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BRST COHOMOLOGY AND PHYSICAL STATES IN 2D SUPERGRAVITY COUPLED TO $\hat{c} \leq 1$ MATTER

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Abstract

We study the BRST cohomology for two-dimensional supergravity coupled to $\hat{c} \leq 1$ superconformal matter in the conformal gauge. The super-Liouville and superconformal matters are represented by free scalar fields ϕ^L and ϕ^M and fermions ψ^L and ψ^M , respectively, with suitable background charges, and these are coupled in such a way that the BRST charge is nilpotent. The physical states of the full theory are determined for NS and R sectors. It is shown that there are extra states with ghost number $N_{FP} = 0, \pm 1$ for discrete momenta other than the degree of freedom corresponding to the "center of mass", and that these are closely related to the "null states" in the minimal models with $\hat{c} < 1$.



1 Introduction

There has recently been significant progress in attempts to find nonperturbative treatment of two-dimensional gravity and string theory. In the matrix models, conformal field theories with central charges $c \leq 1$ are successfully coupled to quantum gravity [1, 2, 3] and partition functions as well as correlation functions have been computed [4]. In particular, it has been found that there are infinite number of extra states at discrete values of momenta other than the degree of freedom corresponding to the “center of mass” or “tachyon”, but the nature and the role of these states in the theory have not been fully understood yet.

To get more insight into the theory, it is clearly necessary to understand these results from the viewpoint of the usual continuum approach to the two-dimensional gravity: the Liouville theory [5]. Most of the results in the matrix models have been confirmed in the recent study of the Liouville theory [5, 6]. In particular, several groups have computed correlation functions and partition functions to find again discrete states [7, 8, 9, 10, 11]. Some attempts to clarify the properties of these states have been made in [12, 13, 14, 15].

A first step toward full understanding of these states has recently been taken in the BRST approach [16, 17, 18, 19]. In this approach, physical states are characterized as nontrivial cohomology classes of the BRST charge. Indeed, recent complete analysis of the BRST cohomology has revealed that there are nontrivial physical states with ghost numbers $N_{FP} = 0, \pm 1$ at special values of momenta, corresponding to the extra states found in the matrix models [16, 17].

On the other hand, little is known for the supersymmetric case except for Marinari and Parisi’s proposal for supersymmetric matrix models [20]. The two-dimensional supergravity coupled to $\hat{c} (\equiv \frac{2}{3}c) \leq 1$ superconformal matter reduces to coupled super-Liouville theory and it correctly reproduces the scaling dimensions [21]. However, no correlation functions have been computed and the physical spectrum has not been clarified. In view of the potential significance of superstring theory, it is very important to understand the physical spectrum in two-dimensional supergravity coupled to $\hat{c} \leq 1$ superconformal

matter.

The purpose of this paper is to compute the BRST cohomology and identify the physical spectrum for such a system. Our computation of the BRST cohomology is quite analogous to the bosonic case [16, 17]. One first decomposes the BRST charge [22, 23] with respect to the ghost zero modes. These are further decomposed according to a grading of the Fock space, and we sort out the nontrivial cohomology classes of the operator with lowest degree. In the critical superstring [22] as well as in the bosonic Liouville theory [16, 17], there is a one-to-one correspondence between this nontrivial cohomology and the cohomology of the total BRST charge. It turns out that this is not true in general in the super-Liouville theory. We have carefully examined which of these nontrivial states can be promoted to the nontrivial cohomology classes of the full BRST charge for the Neveu-Schwarz (NS) and Ramond (R) sectors. In this way, we show that there are indeed nontrivial physical states for discrete values of momenta, corresponding precisely to the “null states” in the minimal superconformal models [24, 25]. The same results are also obtained by using the decoupling mechanism based on the quartet structure with respect to the BRST charge.

In sect. 2, we start by reviewing briefly the super-Liouville theory coupled to $\hat{c} \leq 1$ matter. We use the free field realization with a background charge for the matter sector [25, 26, 27]. We then study the BRST cohomology for the NS sector in sect. 3 and for R sector in sect. 4, and identify the extra physical discrete states. Sect. 5 is devoted to discussions. In particular, we point out the close relationship of these extra states to the “null states” in the $\hat{c} < 1$ minimal models.

2 Super-Liouville theory coupled to $\hat{c} \leq 1$ matter

In this section, we briefly summarize the super-Liouville theory coupled to the $\hat{c} \leq 1$ matter theory for completeness. This also serves to establish our notations and conventions.

In the conformal gauge, the matter and super-Liouville theories can be realized by

free superfields Φ^M and Φ^L which contains scalar ϕ and fermionic fields ψ

$$\Phi(\theta, z) = \phi(z) - i\theta\psi(z) \quad (2.1)$$

with the two-point functions

$$\langle \phi(z)\phi(w) \rangle = -\ln(z-w) \quad , \quad \langle \psi(z)\psi(w) \rangle = \frac{1}{z-w}. \quad (2.2)$$

For the moment we will suppress the superscripts and describe a free superfield realization, which is applicable to both the matter and gravity sectors.

The super-stress tensor is given by

$$\begin{aligned} T(\theta, z) &= -\frac{1}{2}D\Phi D^2\Phi - i\lambda D^3\Phi \\ &\equiv \frac{1}{2}T_F + \theta T_B \end{aligned} \quad (2.3)$$

where $D \equiv \partial_\theta + \theta\partial$ is the covariant derivative. In terms of the component fields defined in eq. (2.1), the stress-energy tensor T_B and super-current T_F are expressed as

$$\begin{aligned} T_B &= -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\psi\partial\psi - i\lambda\partial^2\phi, \\ T_F &= i\psi\partial\phi - 2\lambda\partial\psi \end{aligned} \quad (2.4)$$

which satisfy the $N = 1$ superconformal operator product with the central charge $c = 1 + \frac{1}{2} - 12\lambda^2$ or $\hat{c} = 1 - 8\lambda^2$.

The mode expansions are defined by

$$\phi(z) = q - i(p - \lambda)\ln z + i \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n}, \quad (2.5a)$$

$$\psi(z) = \sum_n \psi_n z^{-n-\frac{1}{2}} \quad (2.5b)$$

with the commutation relations

$$\begin{aligned} [\alpha_n, \alpha_m] &= n\delta_{n+m,0}, \quad [q, p] = i \quad , \\ \{\psi_n, \psi_m\} &= \delta_{n+m,0} \quad . \end{aligned} \quad (2.6)$$

In eq. (2.5b), the sum over n is to be taken over half-odd-integers for the NS sector and integers for the R sector.

The super-Virasoro generators are defined to be the Laurent coefficients of the super-stress tensor. In terms of the mode operators, they are given by

$$\begin{aligned} L_n &= \frac{1}{2} \sum_m : \alpha_m \alpha_{n-m} : + \frac{1}{4} \sum_m (2m - n) : \psi_{n-m} \psi_m : + (n+1) \lambda \alpha_n, \\ G_n &= \sum_m \psi_{n-m} \alpha_m + (2n+1) \lambda \psi_n \end{aligned} \quad (2.7)$$

where $\alpha_0 \equiv p - \lambda$. Note that the subscript n to G is half-odd-integer or integer depending whether it is for the NS or R sector.

In the present case of super-Liouville theory with the background charge λ^L coupled to the matter with λ^M , we have two sets of the above system. The total BRST charge is then

$$\begin{aligned} Q_B &= \sum_n c_{-n} (L_n^M + L_n^L) - \frac{1}{2} \sum_n \gamma_{-n} (G_n^M + G_n^L) - \frac{1}{2} \sum_{n,m} (n-m) : c_{-n} c_{-m} b_{n+m} : \\ &\quad + \sum_{n,m} \left(\frac{3}{2} n + m \right) : c_{-n} \gamma_{n+m} \beta_{-m} : - \frac{1}{4} \sum_{n,m} \gamma_n \gamma_m b_{-n-m} \end{aligned} \quad (2.8)$$

where the sum is to be taken such that the subscripts to G, γ and β are half-odd-integers (integers) for NS (R) case and others are integers. The commutation relations for the ghosts are

$$\{c_n, b_m\} = [\gamma_n, \beta_m] = \delta_{n+m,0}. \quad (2.9)$$

The central charges for the matter and super-Liouville systems are given by $\hat{c}^M = 1 - 8(\lambda^M)^2$ and $\hat{c}^L = 1 - 8(\lambda^L)^2$, respectively. Requiring that the total central charge add up to zero or the BRST charge be nilpotent give $\hat{c}^M + \hat{c}^L - 10 = 0$ or

$$(\lambda^M)^2 + (\lambda^L)^2 = -1. \quad (2.10)$$

Note that the conditions $\hat{c}^M \leq 1$ and (2.10) mean that λ^M is real whereas λ^L is pure imaginary.

