c}{Cr} + \frac{S}{1+S^2} \ln(r+h) \Big] \Big|_{q=q_1}^{q=q_2} \Big|_{h=h_1}^{h=h_2}. \tag{55}$$

Next

$$\Delta\omega_{yz} = -\int h dh \oint_c \frac{1}{r^3} dx = \Sigma_{a,b,c..} \Delta\omega_{yz}(\text{side}) , \tag{56}$$

$$\Delta\omega_{yz}(\text{side}) = -\int dx \int \frac{h}{r^3} dh = \int dx \int \frac{\partial}{\partial h}\Big[\frac{1}{r} \Big] dh . \tag{57}$$

Utilizing Eq.(21)

$$\Delta\omega_{yz}(\text{side}) = \int_{q_1}^{q_2} \frac{1}{r} dx \Big|_{h=h_1}^{h=h_2}, \tag{58}$$

which on comparing with Eq.(45) gives

$$\Delta\omega_{yz}(\text{side}) = -\frac{1}{S} \Delta\omega_{xz}(\text{side}) . \tag{59}$$

Or, from Eq.(49)

$$\Delta\omega_{yz}(\text{side}) = \frac{1}{\sqrt{1+S^2}} \ln(\, r + \text{sgnb}\ r_c \,) \Big|_{q=q_1}^{q=q_2} \Big|_{h=h_1}^{h=h_2}. \tag{60}$$

This completes the evaluation of contributions from the sides of the prism.

IV. Contributions to $\overline{\overline{\omega}}$ from the Ends (e,f) of the Prism .

$$\overline{\overline{\omega}} = \iint \frac{1}{r^3} [\pm \overline{k}\ \overline{r}]\ dxdy \quad [+ \text{ sign for (e)}, \ - \text{sign for (f)}]. \tag{61}$$

Since the contribution from (e) and from (f) are to be added the following notation includes both ends.

$$\Delta\overline{\overline{\omega}}(\text{ends}) = \Big[\iint \frac{1}{r^3} [\overline{k}\ \overline{r}]\ dxdy \Big] \Big|_{h_1}^{h_2}. \tag{62}$$

The individual components are

$$\Delta\omega_{xx}(\text{ends}) = 0\,, \qquad \Delta\omega_{xy}(\text{ends}) = 0\,, \qquad \Delta\omega_{xz}(\text{ends}) = 0\,, \tag{63}$$

$$\Delta\omega_{yx}(\text{ends}) = 0\,, \qquad \Delta\omega_{yy}(\text{ends}) = 0\,, \qquad \Delta\omega_{yz}(\text{ends}) = 0\,, \tag{64}$$

$$\Delta\omega_{zx}(\text{ends}) = \left[\iint \frac{x}{r^3}\,dxdy\right]\Bigg|_{h_1}^{h_2} = -\left[\int dy \int \frac{\partial}{\partial x}\left[\frac{1}{r}\right] dx\right]\Bigg|_{h_1}^{h_2}, \tag{65}$$

$$\Delta\omega_{zy}(\text{ends}) = \left[\iint \frac{y}{r^3}\,dxdy\right]\Bigg|_{h_1}^{h_2} = -\left[\int dx \int \frac{\partial}{\partial y}\left[\frac{1}{r}\right] dy\right]\Bigg|_{h_1}^{h_2}, \tag{66}$$

$$\Delta\omega_{zz}(\text{ends}) = \left[h\iint \frac{1}{r^3}\,dxdy\right]\Bigg|_{h_1}^{h_2}. \tag{67}$$

Equation (67) is the only contribution to ω_{zz} and will be obtained from the from the solid angle; hence it need not be evaluated here.

Comparing Eq.(65) with Eq.(45) and Eq.(49) shows that

$$\Delta\omega_{zx}(\text{ends}) = -\frac{S}{\sqrt{1+S^2}}\ln(\, r + \text{sgnb}\, r_c\,)\,\Bigg|_{q=q_1}^{q=q_2}\Bigg|_{h=h_1}^{h=h_2}. \tag{68}$$

Similarly, comparing Eq.(66) with Eq.(57) and Eq.(60) shows that

$$\Delta\omega_{zy}(\text{ends}) = \frac{1}{\sqrt{1+S^2}}\ln(\, r + \text{sgnb}\, r_c\,)\,\Bigg|_{q=q_1}^{q=q_2}\Bigg|_{h=h_1}^{h=h_2}. \tag{69}$$

This completes the evaluation of the contributions from the ends of the prism.

