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AUTOMORPHISM FIXED POINTS AND EXCEPTIONAL MODULAR INVARIANTS

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ABSTRACT

Exceptional modular invariants for Kac-Moody algebras are obtained systematically by exploiting the relation with the fixed point conformal field theory.

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1. Introduction

The construction of modular invariants for Kac-Moody (KM) algebras is a problem interesting in itself as well as in its connections to 2-dimensional conformal field theories (CFT's). A complete classification of positive modular invariants only exists for the $\widehat{SU}(2)$ algebra at any level[1] and for $\widehat{SU}(N)$ at level one[2]. For other algebras considerable progress has been made recently and many positive invariants have been obtained[3,4,5,6,7,8,9,10,11]. Besides series of invariants that occur at generic level there are invariants that only occur at particular values of the level. This latter class includes invariants, such as the E_7-type for $\widehat{SU}(2)$, due to automorphisms of the fusion rules of the extended algebra[12,13,8]. In this note we point out how sporadic invariants of this type can be found systematically. We will focus on $\widehat{SU}(N)$ but similar considerations apply to other algebras.

A key feature of KM algebras is the existence of automorphisms σ that lead to a systematic construction of infinite series of modular invariants[3,4,5]. Primary states Λ transform among themselves under the action of σ. For certain values of the level there exist *fixed points*, i.e. states with $\sigma(\Lambda) = \Lambda$. In the following these fixed points will play an important rôle.

To explain our basic idea let us recall some known facts. As noticed in Ref. [6] the E_7 invariant of $\widehat{SU}(2)_{16}$ is part of a short sequence that includes the exceptional invariants of $\widehat{SU}(3)_9$ [12] and $\widehat{SU}(5)_5$ [6]. The common property of these N, k pairs is that the only fixed point has $(h - \frac{c}{24}) = 1$. Furthermore, in the three cases the exceptional invariant can be found by substracting

$$|\tilde{\chi}_0|^2 = |\chi_{orbit} - \chi_{fixed}|^2 \tag{1}$$

from the D-type invariant at the same level which includes terms $\cdots + |\chi_{orbit}|^2 + N|\chi_{fixed}|^2 + \cdots$. Here χ_{orbit} is a sum of characters of fields connected by the automorphism and having $(h - \frac{c}{24})$ =integer. $|\tilde{\chi}_0|^2$ is by itself an invariant, it is in fact a constant. For instance, for $\widehat{SU}(2)_{16}$ one can easily verify $\chi_2 + \chi_{14} - \chi_8 = 3$ (χ_l is the character of the $(l+1)$-dimensional representation).

The main lesson to learn from the above discussion is that $\tilde{\chi}_0$ can be regarded as the character of a trivial CFT. In this letter we will show that E_7-type invariants can be derived from D-type invariants by substracting combinations that can be understood as modular invariants of a CFT associated to the fixed points.

This note is organized as follows. In section 2 we review the properties of automorphisms of $\widehat{SU}(N)$, in section 3 we explain how to derive the sporadic invariants and in section 3 we present our final comments.

2. Automorphisms and fixed points

We consider the $\widehat{SU}(N)$ algebra at level k. The primary states are in 1-1 correspondence with heighest weights Λ

$$\Lambda = \sum_{i=1}^{N-1} n_i w_i \tag{2}$$

where w_i are the $SU(N)$ fundamental weights ($w_i \equiv w_{i+N}, w_N \equiv 0$) and $n_i \geq 0$ are the Dynkin labels. Unitarity imposes the constraint

$$\sum_{i=1}^{N-1} n_i \leq k \tag{3}$$

The number of primary fields is then

$$N_P = \frac{(k+N-1)!}{k!(N-1)!} \tag{4}$$

For future reference we introduce the N^{th}-ality of Λ defined as

$$t(\Lambda) = \sum_{i=1}^{N-1} i n_i \tag{5}$$

t is defined mod N.

The affine algebra has automorphisms that form a group isomorphic to Z_N with elements $\sigma^r, r = 1, \cdots, N$. σ^r acts on Λ as

$$\sigma^r(\Lambda) = k w_r + a^r(\Lambda) \tag{6}$$

where a is the Coxeter rotation belonging to the Weyl group whose action on the fundamental weights is given by

$$a^r(w_i) = w_{i+r} - w_r \tag{7}$$

σ^r has order $N(r)$ where $N(r)$ is the least integer such that $rN(r) = 0 \bmod N$. It is actually enough to consider the cases when r divides N. From now on we take $N(r) = N/r =$ integer.