The BRST charge can be decomposed with respect to the ghost zero modes. For the NS sector

$$Q_B = c_0 L_0 - b_0 M_{NS} + d_{NS} \quad (2.11)$$

where¹

$$\begin{aligned}
L_0 &= L_0^M + L_0^L + L_0^G, \\
M_{NS} &= \sum_{n \neq 0} n c_{-n} c_n + \frac{1}{4} \sum_r \gamma_{-r} \gamma_r, \\
d_{NS} &= \sum_{n \neq 0} c_{-n} (L_n^M + L_n^L) - \frac{1}{2} \sum_{nm(n+m) \neq 0} (m-n) : c_{-n} c_{-n} b_{m+n} : \\
&\quad - \frac{1}{2} \sum_r \gamma_{-r} (G_r^M + G_r^L) + \sum_{\substack{n \neq 0 \\ m}} \left(\frac{3}{2}n + r \right) : c_{-n} \gamma_{n+r} \beta_{-r} : - \frac{1}{4} \sum_{r+s \neq 0} \gamma_r \gamma_s b_{-r-s}.
\end{aligned} \tag{2.12}$$

The nilpotency of Q_B is equivalent to the following set of identities:

$$d_{NS}^2 = M_{NS} L_0, \quad [d_{NS}, L_0] = [d_{NS}, M_{NS}] = [L_0, M_{NS}] = 0. \tag{2.13}$$

For our purpose, it is convenient to define a set of generalized momenta

$$P^\pm(n) = \frac{1}{\sqrt{2}} [(p^M + n\lambda^M) \pm i(p^L + n\lambda^L)]. \tag{2.14}$$

In particular, these give the ‘‘lightcone-like’’ momenta for $n = 0$

$$p^\pm \equiv P^\pm(0) = \frac{1}{\sqrt{2}} (p^M \pm ip^L). \tag{2.15}$$

We also define other lightcone-like variables by

$$\begin{aligned}
q^\pm &= \frac{1}{\sqrt{2}} (q^M \pm iq^L), \quad \alpha_n^\pm = \frac{1}{\sqrt{2}} (\alpha_n^M \pm i\alpha_n^L), \\
\psi_r^\pm &= \frac{1}{\sqrt{2}} (\psi_r^M \pm i\psi_r^L),
\end{aligned} \tag{2.16}$$

which satisfy the commutation relations

$$[q^\pm, p^\mp] = i, \quad [\alpha_m^\pm, \alpha_n^\mp] = m\delta_{n+m,0}, \quad \{\psi_r^\pm, \psi_s^\mp\} = \delta_{r+s,0}. \tag{2.17}$$

Using these variables, the operators in eq. (2.12) are cast into the form

$$L_0 = p^+ p^- + \sum_{n \neq 0} : \alpha_{-n}^+ \alpha_n^- : + \sum_r r : \psi_{-r}^+ \psi_r^- : + \sum_{n \neq 0} n : c_{-n} b_n : + \sum_r r : \beta_{-r} \gamma_r :$$

¹In what follows, it is understood that n, m, \dots take integer values whereas r, s, \dots take half-odd-integers unless otherwise specified.

$$= p^+ p^- + \hat{N} \quad (2.18a)$$

$$d_{NS} = \sum_{n \neq 0} c_{-n} [P^+(n) \alpha_n^- + P^-(n) \alpha_n^+] + \sum_{\substack{n, m \neq 0 \\ n+m \neq 0}} : c_{-n} [\alpha_{-m}^+ \alpha_{m+n}^- + \frac{1}{2}(m-n) c_{-m} b_{m+n}] : \\ - \frac{1}{2} \sum_{n, r} (2r+n) : c_{-n} \psi_{n+r}^+ \psi_{-r}^- : - \frac{1}{2} \sum_r \gamma_{-r} [P^+(2r) \psi_r^- + P^-(2r) \psi_r^+] \\ - \frac{1}{2} \sum_{\substack{n \neq 0 \\ r \neq 0}} \gamma_{-r} (\psi_{r-n}^+ \alpha_n^- + \psi_{r-n}^- \alpha_n^+) + \sum_{\substack{n \neq 0 \\ r \neq 0}} (\frac{3}{2}n+r) : c_{-n} \gamma_{n+r} \beta_{-r} : - \frac{1}{4} \sum_{r+s \neq 0} \gamma_r \gamma_s b_{-r-s} \quad (2.18b)$$

In writing down eq. (2.18a), we have subtracted the intercept $\frac{1}{2}$ which appears in rewriting the zero mode part of L_0 in terms of p^+ and p^- using (2.10). This is necessary in order to make the BRST charge (2.8) nilpotent [22].

Similarly the BRST charge for the R sector is decomposed as

$$Q_B = c_0 L_0 - b_0 M_R - \frac{1}{2} \gamma_0 F + 2\beta_0 K + d_R - \frac{1}{4} b_0 \gamma_0^2 \quad (2.19)$$

where L_0 is the same as (2.18a) with all the sum over integers and

$$M_R = \sum_{n \neq 0} (n c_{-n} c_n + \frac{1}{4} \gamma_{-n} \gamma_n) \\ F = p^+ \psi_0^- + p^- \psi_0^+ + \sum_{n \neq 0} (\psi_{-n}^+ \alpha_n^- - \psi_{-n}^- \alpha_n^+ - n c_{-n} \beta_n + \gamma_n b_{-n}) \\ K = \frac{3}{4} \sum_{n \neq 0} n c_{-n} \gamma_n \\ d_R = \sum_{n \neq 0} c_{-n} [P^+(n) \alpha_n^- + P^-(n) \alpha_n^+] + \sum_{\substack{n, m \neq 0 \\ n+m \neq 0}} : c_{-n} [\alpha_{-m}^+ \alpha_{m+n}^- \\ + \frac{1}{2}(m-n) c_{-m} b_{m+n} - \frac{1}{2}(2m+n) \psi_{n+m}^+ \psi_{-m}^-] : \\ - \frac{1}{2} \sum_{n \neq 0} \gamma_{-n} [P^+(2n) \psi_n^- + P^-(2n) \psi_n^+] - \frac{1}{2} \sum_{n, m \neq 0} \gamma_{-m} (\psi_{m-n}^+ \alpha_n^- + \psi_{m-n}^- \alpha_n^+) \\ + \sum_{\substack{n, m \neq 0 \\ n+m \neq 0}} [(\frac{3}{2}n+m) : c_{-n} \gamma_{n+m} \beta_{-m} : - \frac{1}{4} \gamma_n \gamma_m b_{-n-m}] \quad (2.20)$$

In the R sector, there is no subtraction in L_0 [22]. Instead, the $\frac{1}{2}$ coming from $p^+ p^-$ is here absorbed in the normal ordering of the zero modes, as discussed in ref. [23]. The nilpotency of Q_B is rewritten as

$$L_0 = F^2, \quad d_R^2 = M_R L_0 + K F, \quad 2K = [M_R, F]$$

$$\begin{aligned}
K^2 &= [L_0, M_R] = [L_0, F] = [L_0, K] = [L_0, d_R] \\
&= [M_R, K] = [M_R, d_R] = \{F, K\} = \{F, d_R\} = \{K, d_R\} = 0
\end{aligned} \tag{2.21}$$

The Hilbert space \mathcal{H} of the full theory is the direct sum

$$\mathcal{H} = \oplus_{p^M, p^L} \left(\mathcal{H}_{(p^M)}^{(M)} \otimes \mathcal{H}_{(p^L)}^{(L)} \otimes \mathcal{H}^{(G)} \right) \tag{2.22}$$

where $\mathcal{H}_{(p^M)}^{(M)}$ ($\mathcal{H}_{(p^L)}^{(L)}$) is the Fock space of matter (Liouville) oscillators acting on a Fock vacuum with momentum p^M (p^L) and $\mathcal{H}^{(G)}$ is the ghost Hilbert space.

The physical state conditions in both sectors are given by

$$Q_B | \text{phys} \rangle = 0. \tag{2.23}$$

Since in both cases $L_0 = \{b_0, Q_B\}$, these physical states satisfy

$$L_0 | \text{phys} \rangle = Q_B b_0 | \text{phys} \rangle. \tag{2.24}$$

Hence, any physical states are BRST-exact unless they satisfy the on-shell condition $L_0 = 0$.

It is convenient to reduce the zero eigenspace of L_0 by restricting to the states annihilated by b_0 (and also by β_0 in the R sector). In this space the physical state conditions (2.23) reduce to

$$L_0 | \text{phys} \rangle = b_0 | \text{phys} \rangle = d_{NS} | \text{phys} \rangle = 0 \tag{2.25}$$

for the NS sector and to

$$F | \text{phys} \rangle = b_0 | \text{phys} \rangle = \beta_0 | \text{phys} \rangle = d_R | \text{phys} \rangle = 0 \tag{2.26}$$

for the R sector. Note that the condition $L_0 | \text{phys} \rangle = 0$ for the R sector is satisfied due to the relation $L_0 = F^2$. Notice also $d^2 = 0$ when acting on this space because of the relations (2.13) and (2.21).

3 Physical states in the NS sector

In this section, we discuss the relative cohomology (2.25) for the NS sector and identify physical states.