V. Summary

a) Composition of tensor.

$$\bar{\bar{\omega}} = \sum_{a,b,c,\cdots} \begin{bmatrix} xx & xy & xz \\ yx & yy & yz \\ 0 & 0 & 0 \end{bmatrix}_{\text{sides}} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ zx & zy & zz \end{bmatrix}_{\text{ends}}. \tag{70}$$

b) Symmetry of tensor.

$$\omega_{yx} = \omega_{xy}\,, \qquad \omega_{zx} = \omega_{xz}\,, \qquad \omega_{zy} = \omega_{yz}\,. \tag{71}$$

c) Irreducible components in order used in GFUN.

$$\omega(1) = \omega_{xx}\,, \qquad \omega(2) = \omega_{xy}\,, \qquad \omega(3) = \omega_{xz}\,,$$
$$\omega(4) = \omega_{yy}\,, \qquad \omega(5) = \omega_{yz}\,. \tag{72}$$

d) Solid Angle(trace of matrix).

$$\omega(6) = \omega_{zz} = \begin{bmatrix} -\omega_{xx} - \omega_{yy} + 4\pi & \text{P inside element} \\ -\omega_{xx} - \omega_{yy} & \text{P outside element} \end{bmatrix}. \tag{73}$$

e) Terms in TERMS

Let

$$\omega(i) = O(i) \Big|_{q=q_1}^{q=q_2} \Big|_{h=h_1}^{h=h_2}. \qquad (i=1,5) \tag{74}$$

And

$$T_1 = \operatorname{sgnb} \tan^{-1}\left[\frac{h\sqrt{1+S^2}\ r_c}{Cr} \right], \tag{75}$$

$$T_2 = \ln\left(r + h \right), \tag{76}$$

$$T_3 = \ln\left(r + \operatorname{sgnb} r_c \right), \tag{77}$$

$$G = \frac{1}{\sqrt{1+S^2}}. \tag{78}$$

Then

$$O(1) = -S^2 G^2 T_1 - S G^2 T_2\,, \tag{79}$$

$$O(2) = S G^2 T_1 - S^2 G^2 T_2\,, \tag{80}$$

$$O(3) = -S G T_3\,, \tag{81}$$

$$O(4) = -G^2 T_1 + S G^2 T_2\,, \tag{82}$$

$$O(5) = G T_3\,. \tag{83}$$

VI. Appendix A. Algebraic Equivalences.

All equivalences relate to points (x,y) on the line specified by

$$y = C + S\,x\,. \tag{A1}$$

From Eq.(23)

$$r^2 = (1+S^2)\left[\, x^2 + \frac{C^2+h^2}{1+S^2} + 2\frac{CS}{1+S^2} + \frac{C^2S^2}{(1+S^2)^2} - \frac{C^2S^2}{(1+S^2)^2} \,\right]$$

Or

$$= (1+S^2)\left[\left[\, x + \frac{CS}{1+S^2}\right]^2 + \frac{C^2+(1+S^2)h^2}{(1+S^2)^2}\right]. \tag{A2}$$

Concerning Eq.(28):

$$r_c = \sqrt{\;x^2 \;+\; y^2 \;-\; \frac{C^2}{1+S^2}}\,. \tag{A3}$$

Note that $x^2 + y^2$ is the square of the distance from the origin to a point on the line. Note also that $C^2/(1+S^2)$ is the square of the perpendicular distance to the line. Thus, geometrically

$$x^2 + y^2 \geq \frac{C^2}{1+S^2}\,. \tag{A4}$$

The utility of r_c becomes evident from the following observation:

$$(1+S^2)\, r_c^2 = (1+S^2)(x^2 + y^2) - (y - Sx)^2$$

$$= (x + Sy)^2. \tag{A5}$$

But

$$x + Sy = (1+S^2)\,x + CS = \frac{1}{S}\left[\, (1+S^2)\,y - C \,\right]. \tag{A6}$$

Hence, defining a and b for convenience,

$$b \equiv (1+S^2)\,x + CS\,, \qquad\qquad a \equiv (1+S^2)\,y - C\,. \tag{A7}$$

From Eq.(A5)

$$(1+S^2)\, r_c^2 = (x+Sy)^2 = b^2\,.$$

Or

$$b = \pm\sqrt{1+S^2}\; r_c\,, \qquad\qquad a = Sb\,. \tag{A8}$$

The ambiguous sign is to be resolved by computing the sign of b in Eq.(A7) or its equivalent the sign of $x + Sy$ in Eq.(A6). Thus

$$\mathrm{sgnb} \equiv \mathrm{sign}(b) = \mathrm{sign}(x+Sy). \tag{A9}$$

Hence

$$x + \frac{CS}{1+S^2} = \frac{r_c}{\sqrt{1+S^2}}\,\mathrm{sgnb}\,, \tag{A10}$$

and

$$y - \frac{C}{1+S^2} = \frac{S\,r_c}{\sqrt{1+S^2}}\,\mathrm{sgnb}\,. \tag{A11}$$

All slopes S that are either infinite or zero must be replaced by finite numbers such as 10^{+6} or 10^{-6} to avoid numerical difficulties. The new FORTRAN subroutine TERMS is available on request.

VII. References.

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Fig. 1. Prism Element

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