We now study the fixed points of σ^r. From (6) and (7) we readily find that $\sigma^r(\Lambda) = \Lambda$ implies

$$\begin{gathered} n_i = n_{i+r} \\ k - \sum_{i=1}^{N-1} n_i = n_r \end{gathered} \tag{8}$$

which in turn give

$$n_1 + \cdots + n_r = \frac{rk}{N} \equiv \tilde{k} \tag{9}$$

This shows that fixed points only exist when $k = 0 \bmod N/r$. Moreover, when this is the case the number of fixed points is

$$N_F = \frac{(\tilde{k} + r - 1)!}{\tilde{k}!(r-1)!} \tag{10}$$

Comparing with (4) we see that this is exactly the number of primary fields of $\widehat{SU}(r)$ at level $\tilde{k}$. Thus, to each fixed point Λ_F of σ^r we can associate a heighest weight λ of $SU(r)$. More precisely we have (in their respective Dynkin basis)

$$\lambda = (n_1, \cdots, n_{r-1}) \leftrightarrow \Lambda_F = (n_1, n_2, \cdots, n_{r-1}, n_r, n_1, n_2, \cdots) \tag{11}$$

where $n_1 + \cdots n_{r-1} \leq \tilde{k}$ and n_r is given in (9) .

The above correspondence was first derived in Ref. [6] where a relation between the conformal dimensions of Λ_F and λ was also established. This relation reads

$$\Delta(\Lambda_F, \lambda) = \big(h - \frac{c}{24}\big)_{\Lambda_F} - \big(h - \frac{c}{24}\big)_{\lambda} = \frac{\tilde{k}r}{24}[(N/r)^2 - 1] \tag{12}$$

Notice that $\Delta = m/12$ with m integer. This property truly identifies $\widehat{SU}(r)_{\tilde{k}}$ as the fixed point CFT as explained in Ref. [6]. The proof of (12) follows from (11) together with the definitions

$$h(W) = \frac{W \cdot (W + 2\rho)}{2(k+g)} \quad , \quad c = \frac{k\dim G}{k+g} \tag{13}$$

where ρ is the sum of the fundamental weights of G and the Coxeter number g is N for $SU(N)$. When $r = 1$, $\lambda = 0$, $\Lambda_F = \frac{k}{N}\rho$ and (12) is easily found using the "strange" formula $\rho^2 = g\dim G/12$.

To end this section we discuss briefly a particular infinite series of modular invariants associated to σ^r found by Bernard [3]. A D-type invariant can be associated to σ^r when $k = 0 \bmod N/r$ and both N, r are either odd or even. This invariant can be written as

$$[D^r(N,k)]_{\Lambda',\Lambda} = \begin{cases} \sum_{p=1}^{N/r} \delta_{\Lambda',\sigma^{rp}(\Lambda)} & \text{if } t(\Lambda) = 0 \bmod N/r \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

For N even and $k = 0 \bmod 2N$ there is an invariant associated to σ given by $D^1 \equiv D$. These D^r invariants also exist in less generic cases. For instance, for $N = 8$, any k, there is a D invariant and for $N = 16$, $k = 0 \bmod 2$ there is a D^4 invariant. Other series can be constructed but they are not relevant for our purposes. D^r is the "integer spin" invariant generated by the simple current $J_r = kw_r$ with spin $h = kr(N-r)/2N$ [5]. D^r can be written as a sum of squares, one for each orbit of σ^r with $t = 0 \bmod N/r$. Most orbits are of length N/r but orbits including fixed points are shorter. D^r can be understood as a diagonal invariant with respect to the original algebra extended by J_r [12,5].