3.1 Cohomology of d_0

From the first condition $L_0 | \text{phys} \rangle = 0$ in (2.25) and (2.18a), we see that this space is nontrivial only if $p^+ p^-$ takes a non-positive half-integer or integer value. For $p^+ p^- = 0$, there is a unique state $| p^M, p^L \rangle$.

In order to examine the cohomology of d_{NS} , we introduce the degree for the oscillators [18, 17] as

$$\begin{aligned} \text{deg}(\alpha_n^+, \psi_n^+, c_n, \gamma_r) &= +1 \\ \text{deg}(\alpha_n^-, \psi_n^-, b_n, \beta_r) &= -1 \end{aligned} \quad (3.1)$$

and define the degree of $| p^M, p^L \rangle$ to be zero. The cohomology operator d_{NS} is then decomposed into components with definite degrees:

$$d_{NS} = d_0 + d_1 + d_2. \quad (3.2)$$

Here

$$d_0 = \sum_{n \neq 0} P^+(n) c_{-n} \alpha_n^- - \frac{1}{2} \sum_r P^+(2r) \gamma_{-r} \psi_r^-, \quad (3.3a)$$

$$\begin{aligned} d_1 = \sum_{nm(n+m) \neq 0} : c_{-n} \left[\alpha_{-m}^+ \alpha_{m+n}^- + \frac{1}{2}(m-n) c_{-m} b_{m+n} \right] : &- \frac{1}{2} \sum_{n \neq 0} \gamma_{-r} (\psi_{r-n}^+ \alpha_n^- + \psi_{r-n}^- \alpha_n^+), \\ -\frac{1}{2} \sum_{n \neq 0} (2r+n) : c_{-n} \psi_{n+r}^+ \psi_{-r}^- : &+ \sum_{n \neq 0} \left(\frac{3}{2}n + r \right) : c_{-n} \gamma_{n+r} \beta_{-r} : - \frac{1}{4} \sum_{r+s \neq 0} \gamma_r \gamma_s b_{-r-s} \end{aligned} \quad (3.3b)$$

$$d_2 = \sum_{n \neq 0} P^-(n) c_{-n} \alpha_n^+ - \frac{1}{2} \sum_r P^-(2r) \gamma_{-r} \psi_r^+ \quad (3.3c)$$

satisfy in the on-shell subspace

$$d_0^2 = d_2^2 = 0, \quad \{d_0, d_1\} = \{d_1, d_2\} = 0, \quad d_1^2 + \{d_0, d_2\} = 0. \quad (3.4)$$

Note that the argument in $P^+(2r)$ in the second term in (3.3a) is an odd integer.

Our strategy is first to consider the nontrivial cohomology of d_0 and then examine if it can be extended to the cohomology of d_{NS} . In the critical string [18, 22] and the Liouville theory coupled to matter [16, 17], it has been shown that there is an isomorphism between the nontrivial cohomology classes of d_0 and d_{NS} . In our case of the super-Liouville coupled to superconformal matters, it turns out that this is no longer true in general. Nevertheless, we will find it useful to examine the cohomology of d_0 .

To compute the cohomology of d_0 , we must consider the following two cases:

- I. $P^+(n) \neq 0, P^-(n) \neq 0$ for all integers $n \neq 0$.
- II. There exist integers j, k such that $P^+(j) = P^-(k) = 0$.

We will see later that if $P^+(n)$ or $P^-(n)$ vanishes for some integer at all, the other must also vanish at some integer due to the on-shell condition, and hence there is no other case than these two. The case II is further divided into four possibilities: (i) even j and odd k ; (ii) even j and k ; (iii) odd j and k ; (iv) odd j and even k .

Let us examine the cohomology of d_0 in each case.

Case I. $P^+(n) \neq 0, P^-(n) \neq 0$ for all $n \neq 0$

If we define

$$K_{NS} \equiv \sum_{n \neq 0} \frac{1}{P^+(n)} \alpha_{-n}^+ b_n + \sum_r \frac{2r}{P^+(2r)} \psi_{-r}^+ \beta_r \quad (3.5)$$

the number operator \hat{N} may be written as $\hat{N} = \{d_0, K_{NS}\}$. This implies that any d_0 -closed state of nonzero level is d_0 -exact, *i.e.* cohomologically trivial. Hence the only nontrivial cohomology is obtained for $\hat{N} = 0$, *i.e.*

$$|p^M, p^L\rangle \quad \text{with} \quad p^+ p^- = 0. \quad (3.6)$$

which is the state we called the degree of freedom corresponding to the ‘‘center of mass’’.

Case II. $P^+(j) = P^-(k) = 0$

Since $P^\pm(n)$ are linear in n , it follows from eq. (2.14) that

$$\begin{aligned} P^+(n) &= \frac{1}{\sqrt{2}} (\lambda^M + i\lambda^L) (n - j) \\ P^-(m) &= \frac{1}{\sqrt{2}} (\lambda^M - i\lambda^L) (m - k) \end{aligned} \quad (3.7)$$

In particular, this implies that these are nonzero for other values of n and m . From (3.7) we see that

$$p^+ p^- = P^+(0)P^-(0) = \frac{1}{2} [(\lambda^M)^2 + (\lambda^L)^2] jk = -\frac{1}{2} jk. \quad (3.8)$$

Combined with the on-shell condition, we find the level is given by

$$\hat{N} = \frac{1}{2} jk \quad . \quad (3.9)$$

Hence we have either $j, k > 0$ or $j, k < 0$.

(i) Even j and odd k

If we define

$$K_j = \sum_{n \neq 0, j} \frac{1}{P^+(n)} \alpha_{-n}^+ b_n + \sum_r \frac{2r}{P^+(2r)} \psi_{-r}^+ \beta_r \quad (3.10)$$

then $\hat{N}_{0,j} = \{d_0, K_j\}$ is the level operator for all the oscillators except α_{-j}^+ and c_{-j} (α_j^- and b_j) when $j, k > 0$ ($j, k < 0$). The cohomology of d_0 is thus constructed from these mode operators. It turns out that no such state can satisfy the on-shell condition. If $j = 2(2l + 1)$ for some integer l , the level $\frac{1}{2}jk$ is an odd integer whereas all the available mode operators have even levels. If $j = 2 \cdot 2l$, on the other hand, the level is $2lk$ which cannot be made from the mode operators α_{-4l}^+ and c_{-4l} . Hence we conclude that the cohomology of d_0 is trivial.

(ii) Even j and k

We can define the same level operator as in (i). The nontrivial cohomology of d_0 is represented by the states

$$\left(\alpha_{-j}^+\right)^{k/2} |p^M, p^L\rangle, \quad c_{-j} \left(\alpha_{-j}^+\right)^{k/2-1} |p^M, p^L\rangle \quad (3.11)$$

for $j, k > 0$ and

$$\left(\alpha_j^-\right)^{-k/2} |p^M, p^L\rangle, \quad b_j \left(\alpha_j^-\right)^{-k/2-1} |p^M, p^L\rangle \quad (3.12)$$

for $j, k < 0$. By inspection, we see that these are indeed nontrivial cohomology states. This is similar to the bosonic case [16, 17].

(iii) Odd j and k

In this case, we have to modify K_j to

$$K'_j = \sum_{n \neq 0, j} \frac{1}{P^+(n)} \alpha_{-n}^+ b_n + \sum_{r \neq j/2} \frac{2r}{P^+(2r)} \psi_{-r}^+ \beta_r \quad (3.13)$$

and then $\hat{N}'_{0,j} = \{d_0, K'_j\}$ is the level operator except for $\alpha_{-j}^+, c_{-j}, \psi_{-j/2}^+$ and $\gamma_{-j/2}$ ($\alpha_j^-, b_j, \psi_{j/2}^-$ and $\beta_{j/2}$) when $j, k > 0$ ($j, k < 0$). We find that the nontrivial cohomology of d_0 is represented by the states listed below according to their degrees.

For $j, k > 0$:

degree

$$\begin{array}{rcl}
 k : & & (\gamma_{-j/2})^k, \quad \psi_{-j/2}^+ (\gamma_{-j/2})^{k-1}, \\
 & \nearrow & \nearrow \\
 k-1 : & c_{-j} (\gamma_{-j/2})^{k-2}, & \left\{ \begin{array}{l} \alpha_{-j}^+ (\gamma_{-j/2})^{k-2}, \\ c_{-j} \psi_{-j/2}^+ (\gamma_{-j/2})^{k-3}, \end{array} \right. \quad \psi_{-j/2}^+ \alpha_{-j}^+ (\gamma_{-j/2})^{k-3} \\
 & \nearrow & \nearrow \\
 k-2 : & c_{-j} \alpha_{-j}^+ (\gamma_{-j/2})^{k-4}, & \left\{ \begin{array}{l} (\alpha_{-j}^+)^2 (\gamma_{-j/2})^{k-4}, \\ c_{-j} \psi_{-j/2}^+ \alpha_{-j}^+ (\gamma_{-j/2})^{k-5}, \end{array} \right. \quad \psi_{-j/2}^+ (\alpha_{-j}^+)^2 (\gamma_{-j/2})^{k-5} \\
 & \nearrow & \nearrow \\
 \dots & \dots & \dots \\
 & \nearrow & \nearrow \\
 \frac{k+3}{2} : & c_{-j} (\alpha_{-j}^+)^{(k-5)/2} (\gamma_{-j/2})^3, & \left\{ \begin{array}{l} (\alpha_{-j}^+)^{(k-3)/2} (\gamma_{-j/2})^3, \\ c_{-j} \psi_{-j/2}^+ (\alpha_{-j}^+)^{(k-5)/2} (\gamma_{-j/2})^2, \end{array} \right. \quad \psi_{-j/2}^+ (\alpha_{-j}^+)^{(k-3)/2} (\gamma_{-j/2})^2 \\
 & \nearrow & \nearrow \\
 \frac{k+1}{2} : & c_{-j} (\alpha_{-j}^+)^{(k-3)/2} \gamma_{-j/2}, & \left\{ \begin{array}{l} (\alpha_{-j}^+)^{(k-1)/2} \gamma_{-j/2}, \\ c_{-j} \psi_{-j/2}^+ (\alpha_{-j}^+)^{(k-3)/2}, \end{array} \right. \quad \psi_{-j/2}^+ (\alpha_{-j}^+)^{(k-1)/2}
 \end{array} \quad (3.14)$$

For $j, k < 0$:

$$\begin{array}{rcccl}
k : & & (\beta_{j/2})^{-k}, & & \psi_{j/2}^{-}(\beta_{j/2})^{-k-1}, \\
& & \swarrow & & \swarrow \\
k+1 : & b_j(\beta_{j/2})^{-k-2}, & \left\{ \begin{array}{l} \alpha_j^{-}(\beta_{j/2})^{-k-2}, \\ b_j\psi_{j/2}^{-}(\beta_{j/2})^{-k-3}, \end{array} \right. & & \psi_{j/2}^{-}\alpha_j^{-}(\beta_{j/2})^{-k-3} \\
& & \swarrow & & \swarrow \\
\dots & \dots & \dots & & \dots \\
& & \swarrow & & \swarrow \\
\frac{k-3}{2} : & b_j(\alpha_j^{-})^{-(k+5)/2}(\beta_{j/2})^3, & \left\{ \begin{array}{l} (\alpha_j^{-})^{-(k+3)/2}(\beta_{j/2})^3, \\ b_j\psi_{j/2}^{-}(\alpha_j^{-})^{-(k+5)/2}(\beta_{j/2})^2, \end{array} \right. & & \psi_{j/2}^{-}(\alpha_j^{-})^{-(k+3)/2}(\beta_{j/2})^2 \\
& & \swarrow & & \swarrow \\
\frac{k-1}{2} : & b_j(\alpha_j^{-})^{-(k+3)/2}\beta_{j/2}, & \left\{ \begin{array}{l} (\alpha_j^{-})^{-(k+1)/2}\beta_{j/2}, \\ b_j\psi_{j/2}^{-}(\alpha_j^{-})^{-(k+3)/2}, \end{array} \right. & & \psi_{j/2}^{-}(\alpha_j^{-})^{-(k+1)/2}
\end{array} \tag{3.15}$$

Here the ground state $|p^M, p^L\rangle$ with $p^+p^- = -\frac{1}{2}jk$ is not exposed explicitly and the arrows indicate flows under the action of d_{NS} to be discussed in the next subsection. We thus see that there are many states with various ghost numbers.

(iv) Odd j and even k

The number operator is the same as in (iii). The states to span the nontrivial space are, however, a little different from the case (iii) in the bottom parts of the tables:

For $j, k > 0$:

$$\begin{array}{rcccl}
k : & & (\gamma_{-j/2})^k, & & \psi_{-j/2}^+(\gamma_{-j/2})^{k-1}, \\
& & \nearrow & & \nearrow \\
k-1 : & c_{-j}(\gamma_{-j/2})^{k-2}, & \left\{ \begin{array}{l} \alpha_{-j}^+(\gamma_{-j/2})^{k-2}, \\ c_{-j}\psi_{-j/2}^+(\gamma_{-j/2})^{k-3}, \end{array} \right. & & \psi_{-j/2}^+\alpha_{-j}^+(\gamma_{-j/2})^{k-3} \\
& & \nearrow & & \nearrow \\
\dots & \dots & \dots & & \dots \\
& & \nearrow & & \nearrow \\
\frac{k+2}{2} : & c_{-j}(\alpha_{-j}^+)^{(k-4)/2}(\gamma_{-j/2})^2, & \left\{ \begin{array}{l} (\alpha_{-j}^+)^{(k-2)/2}(\gamma_{-j/2})^2, \\ c_{-j}\psi_{-j/2}^+(\alpha_{-j}^+)^{(k-4)/2}\gamma_{-j/2}, \end{array} \right. & & \psi_{-j/2}^+(\alpha_{-j}^+)^{(k-2)/2}\gamma_{-j/2} \\
& & \nearrow & & \nearrow \\
\frac{k}{2} : & c_{-j}(\alpha_{-j}^+)^{(k-2)/2}, & & & (\alpha_{-j}^+)^{k/2}
\end{array} \tag{3.16}$$

For $j, k < 0$:

$$\begin{array}{rcccl}
k : & & (\beta_{j/2})^{-k}, & & \psi_{j/2}^-(\beta_{j/2})^{-k-1}, \\
& & \swarrow & & \swarrow \\
k+1 : & b_j(\beta_{j/2})^{-k-2}, & \left\{ \begin{array}{l} \alpha_j^-(\beta_{j/2})^{-k-2}, \\ b_j\psi_{j/2}^-(\beta_{j/2})^{-k-3}, \end{array} \right. & & \psi_{j/2}^-\alpha_j^-(\beta_{j/2})^{-k-3} \\
& & \swarrow & & \swarrow \\
\dots & \dots & \dots & & \dots \\
& & \swarrow & & \swarrow \\
\frac{k-2}{2} : & b_j(\alpha_j^-)^{-(k+4)/2}(\beta_{j/2})^2, & \left\{ \begin{array}{l} (\alpha_j^-)^{-(k+2)/2}(\beta_{j/2})^2, \\ b_j\psi_{j/2}^-(\alpha_j^-)^{-(k+4)/2}\beta_{j/2}, \end{array} \right. & & \psi_{j/2}^-(\alpha_j^-)^{-(k+2)/2}\beta_{j/2} \\
& & \swarrow & & \swarrow \\
\frac{k}{2} : & b_j(\alpha_j^-)^{-(k+2)/2}, & & & (\alpha_j^-)^{-k/2},
\end{array} \tag{3.17}$$

Here again the ground state is suppressed.

Finally let us show that we have exhausted all possible cases. Suppose that $P^+(j) = 0$ but $P^-(n) \neq 0$ for all nonzero integers n . Owing to the linearity in n , $P^+(n)$ is rewritten

like eq. (3.7)

$$P^-(m) = \frac{1}{\sqrt{2}} (\lambda^M - i\lambda^L) (m - \alpha) \quad (3.18)$$

but with α being not an integer. This gives us

$$p^+ p^- = P^+(0) P^-(0) = -\frac{1}{2} j \alpha \quad (3.19)$$

which implies that the nontrivial cohomology is possible at the level $\frac{1}{2} j \alpha$. However, as is evident from the above analysis, the only available mode operators for creating cohomologically nontrivial states have the level either j or $j/2$, which cannot give the level $\frac{1}{2} j \alpha$. Therefore there is no nontrivial cohomology of d_0 in this case. The same argument excludes the case when $P^-(k) = 0$ but $P^+(n) \neq 0$.

3.2 Cohomology of d_{NS}

In order to construct a state representing nontrivial cohomology of d ,² we may start from a state nontrivial with respect to d_0 and add terms of higher degrees. This is due to the following fact. The lowest degree term in a state nontrivial with respect to d may be always chosen to represent a nontrivial cohomology of d_0 ; or if the cohomology of d_0 is trivial, then the cohomology of d is also trivial.

The proof is simple [17]. Suppose that $\psi = \psi_k + \psi_{k+1} + \dots$ represents a cohomology of d , where the subscripts $k, k+1, \dots$ stand for the degrees. Then $d\psi = 0$ means $d_0\psi_k = 0$ and thus $\psi_k = d_0\chi_k$ by assumption. If we consider $\psi' = \psi - d\chi_k$ which also belongs to the same cohomology class as ψ , the lowest degree term in ψ' has degree at least $k+1$. Repeating the same procedure as above, we arrive at $\psi = d(\chi_k + \chi_{k+1} + \dots)$, which is the desired result.

Furthermore, if for each ghost number N_{FP} , the cohomology of d_0 is nontrivial for at most one fixed degree k independent of N_{FP} , then the nontrivial cohomology classes of d_0 and d are isomorphic. We only sketch the proof briefly. (See ref. [17] for the details.)

²We suppress the subscript NS to d in what follows since most of the following discussions are valid for d_R as well.