3. Sporadic Invariants

In the previous section we learned that the fixed points of the automorphism σ^r of $\widehat{SU}(N)$, level k, are naturally associated with $\widehat{SU}(r)$, level $\tilde{k} = rk/N$. We are then led to expect a relation between characters of the fixed points χ_F and characters $\tilde{\chi}_\lambda$ of the associated weights λ. From the generic character expansion

$$\chi_W = q^{h(W) - \frac{c}{24}} \sum_{n=0}^{\infty} d_n q^n \tag{15}$$

we see that such a relation can arise if the shift $\Delta(\Lambda_F, \lambda)$ in eq. (12) is an integer. It is also necessary that there exists another $\widehat{SU}(N)_k$ field Λ_0 (belonging to some orbit) such that $\Delta(\Lambda_0, \lambda) = 0$ and $h(\Lambda_F) - h(\Lambda_0)$ =integer. We have verified that these conditions are satisfied when $\Delta(\Lambda_F, \lambda) = 1$ and that it is only in this case when a modular invariant of $\widehat{SU}(r)_{\tilde{k}}$ leads to a new *positive* invariant of $\widehat{SU}(N)_k$ when substracted from a D-type invariant. The empirical reason for this is that in the expansion of $\tilde{\chi}_\lambda$ when $\Delta(\Lambda_F, \lambda) > 1$ χ_F will appear with opposite relative signs with respect to characters of other orbits.

It is also possible that the relation between χ_F and $\tilde{\chi}_\lambda$ involves multiplying by an appropriate rational function of the absolute modular invariant $j(q)$. For instance, for a single fixed point it is conceivable that $(\chi_{orbit} - \chi_F) = j^{n/3}P(j)$ where P is a polynomial and $n > 0$ is an integer. The reason for this is that a CFT with a single (identity) field has $c = 0$ mod 8 [14] and its character is necessarily of the form $j^{n/3}P(j)$ (recall that $j^{1/3} = q^{-1/3}(1 + 248q + 4124q^2 + \cdots)$ is the famous $\widehat{E_8}$ level 1 character). Thus, in principle we can also allow for $\Delta = n/3$ mod integer. We have checked that for $\widehat{SU}(N)$ this broader requirement does not lead to new invariants.

The condition $\Delta(\Lambda_F, \lambda) = 1$ (together with $\tilde{k}$ = integer) is only satisfied for the particular values of N, r, k shown in Table 1. We conjecture that only for those $\widehat{SU}(N)_k$ there exist E_7-type invariants, i.e. due to automorphisms of the fusion rules of the algebra extended by a simple current. This conjecture is partially proven since in all cases the invariants are known. Those for $N = 2, 3, 5$ were mentioned before and those for $N = 4, 8, 9, 16$ were found in Ref. [8] where a numerical search for invariants of this type failed for larger N. The new element contributed here is a clear criterium for when these invariants exist. Notice that Table 1 is "dual" under $N \leftrightarrow k$ as it should since invariants of $\widehat{SU}(N)_k$ and $\widehat{SU}(k)_N$ are naturally related[9,10,15].

Let us now show how modular invariants of the fixed point CFT enter into the construction ot the sporadic invariants. In the introduction we already remarked that for $N = 2, 3, 5$, $(\chi_{orbit} - \chi_F) = \tilde{\chi}_0$ where $\tilde{\chi}_0$ =constant is the character of the trivial CFT associated to the single fixed point. To see how this generalizes to the other N we take the $\widehat{SU}(4)_8$ example for concreteness. We find

$$
\begin{aligned}
\chi_{101} + \chi_{610} + \chi_{161} + \chi_{016} - \chi_{040} - \chi_{404} &= 15(\tilde{\chi}_0 + \tilde{\chi}_4) \\
\chi_{012} + \chi_{501} + \chi_{250} + \chi_{125} - \chi_{222} &= 15\tilde{\chi}_2 \qquad (16) \\
\chi_{210} + \chi_{521} + \chi_{052} + \chi_{105} - \chi_{222} &= 15\tilde{\chi}_2
\end{aligned}
$$

Here characters of $\widehat{SU}(N)_k$ are represented as $\chi_{n_1 n_2,\ldots}$ and those of $\widehat{SU}(r)_{\tilde{k}}$ as $\tilde{\chi}_{m_1 m_2,\ldots}$ where n_i, m_i are the respective Dynkin labels. Notice that $(2,2,2)$ is a fixed point of order 4 whereas $(0,4,0)$ and $(4,0,4)$ are fixed points of order 2.