Let ψ_k be a nontrivial element in the cohomology of d_0 . Then $d\psi_k = (d_1 + d_2)\psi_k$ has the lowest degree at least $k + 1$. Using (3.4), we find $d_0(d_1\psi_k) = 0$. Since there is no nontrivial cohomology at degree $k + 1$, we get $d_1\psi_k = d_0\chi_{k+1}$. Then $d(\psi_k - \chi_{k+1})$ has terms of degree at least $k + 2$, and the use of eq. (3.4) tells us that $d_0[d(\psi_k - \chi_{k+1})]_{k+2} = 0$. Repeating the above procedure, we construct $\psi \equiv \psi_k - \chi_{k+1} - \chi_{k+2} - \dots$ such that ψ is closed under d . Moreover, one can show that this map gives a unique element in d -cohomology. Conversely, it can be shown that each element of cohomology of d projects onto a unique element of d_0 -cohomology. This completes the proof.

Since the assumption here is satisfied for cases I and II (i) and (ii), we find nontrivial cohomology of d only for $\hat{N} = 0$ and $\frac{1}{2}jk$ for even j and k with ghost number $N_{FP} = 0, \pm 1$.

In the other cases, the above statement does not apply because there are many nontrivial cohomology classes of d_0 . In fact, when d acts on the states in the tables (3.14)–(3.17), there appear other states of the nontrivial cohomology of d_0 as indicated by the arrows in (3.14)–(3.17). For example, for the case (iii) the states transform under the action of d as

$$\begin{aligned}
c_{-j}(\gamma_{-j/2})^{k-2}|p^M, p^L \rangle &\rightarrow (\gamma_{-j/2})^k|p^M, p^L \rangle \rightarrow 0 \\
c_{-j}\alpha_{-j}^+(\gamma_{-j/2})^{k-4}|p^M, p^L \rangle &\rightarrow [-\frac{1}{4}\alpha_{-j}^+(\gamma_{-j/2})^{k-2} + \frac{j}{2}c_{-j}\psi_{-j/2}^+(\gamma_{-j/2})^{k-3}]|p^M, p^L \rangle \\
&\rightarrow 0
\end{aligned} \tag{3.20}$$

where we have written only the terms nontrivial in the d_0 -cohomology. It is important to note that these states always vanish under the second action of d and that states nontrivial with respect to d_0 are produced only at the next degree and ghost number. The first fact is a reflection of the nilpotency of the BRST charge and the second is due to the fact that the states are created by the action of d_1 . If we call the initial states “parents” and the resulting states “daughters”, we can check that most of these states in the tables are either parents or daughters.

Now we prove that neither parents nor daughters can give rise to nontrivial cohomology of the total d even if higher degree terms are added.

To show that the parents do not produce any, assume that we succeed in constructing

a state $\psi = \psi_k + \psi_{k+1} + \dots$ such that

$$d(\psi_k + \psi_{k+1} + \dots) = 0 \quad (3.21)$$

starting from a parent ψ_k . Since $d_0\psi_k = 0$, the lowest degree terms in (3.21) give

$$d_1\psi_k + d_0\psi_{k+1} = 0. \quad (3.22)$$

According to (3.4), $d_0(d_1\psi_k) = -d_1d_0\psi_k = 0$ and thus we get

$$d_1\psi_k = \eta_{k+1} + d_0\chi_{k+1} \quad (3.23)$$

where η_{k+1} denote a state nontrivial with respect to d_0 . (From parents, states corresponding to nontrivial cohomology classes of d_0 are always produced by the action of d_1 .)

But if we substitute (3.23) into (3.22), we obtain

$$\eta_{k+1} = -d_0(\chi_{k+1} + \psi_{k+1}) \quad (3.24)$$

in contradiction to the fact that η_{k+1} is nontrivial. Therefore the parents are excluded.

Let us next consider a daughter state, η_{k+1} , obtained from a parent state, ψ_k , as given in (3.23). This relation can be rewritten as

$$d(\psi_k - \chi_{k+1}) = \eta_{k+1} + [-d_1\chi_{k+1} + d_2\psi_k] - d_2\chi_{k+1} \quad (3.25)$$

Thus we obtain a d -trivial state by adding degree $k+2$ and $k+3$ terms to η_{k+1} . Let us write this as $\eta_{k+1} + \eta_{>}$. Suppose that there is a d -nontrivial state with η_{k+1} as the lowest degree term and write it as $\eta_{k+1} + \eta'_{>}$. We may take a representative of the same cohomology class as

$$\eta_{k+1} + \eta'_{>} - d(\psi_k - \chi_{k+1}) \quad (3.26)$$

However this does not contain η_{k+1} . Thus there is no nontrivial cohomology class represented by a state with η_{k+1} as the lowest degree term. Hence the above statement is proved.

Using the above results, we can discard all the parents and daughters for constructing the cohomology of d . In particular, for case II (iv) all the states are either parents or

daughters. We find that the only exceptions are

$$\begin{aligned} & \psi_{-j/2}^+(\alpha_{-j}^+)^{(k-1)/2} |p^M, p^L \rangle, \\ & [(\alpha_{-j}^+)^{(k-1)/2} \gamma_{-j/2} - j(k-1)c_{-j} \psi_{-j/2}^+(\alpha_{-j}^+)^{(k-3)/2}] |p^M, p^L \rangle \end{aligned} \quad (3.27)$$

for odd $j, k > 0$ and

$$\begin{aligned} & \psi_{j/2}^-(\alpha_j^-)^{-(k+1)/2} |p^M, p^L \rangle, \\ & [(\alpha_j^-)^{-(k+1)/2} \beta_{j/2} - \frac{1}{2} b_j \psi_{j/2}^-(\alpha_j^-)^{-(k+3)/2}] |p^M, p^L \rangle \end{aligned} \quad (3.28)$$

for odd $j, k < 0$. The linear combinations are singled out by the requirement that they are neither parents nor daughters.

It can be shown that these states can be promoted to the cohomology of d by adding higher degree terms by a procedure described before for d_0 -nontrivial states. It is also easy to see that $\psi_k - \chi_{k+1} - \chi_{k+2} - \dots$ thus constructed is not trivial. (If we assume that it is trivial, it leads to a contradiction that ψ_k is d_0 -trivial.)

To summarize, we have found that there are nontrivial states for $p^+ p^- = 0$ and for $p^+ p^- = -\frac{1}{2} j k$ with $j - k = \text{even}$. In the latter case, the states can be constructed from those in eqs. (3.11) and (3.12) for even j and k , and from those in eqs. (3.27) and (3.28) for odd j and k .

3.3 Quartet mechanism

In the previous subsections we have examined the nontrivial cohomology classes for d_{NS} . We have encountered quite a different situation from the bosonic case where there is a one-to-one correspondence between the cohomologies of d_0 and d_{NS} , and yet have succeeded in identifying the nontrivial classes of d_{NS} . Here we will give an alternative derivation of this result by showing that the states, which are found to be d -trivial but d_0 -nontrivial in the previous subsection, actually fall into the so-called BRST quartet representations and hence decouple from the system [28]. Although the final results are the same as in the previous subsection, we believe that this reformulation shows the essence of the decoupling mechanism of the states and also it is more accessible to physicists. Since the

other cases are essentially the same, let us consider the NS sector with both j and k odd integers.

The Fock vacuum is the direct product of the vacua for the matter, Liouville and ghost systems

$$|p^M, p^L\rangle \equiv |p^M\rangle \otimes |p^L\rangle \otimes |0\rangle_{gh}. \quad (3.29)$$

From the conditions $P^+(j) = P^-(k) = 0$, we find the momenta are given as

$$p^{M,L} = \frac{1}{2}(jt_+^{M,L} + kt_-^{M,L}) = t_{(j,k)}^{M,L} + \lambda^{M,L} \quad (3.30)$$

where $t_{\pm}^M = -\lambda^M \mp i\lambda^L$, $t_{\pm}^L = -\lambda^L \pm i\lambda^M$ and $t_{(j,k)}^{M,L} \equiv \frac{1+j}{2}t_+^{M,L} + \frac{1+k}{2}t_-^{M,L}$. Thus the vacuum in (3.29) may also be labelled by these integers as $|(j, k)\rangle$.

Let us first observe the following pattern in the d_0 -nontrivial states in (3.14) and (3.15).