To establish (16) we first determine the fixed points Λ_F and their associated λ weights. Second we find the states with $h(\Lambda_F) - h(\Lambda)$ =integer. Notice that these states must organize into orbits of σ^r since $h(\Lambda) - h(\sigma(\Lambda))$ =integer. At this point we can guess how the χ_{orbit}, χ_F will arrange into $\tilde{\chi}_\lambda$. Next the guess is checked by computing the first terms in the expansion of the characters. To this purpose we use the Weyl-Kac formula[16] that for the $\widehat{SU}(N)$ specialized characters reads

$$
\chi_W = \frac{1}{\eta^{N^2-1}} \sum_{y \in M} \left[\prod_{\alpha+} \frac{((k+N)y + W + \rho)\cdot\alpha}{\rho\cdot\alpha} \right] q^{\frac{k+N}{2}[y + \frac{W+\rho}{k+N}]^2} \qquad (17)
$$

where η is the Dedekind function, M is the root lattice and the product is over the positive roots.

Now, we know that for $\widehat{SU}(2)_4$ there is a D-invariant

$$
|\tilde{\chi}_0 + \tilde{\chi}_4|^2 + |\tilde{\chi}_2|^2 + |\tilde{\chi}_2|^2 \qquad (18)
$$

Writing this invariant in terms of the $\widehat{SU}(4)$ characters as indicated in (16) (multiplying by an irrelevant factor) and substracting from the $D(4,8)$ invariant derived from (14) gives the exceptional invariant

$$
\begin{aligned}
\mathcal{E}(4,8) = &|\chi_{000}+\chi_{800}+\chi_{080}+\chi_{008}|^2+|\chi_{400}+\chi_{440}+\chi_{044}+\chi_{004}|^2 \\
&+|\chi_{020}+\chi_{602}+\chi_{060}+\chi_{206}|^2+|\chi_{024}+\chi_{202}+\chi_{420}+\chi_{242}|^2 \\
&+|\chi_{032}+\chi_{303}+\chi_{230}+\chi_{323}|^2+|\chi_{311}+\chi_{331}+\chi_{133}+\chi_{113}|^2 \\
&+|\chi_{121}+\chi_{412}+\chi_{141}+\chi_{214}|^2+|\chi_{040}+\chi_{404}|^2+2|\chi_{222}|^2 \\
&+\big[(\chi_{012}+\chi_{501}+\chi_{250}+\chi_{125})\chi_{222}^*+(\chi_{210}+\chi_{521}+\chi_{052}+\chi_{105})\chi_{222}^* \\
&+(\chi_{101}+\chi_{610}+\chi_{161}+\chi_{016})(\chi_{040}+\chi_{404})^*+c.c.\big]
\end{aligned}
\tag{19}
$$

$\mathcal{E}(4,8)$ clearly arises from acting with an automorphism of the fusion rules of the extended algebra whose (diagonal) invariant is $D(4,8)$. This algebra has primary fields that can be denoted by an orbit representative, i.e. $(0,0,0)$, $(4,0,0)$, $(0,2,0)$, $(0,2,4)$, $(0,3,2)$, $(3,1,1)$, $(1,2,1)$, $(0,1,2)$, $(2,1,0)$, $(1,0,1)$, $(0,4,0)_{i=1,2}$, $(2,2,2)_{I=1,\cdots,4}$. Comparing $\mathcal{E}(4,8)$ with $D(4,8)$ we see that one is related to the other by the transformation that takes $(0,4,0)_1 \leftrightarrow (1,0,1)$, $(2,2,2)_1 \leftrightarrow (0,1,2)$, $(2,2,2)_2 \leftrightarrow (2,1,0)$, and leaves all the other fields fixed.

The results for other (N,k) can be summarized as follows:

$$
\begin{aligned}
\mathcal{E}(8,4) &= D^2(8,4) - 63^2\tilde{D}^2(4,2) \\
\mathcal{E}(9,3) &= D^3(9,3) - 80^2\tilde{A}(3,1) \\
\mathcal{E}(16,2) &= D^4(16,2) - 255^2\tilde{D}^4(8,1)
\end{aligned}
\tag{20}
$$

where D invariants are computed using (14) and $A = D^N$. The tilde invariants of $\widehat{SU}(r)_{\tilde{k}}$ are written in terms of characters of $\widehat{SU}(N)_k$ as shown below:

$$
\begin{aligned}
63^2\tilde{D}^2(4,2) = &|(\chi_{1000001}+\cdots)-\chi_{0002000}-\chi_{0200020}|^2 \\
&+|(\chi_{2000010}+\cdots)-\chi_{1010101}|^2+|(\chi_{0100002}+\cdots)-\chi_{1010101}|^2 \\
&+|(\chi_{1010000}+\cdots)-\chi_{0101010}|^2+|(\chi_{0000101}+\cdots)-\chi_{0101010}|^2 \\
&+|(\chi_{0000012}+\cdots)-\chi_{0020002}-\chi_{2000200}|^2 \\
80^2\tilde{A}(3,1) = &|(\chi_{1,8}+\cdots)-\chi_{3,6}|^2+|(\chi_{1,2}+\cdots)-\chi_{1,4,7}|^2+|(\chi_{7,8}+\cdots)-\chi_{2,5,8}|^2 \\
255^2\tilde{D}^4(8,1) = &|(\chi_{3,5}+\cdots)-\chi_{4,12}-\chi_{8}|^2+|(\chi_{1,3}+\cdots)-\chi_{6,14}-\chi_{2,10}|^2
\end{aligned}
\tag{21}
$$

where dots stand for characters of the remaining fields in the orbit. For $\widehat{SU}(9)$ and $\widehat{SU}(16)$ we have adopted the notation $\chi_{a,\cdots,b}$ meaning $n_a = \cdots = n_b = 1$ and all other $n_i = 0$. All the resulting $\mathcal{E}$ invariants can be obtained acting with an automorphism of the fusion rules of the extended algebra[8].

4. Conclusions

In this note we have contributed towards a classification of positive modular invariants of $\widehat{SU}(N)_k$ by determining the particular values of N, k for which this algebra possesses modular invariants of the E_7 type. This was achieved by using properties of the fixed point CFT. In particular, it was argued that these invariants appear when the shift $\Delta(\Lambda_F, \lambda) = 1$. An open interesting problem is to find the deeper significance of this relation.

Our arguments do apply to other algebras. For instance, in Ref. [6] the fixed point CFT of $\widehat{SO}(2n)$ together with $\Delta(\Lambda_F, \lambda)$ were found. From those results we can infer that $\widehat{SO}(2n)_8$ and $\widehat{SO}(4n)_k$ with $nk = 16$, $k \geq 2$ have exceptional invariants. Remarkably, these invariants were first conjectured (based mainly on a numerical analysis) and then found by Verstegen[8].

The fixed point CFTs of $\widehat{E_6}$, $\widehat{E_7}$, $\widehat{Sp}(4n+2)$, $\widehat{Sp}(4n)$, k odd and $\widehat{SO}(2n+1)$, k odd, have also been determined in Ref. [6]. The results for Δ indicate that these algebras do not have E_7-type invariants except possibly when there is a single fixed point. This leaves the possibilities $\widehat{Sp}(2)_{16} \equiv \widehat{SU}(2)_{16}$ and $\widehat{Sp}(32)_1$. This last case is rather interesting as it has $\Delta = 5/3$ and we find $(\chi_2 + \chi_{14} - \chi_8) = 495 j^{1/3}$ (χ_a is the character of the representation with $n_a = 1$ and all other $n_i = 0$). Combining this result with the integer spin invariant of $\widehat{Sp}(32)_1$ gives an exceptional invariant dual to that of $\widehat{SU}(2)_{16}$ as explained in Ref. [8].

For $\widehat{SO}(2n+1)$, k even, and $\widehat{Sp}(4n)$, k even, the fixed point CFTs have only been identified in a few cases. For instance, for $\widehat{SO}(5)_k \equiv \widehat{Sp}(4)_k$ the fixed point CFT turns out to be[6] the non-unitary minimal model with $p = 2, p' = k+3$. When $k = 8$ we find $\Delta = 1$ in agreement with the fact that an exceptional $\widehat{SO}(5)_8$ invariant has been found by Verstegen[8]. In fact, in Ref. [8] exceptional invariants of the type discussed in this note were also found for $\widehat{SO}(2n+1)_8$, $\widehat{Sp}(8)_4$ and $\widehat{Sp}(16)_2$. In these cases our arguments can be reversed to instead determine the associated fixed point CFTs.

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N	k	r
2	16	1
3	9	1
4	8	2
5	5	1
8	4	4
9	3	3
16	2	8

TABLE 1. Values of $\widehat{SU}(N)_k$ and order of the automorphism for which E_7-type invariants occur.