For the momenta satisfying $P^+(j) = P^-(k) = 0$ with $j, k > 0$

$\begin{array}{l} N_{FP} \\ \text{deg.} \end{array}$	$k - (2l + 1)$	$k - (2l + 2)$	$k - (2l + 3)$
$k - l$	$ k - (2l + 1)\rangle$		
$k - l - 1$	**	$ k - (2l + 2)\rangle_+$ $ k - (2l + 2)\rangle_-$	**
$k - l - 2$			$ k - (2l + 3)\rangle$

(3.31)

and for $P^+(-j) = P^-(-k) = 0$ with $j, k > 0$

$\begin{array}{l} N_{FP} \\ \text{deg.} \end{array}$	$-k + (2l + 1)$	$-k + (2l + 2)$	$-k + (2l + 3)$
$-k + l$	$ -k + (2l + 1)\rangle$		
$-k + l + 1$	**	$ -k + (2l + 2)\rangle_+$ $ -k + (2l + 2)\rangle_-$	**
$-k + l + 2$			$ -k + (2l + 3)\rangle$

(3.32)

Here we have defined the states by

$$|k - (2l + 1)\rangle \equiv \frac{l+1}{4} j(\alpha_{-j}^+)^l \psi_{-j/2}^+ (\gamma_{-j/2})^{k-(2l+1)} |(j, k)\rangle,$$

$$\begin{aligned}
|k - (2l + 2) \rangle_{\pm} &\equiv \left[-\frac{1}{4}(\alpha_{-j}^+)^{l+1}(\gamma_{-j/2})^{k-(2l+2)} \right. \\
&\quad \left. \pm \frac{l+1}{2} j c_{-j}(\alpha_{-j}^+)^l \psi_{-j/2}^+(\gamma_{-j/2})^{k-(2l+3)} \right] |(j, k) \rangle, \\
|k - (2l + 3) \rangle &\equiv c_{-j}(\alpha_{-j}^+)^{l+1}(\gamma_{-j/2})^{k-(2l+4)} |(j, k) \rangle, \\
|-k + (2l + 1) \rangle &\equiv \frac{(-1)^{l+1} 4}{j^{l+1}(l+1)!(k-2l-1)!} (\alpha_{-j}^-)^l \psi_{-j/2}^-(\beta_{-j/2})^{k-(2l+1)} |(-j, -k) \rangle, \\
|-k + (2l + 2) \rangle_{\pm} &\equiv \frac{(-1)^{l+1}}{j^{l+1}(l+1)!(k-2l-2)!} \left[-2(\alpha_{-j}^-)^{l+1}(\beta_{-j/2})^{k-(2l+2)} \right. \\
&\quad \left. \mp (k-2l-2) b_{-j}(\alpha_{-j}^-)^l \psi_{-j/2}^-(\beta_{-j/2})^{k-(2l+3)} \right] |(-j, -k) \rangle, \\
|-k + (2l + 3) \rangle &\equiv \frac{(-1)^{l+1}}{j^{l+1}(l+1)!(k-2l-4)!} \\
&\quad \times b_{-j}(\alpha_{-j}^-)^{l+1}(\beta_{-j/2})^{k-(2l+4)} |(-j, -k) \rangle. \tag{3.33}
\end{aligned}$$

In the tables l runs from -1 to $\frac{k-3}{2}$ and the double asterisks mean that there are other states for adjacent values of l . (It is to be understood that when the power of the mode operators becomes negative, there is no corresponding state.)

The proper inner product is to be defined, in an abbreviated form³, as

$$\langle O, O' \rangle \equiv (O | -p^M, -p^L \rangle)^\dagger P c_0 O' | p^M, p^L \rangle \tag{3.34}$$

where P , a parity operator [25], changes the signs of the oscillators including zero modes in the matter sector. It is important to note that the inner product in (3.34) respects the hermiticity of the Virasoro generators

$$P L_n^\dagger(\lambda^{M,L}) P = L_{-n}(\lambda^{M,L}). \tag{3.35}$$

In the following, we will show that there are two sets of quartets associated with the states listed in eqs. (3.31) and (3.32).

As we have already seen in simple examples in eq.(3.20), the states are transformed under the action of d_{NS} as

$$d_{NS} |k - (2l + 3) \rangle = |k - (2l + 2) \rangle_+ + d_0 \chi^{k-(2l+2)}$$

³As discussed in ref. [13], there is another possibility to define an inner product when $\hat{c} = 1$. Here we adopt the convention defined in the text.

$$\begin{aligned}
&\equiv |k - (2l + 2) \rangle'_+, \\
d_{NS}|k - (2l + 2) \rangle_- &= |k - (2l + 1) \rangle + d_0\chi^{k-(2l+1)} \\
&\equiv |k - (2l + 1) \rangle' .
\end{aligned} \tag{3.36}$$

Note that the terms other than d_0 -nontrivial states may be written in d_0 -exact forms, denoted as $d_0\chi^{k-(2l+2)}$ and $d_0\chi^{k-(2l+1)}$. These parent and daughter states in (3.36) form the doublet representations of the BRST charge. For the states in eq. (3.32), we find similar relations

$$\begin{aligned}
d_{NS}| -k + (2l + 1) \rangle &= | -k + (2l + 2) \rangle_+ + d_0\chi^{-k+2l+2}, \\
&\equiv | -k + (2l + 2) \rangle'_+, \\
d_{NS}| -k + (2l + 2) \rangle_- &= | -k + (2l + 3) \rangle + d_0\chi^{-k+2l+3} \\
&\equiv | -k + (2l + 3) \rangle' .
\end{aligned} \tag{3.37}$$

For the following discussion, it is important to realize that the states in eq. (3.33) have nonvanishing inner products

$$- \langle -k + (2l + 3) |'_+ c_0 | k - (2l + 3) \rangle_- = - \langle -k + (2l + 2) | c_0 | k - (2l + 2) \rangle'_+ = 1. \tag{3.38}$$

where we have used the relation $P(\alpha_n^\pm)^\dagger P = -\alpha_{-n}^\pm$ and $P(\psi_n^\pm)^\dagger P = -(\psi_{-n}^\pm)$. Note also that the d_0 -exact terms do not contribute to the inner products.

From eqs. (3.36)-(3.38), we see that those four states in (3.38) form a BRST quartet, a pair of doublets with respect to d_{NS} . The quartet mechanism first considered in non-abelian gauge theories then tells us that these states decouple from the system [28]. To see this, it is enough to note that the projection operator to the states in the quartets in eq. (3.38) is given by

$$\mathcal{P} \equiv \{ d_{NS}, |k-(2l+3) \rangle_- \cdot \langle -k+(2l+2) | c_0 - | -k+(2l+2) \rangle_- \cdot \langle k-(2l+3) | c_0 \} \tag{3.39}$$

due to eqs. (3.36)-(3.38). It follows that the quartet appears only in d_{NS} -exact forms and all of the four states decouple from the system. One may repeat the same argument for the other states in (3.36) and (3.37).

It is not difficult to see that most of the states in the tables (3.14)–(3.17) fall into quartet representations as illustrated above. The only exceptions are the states in (3.27) and (3.28), which therefore contribute to the physical spectrum.

3.4 Absolute cohomology

Having identified the relative cohomology, let us make brief comments on the absolute cohomology specified by (2.23).

The difference between (2.23) and the relative cohomology (2.25) is that in the latter case we choose a special vacuum $|\downarrow\rangle$ annihilated by b_0 . Hence the absolute cohomology is obtained essentially by adding the states built on the other vacuum $|\uparrow\rangle \equiv c_0|\downarrow\rangle$. Indeed, it is not difficult, following the bosonic case [16, 17], to show that the absolute cohomology is isomorphic to the direct product of these two relative cohomologies.

4 Physical states in the R sector

In this section, we proceed to the discussion of the relative cohomology (2.26) in the R sector.

Let us first consider the on-shell condition. The Ramond-Dirac operator F in eq.(2.20) contains the zero modes ψ_0^\pm which are actually two-dimensional gamma matrices. We use the following representation:

$$\psi_0^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi_0^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (4.1)$$

It is then convenient to suppose that \hat{F} , the nonzero-mode part of F , is multiplied by σ_3 so that it automatically anticommutes with (4.1). Any spinor in this representation can be written as

$$\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix} \quad (4.2)$$

where $|A\rangle$ and $|B\rangle$ denote the states spanned by the mode operators and carrying momenta. On this state the condition, $F|\text{phys}\rangle = 0$, becomes

$$\begin{pmatrix} p^-|B\rangle + \hat{F}|A\rangle \\ p^+|A\rangle - \hat{F}|B\rangle \end{pmatrix} = 0. \quad (4.3)$$

The minus sign in the lower column is due to σ_3 .

Since $F^2 = L_0 = p^+p^- + \hat{F}^2$, eq. (4.3) may be solved as follows. When $p^- \neq 0$, we take an on-shell state $|A\rangle$ ($L_0|A\rangle = 0$) and define $|B\rangle = -\frac{1}{p^-}\hat{F}|A\rangle$. Then we have

$$\hat{F}|B\rangle = -\frac{1}{p^-}\hat{F}^2|A\rangle = p^+|A\rangle. \quad (4.4)$$

Thus the condition (4.3) is satisfied. If $p^+ \neq 0$, we can similarly prepare the state $|B\rangle$ and construct a spinor satisfying (4.3). Therefore it is enough to examine if there is any nontrivial cohomology of d_R satisfying the on-shell condition $L_0 = 0$, since we can always construct on-shell spinors from such states. From now on we will consider only the on-shell states.

The cohomology operator d_R is again decomposed as in (3.2):

$$d_R = d_0 + d_1 + d_2 \quad (4.5)$$

where again

$$d_0 = \sum_{n \neq 0} P^+(n) c_{-n} \alpha_n^- - \frac{1}{2} \sum_{n \neq 0} P^+(2n) \gamma_{-n} \psi_n^- \quad (4.6)$$

The explicit forms of other operators are not necessary except that they satisfy (3.4). An important difference from the NS sector is that the argument of $P^+(2n)$ is an even integer.

We again have two different cases I and II in sect. 3. Let us first enumerate the nontrivial cohomology of d_0 .

Case I. $P^+(n) \neq 0, P^-(n) \neq 0$ for all $n \neq 0$

In this case we may define

$$K_R \equiv \sum_{n \neq 0} \frac{1}{P^+(n)} \alpha_{-n}^+ b_n + \sum_{n \neq 0} \frac{2n}{P^+(2n)} \psi_{-n}^+ \beta_n \quad (4.7)$$

and the number operator \hat{N} is given as $\hat{N} = \{d_0, K_R\}$. Hence the only nontrivial cohomology is the same as (3.6):

$$|p^M, p^L\rangle \quad \text{with} \quad p^+ p^- = 0. \quad (4.8)$$

Case II. $P^+(j) = P^-(k) = 0$

Since $p^+ p^- = -\frac{1}{2}jk$ as in (3.8), we have the level $\hat{N} = \frac{1}{2}jk > 0$ from the on-shell condition $L_0 = 0$, and again we have either $j, k > 0$ or $j, k < 0$.

(i) Even j and odd k

Similarly to case (iii) in the NS sector, we should define

$$K'_j \equiv \sum_{n \neq 0, j} \frac{1}{P^+(n)} \alpha_{-n}^+ b_n + \sum_{n \neq 0, j/2} \frac{2n}{P^+(2n)} \psi_{-n}^+ \beta_n \quad (4.9)$$

and the nontrivial cohomology of d_0 is given precisely by the states in (3.14) and (3.15).

(ii) Even j and k

We can use the same level operator as in (i). The nontrivial cohomology of d_0 is given by the states in (3.16) and (3.17).

(iii) Odd j and k

In this case, we can define

$$K_j \equiv \sum_{n \neq 0, j} \frac{1}{P^+(n)} \alpha_{-n}^+ b_n + \sum_{n \neq 0} \frac{2n}{P^+(2n)} \psi_{-n}^+ \beta_n \quad (4.10)$$

and the nontrivial cohomology is possible in terms of α_{-j}^+, c_{-j} (α_j^-, b_j) for $j, k > 0$ ($j, k < 0$). However, we cannot construct states with the level $\frac{1}{2}jk$ (half-odd-integer) from integer mode operators. Thus there is no nontrivial cohomology in this case.

(iv) Odd j and even k

The number operator is the same as in (iii) above and the nontrivial cohomology of d_0 is given by the space spanned by the states (3.11) and (3.12).

Again here is no other possible case that could give rise to nontrivial cohomology in the R sector. The argument is the same as in the NS case (cf. (3.19)).

As proved in sect. 3, parents and daughters under the action of d_R do not correspond to nontrivial cohomology of d_R . By studying how these states transform into each other,

we find that the nontrivial cohomology of d_R is possible only for $j - k = \text{odd}$ in case II. For even j and odd k , it is constructed from the states given in eqs. (3.27) and (3.28). For odd j and even k , the states are obtained from (3.11) and (3.12).

We thus again find that nontrivial cohomology classes are possible only at levels where “null states” in the minimal models with $\hat{c} < 1$ exist [25]. We will discuss why this is so in the next section.

The absolute cohomology is again obtained by considering the vacua of the ghost zero modes. For the $b - c$ ghost, this is essentially the same as NS case. The vacua for the bosonic ghosts $\beta - \gamma$ are infinitely degenerate and we expect this leads to infinite number of such spaces corresponding to these degrees of freedom.

5 Discussions

Using the cohomological terms, we have examined the nontrivial states allowed in the physical state conditions (2.25) and (2.26) for all the cases in the NS and R sectors. Remarkably we have found, apart from the ground state $|p^M, p^L\rangle$ with $p^+ p^- = 0$, there exist nontrivial states at levels $\hat{N} = \frac{1}{2}jk$ and the discrete values of momenta (3.30) with $p^+ p^- = -\frac{1}{2}jk$. These states exist only for $j - k = \text{even}$ (odd) in the NS (R) sector.

These are precisely the values of momenta at which special states with respect to the Virasoro algebras appear in the Fock spaces of free fields with background charges [25, 26]. Let us discuss why this happens. We consider the bosonic case for simplicity of presentation since the structure of the cohomology states is essentially the same for the bosonic and supersymmetric cases as we have seen in this paper. We must remember that in the bosonic case the extra states are present at levels $N = jk$.

Let us first recall some properties of free field realization of a conformal field theory (the matter or gravity sector). An important quantity for our argument is the C -matrix considered in ref. [25], relating the complete sets of states spanned by the Virasoro

generators and the boson oscillators. At level N it is defined by

$$L^{-I}(\lambda)|t + \lambda \rangle = \sum_J C_{IJ}(p, \lambda) \alpha^{-J} |t + \lambda \rangle \quad (5.1)$$

where $|t + \lambda \rangle$ is a Fock vacuum with the momentum $t + \lambda$, and $L^{-I}(\lambda)$ and α^{-J} stand for all the independent combinations of the Virasoro generators and oscillators at level N , respectively. Thus I and J run from 1 to $P(N)$, the partition number of the integer N . It has been shown in ref. [25] that the C -matrix satisfies

$$\det[C(t + \lambda, \lambda)] = \text{const.} \times \prod_{\substack{j,k > 0 \\ 1 \leq jk \leq N}} (t - t_{(-j, -k)})^{P(N-jk)}, \quad (5.2a)$$

$$\det[C(t + \lambda, -\lambda)] = \text{const.} \times \prod_{\substack{j,k > 0 \\ 1 \leq jk \leq N}} (t - t_{(j, k)})^{P(N-jk)}. \quad (5.2b)$$

The first equation tells us that for particular values of $t = t_{(-j, -k)}$, there are states at the level jk in the Fock space which cannot be constructed by the Virasoro generators; there are linear combinations of the states generated by L^{-J} which identically vanish. At the zeros of (5.2b), we find primary states (“null states” for $c < 1$) at the same level jk , which may be constructed by the singular vertex operators [25]. We note that the vanishing conditions of (5.2a) and (5.2b) coincide with each other for $c = 1$ ($\lambda = 0$).

Let us also note a general feature of the BRST formalism. In sect. 3.3, we discussed the quartet mechanism, where the doublets appeared in pairs. Suppose a state $|G + 1 \rangle$ with a ghost number $G + 1$ is generated from $|G \rangle$ by the action of the BRST charge Q_B :

$$Q_B |G \rangle = |G + 1 \rangle, \quad Q_B |G + 1 \rangle = 0. \quad (5.3)$$

Then there must be a state $|-G - 1 \rangle$ which has a nonzero inner product with $|G + 1 \rangle$; otherwise $|G + 1 \rangle$ does not appear in the theory as poles in Green functions. A state $|-G \rangle$ defined by the sequence

$$Q_B |-G - 1 \rangle = |-G \rangle, \quad Q_B |-G \rangle = 0 \quad (5.4)$$

then has a nonzero inner product with $|G \rangle$ since

$$\langle -G | c_0 | G \rangle = \langle -G - 1 | Q_B c_0 | G \rangle = - \langle -G - 1 | c_0 | G + 1 \rangle = -1. \quad (5.5)$$

Therefore the BRST-doublets always appear in pairs. One can prove that these quartets do not contribute to the physical spectrum, as in subsection 3.3.

Combining the properties described in the last paragraphs, we may understand the origin of discrete states. The idea is as follows. Suppose that we have a quartet satisfying (5.3)-(5.5). Let us assume that in the relation (5.4) the momenta for both matter and gravity sectors take the values at the zeros of (5.2a). Then one can choose a state $|-G-1\rangle$ at the level jk such that the vanishing combinations of Virasoro generators appear in $|-G\rangle$. The state $|-G\rangle$ vanishes because it is multiplied by a coefficient which has a zero at the particular values of momenta. We may find a Fock state from $|-G\rangle$ by dividing out the coefficient (See the examples given below). These two states no longer belong to a BRST-doublet and do not necessarily decouple from the physical spectrum; both $|-G-1\rangle$ and $|-G\rangle$ (or precisely speaking, the corresponding Fock states) are in $\text{Ker } Q_B$. In order to find them in the physical spectrum, there must be states which have nonzero inner products with them. From the quartet structure in our formulation, $|G\rangle$ and $|G+1\rangle$ must be these states. From the consistency with (5.5), we expect $|G\rangle$ contains the inverse of the vanishing coefficient. Furthermore we will see that $|G\rangle$ is essentially a primary state corresponding to a zero of (5.2b). This is to be expected because any state constructed from the primary states both in the matter and gravity sectors is in $\text{Ker } Q_B$ since Q_B contains only Virasoro generators for the matter and gravity. We will explain this point further in our examples.

Let us study examples at levels one and two. In order to see how physical states emerge from the quartet structure, we start from general momenta and later put them to the values of interest. To do this, we should first note that the momenta $p^{M,L} = t^{M,L} + \lambda^{M,L}$ and $\lambda^{M,L}$ must satisfy two constraints from the nilpotency of the BRST charge and the on-shell condition

$$\begin{aligned}
-\frac{1}{2}[(\lambda^M)^2 + (\lambda^L)^2] &= 1, \\
\hat{N} + \frac{1}{2}(t^M + \lambda^M)^2 + \frac{1}{2}(t^L + \lambda^L)^2 &= 0.
\end{aligned}
\tag{5.6}$$

At level one $\hat{N} = 1$, we have the relations

$$\begin{aligned} Q_B b_{-1} |t + \lambda\rangle &= [L_{-1}^M(\lambda^M) + L_{-1}^L(\lambda^L)] |t + \lambda\rangle \\ &= t^M (\alpha_{-1}^M + \frac{t^L}{t^M} \alpha_{-1}^L) |t + \lambda\rangle \end{aligned} \quad (5.7)$$

$$\begin{aligned} Q_B [t^M (1 + (\frac{t^L}{t^M})^2)]^{-1} (\alpha_{-1}^M + \frac{t^L}{t^M} \alpha_{-1}^L) |-(t + \lambda)\rangle \\ = -c_{-1} |-(t + \lambda)\rangle. \end{aligned} \quad (5.8)$$

where we have denoted the vacua with momenta $t + \lambda$ by

$$|t + \lambda\rangle \equiv |t^M + \lambda^M\rangle_M \otimes |t^L + \lambda^L\rangle_L \otimes |0\rangle_{gh}. \quad (5.9)$$

Note that the states in (5.7) and (5.8) have the opposite momenta to give the nonzero inner product (see the definition in (3.34)). We note that for general momenta the four states form a quartet. Let us now choose the momenta t^M close to a zero in (5.2a):

$$t^M = t_{(-1,-1)}^M + \epsilon = \epsilon. \quad (5.10)$$

From the relation (5.6) t^L is determined; we choose a solution so that $t^L = t_{(-1,-1)}^L$ for $t^M = t_{(-1,-1)}^M$. The ratio t^L/t^M in (5.7) and (5.8) then takes a finite value

$$\frac{t^L}{t^M} = -\frac{\lambda^M}{\lambda^L} + O(\epsilon). \quad (5.11)$$

Now it is easy to see that the quartet *decomposes* into singlets when $\epsilon = 0$, in a manner we explained earlier. Note that the Fock states with $N_{FP} = 0$ in (5.7) and (5.8) contain only matter oscillators for $\lambda^M = 0$ ($c = 1$), a general feature discussed in refs.[13, 17].

We have emphasized the importance of the vanishing combinations of Virasoro generators on the particular momentum states. Here let us point out that they are closely related to the “null states” corresponding to the zeros in (5.2b). In general, the states which have nonzero inner products must have opposite momenta. In fact, the state in (5.8) has opposite momenta $t_{(1,1)}^M$ and $t_{(1,1)}^L$ to those of the state in (5.7), $t_{(-1,-1)}^M$ and $t_{(-1,-1)}^L$, owing to the relation $t_{(j,k)} + \lambda = -t_{(-j,-k)} - \lambda$. These opposite momenta precisely correspond to zeros in (5.2b) where we find primary states, which actually give rise to the state on the LHS of (5.8).

At level two, one can observe the same phenomena. We report only the result for $\lambda^M \sim 0$. We take the momenta close to the relevant zeros at $t = t_{(-1,-2)}$ and $t = t_{(-2,-1)}$:

$$\epsilon \equiv t^M - t_{(-1,-2)}^M \quad \text{or} \quad t^M - t_{(-2,-1)}^M.$$

Expanding the states with respect to ϵ and $\lambda^M (\sim 0)$, we find the quartet structure

$$\begin{aligned} & Q_B [b_{-2} + b_{-1}(A^M L_{-1}^M + A^L L_{-1}^L)] |t + \lambda \rangle \\ &= \frac{1 + (t_+^M)^2}{1 + (t_+^M)^2} \epsilon \{ [\alpha_{-2}^M \mp \sqrt{2}(\alpha_{-1}^M)^2] + O(\epsilon, \lambda^M) \} |t + \lambda \rangle \end{aligned} \quad (5.12)$$

and

$$\begin{aligned} & Q_B \frac{1 + (t_+^M)^2}{6\epsilon(1 + (t_+^M)^2/2)} \{ [\alpha_{-2}^M \pm \sqrt{2}(\alpha_{-1}^M)^2] + O(\epsilon, \lambda^M) \} |-(t + \lambda) \rangle \\ &= [-\frac{1}{2}(c_{-2} \pm \sqrt{2}c_{-1}\alpha_{-1}) + O(\epsilon, \lambda^M)] |-(t + \lambda) \rangle \end{aligned} \quad (5.13)$$

for the choice of the solution $t^L = \frac{i}{\sqrt{2}} + O(\epsilon, \lambda^M)$, where the upper (lower) sign is for $t_{(-1,-2)}$ ($t_{(-2,-1)}$). The coefficients in (5.12) are given by

$$A^{M,L} = -\frac{1 + 2t_+^{M,L}(t^{M,L} - \lambda^{M,L})}{2t^{M,L}(t^{M,L} + t_+^{M,L})} = \mp 1 + O(\epsilon, \lambda^M), \quad (5.14)$$

where the upper (lower) sign is for matter (gravity) sector. From (5.12) and (5.13), we again observe the *decomposition* of a quartet into singlets at level two. On the LHS of (5.13) we find the primary states corresponding to the zeros $t^M = t_{(1,2)}, t_{(2,1)}$ of (5.2b).⁴

The coefficients given in (5.14) may be used for general c . So one may find the similar structure for any c , starting from the state on the LHS of (5.12).

It is known that $N_{FP} = 0$ discrete states are classified according to $SU(2)$ generated by $\int dz : e^{\pm i\sqrt{2}\phi} :$ and $\int dz i\partial\phi$ (ϕ is the scalar field for the matter). From the ‘‘quartet’’ structure, we expect that $N_{FP} \neq 0$ discrete states form $SU(2)$ multiplets as well. Let us study our examples whether this is the case for $c^M = 1(\lambda^M = 0)$. The state $\alpha_{-1}^M |(1, 1) \rangle$ on the RHS of (5.7) is a state in the triplet representation, while on the LHS we find a singlet. In (5.12) and (5.13), we find two states in the quartet consisting of matter

⁴Comparison with (2.95) of [25] may be useful.

oscillators and the doublet of ghost modes. These examples suggest the general pattern: r in $N_{FP} = 0$ goes into $r - 2$ in $N_{FP} \neq 0$ under the action of Q_B .

The extension of all the above discussions to supersymmetric case is straightforward, and the origin of the extra physical states may be explained by the same reasoning.

Returning to the supersymmetric case, we have found extra physical states with ghost numbers $N_{FP} = 0, \pm 1$. We can actually construct these states for $N_{FP} = 0$ as an extension of the bosonic case [13]:

$$\begin{aligned} (W_0^+)^j |(0, j+k) \rangle_M \otimes |(j+k, 0) \rangle_L \otimes |0 \rangle_{gh}, \\ (W_0^-)^j |(j+k, 0) \rangle_M \otimes |(0, -j-k) \rangle_L \otimes |0 \rangle_{gh} \end{aligned} \quad (5.15)$$

where

$$\begin{aligned} W_0^\pm &= \int \frac{dz}{2\pi i} W(t_\pm, z) \\ &= \int \frac{dz}{2\pi i} : t_\pm \psi^M(z) e^{it_\pm \phi^M(z)} : \end{aligned} \quad (5.16)$$

are the charge screening operators with $t_\pm = \pm 1$ for $\hat{c} = 1$. The matter part of the above states are obtained by the use of the so-called singular vertex operators [25].

To see that these states are in the physical spectrum, we first note that they are in $Ker Q_B$ since W_0^\pm commute with Q_B and the rest of the states satisfy the on-shell condition. They cannot be in $Im Q_B$ since it can be shown that they have nonzero inner product by using the algebra satisfied by W_0^\pm and a proper definition of the inner product [13].

Similarly to the bosonic case, the operators in (5.16), together with $J_0(z) = i\partial\phi(z)$, satisfy an $SU(2)$ -like current algebra

$$\begin{aligned} W(t_+, z)W(t_-, w) &\sim -\frac{1}{(z-w)^2} - \frac{1}{z-w} J_0(w), \\ J_0(z)W(t_\pm, w) &\sim \frac{\pm 1}{z-w} W(t_\pm, w). \end{aligned} \quad (5.17)$$

The above discrete states therefore form multiplets with respect to the global algebra satisfied by the zero modes of the currents.

In the course of writing this paper, we received ref. [29] which considerably overlaps with sect. 5 of this paper.

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