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Q-Deformations of the Heisenberg Equations of Motion*

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Abstract

Two formulations of q-quantum mechanics based on quantum deformations of the Heisenberg equations of motion, are discussed. In one, the commutator is replaced by the "quommutator": $[A, B]_q = qAB - (1/q)BA$. The other involves using the quantum bracket of the time derivative of an operator in the equation of motion. Both have advantages and difficulties, which are discussed, along with several simple examples, the conclusion being that these q-quantum mechanics appear to be sensible.

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ABSTRACT

Two formulations of q-quantum mechanics based on quantum deformations of the Heisenberg equations of motion, are discussed. In one the commutator is replaced by the "quommutator": $[A,B]_q = qAB - (1/q)BA$. The other involves using the quantum bracket of the time derivative of an operator in the equation of motion. Both have advantages and difficulties, which are discussed, along with several simple examples, the conclusion being that these q-quantum mechanics appear to be sensible.

1. Introduction and Motivation

This is a report of work done in collaboration with A. Chodos. [1]

Quantum groups [2], or, more properly, a quantized deformation of a universal enveloping algebra, $U_q(g)$, typically for some Lie algebra g, have been found to play important roles in integrable systems, including exactly solvable lattice models, as well as conformal field theories (arising in statistical mechanics systems and in string theories),and in topological field theories and related knot theory. In all of these the Yang-Baxter equation plays a central role. [2,3] In much of this work the role of quantum groups, while crucial in the analysis of these systems, has not been that of a direct symmetry.

However, in the work of Pasquier and Saleur [4] the generators of the quantum Lie algebra $su(2)_q$ actually commute with the Hamiltonian of the one-dimensional spin chain they investigated, and so $su(2)_q$ is a direct symmetry of the system. Yet in the cases when the deforming parameter q is not a root of unity, it has been pointed out [5] that, by making use of the deforming maps constructed by Curtright and Zachos [6], these Hamiltonians are also necessarily invariant under ordinary su(2). In fact there is a general theorem [5] that if the Hamiltonian is invariant under g_q, it is also invariant under g, since U(g) and $U_q(g)$ are isomorphic as algebras. So at least in these $q^n \neq 1$ cases, it seems that these spin chain models do not exhibit any radically new physics. This is confirmed by the identical representation content of the deformed algebra and its classical parent when $q^n \neq 1$. Of course, there are striking and significant differences when q is a root of unity, and, not surprisingly, that is where most of the interest and work has been concentrated.

In connection with the study of quantum groups, there have been various investigations of deformed harmonic oscillator algebras and thereby deformed Heisenberg algebras [7]. For example, one can use the "quommutator" instead of the commutator to get the deformed Heisenberg algebra for a single oscillator: $qbb^\dagger - (1/q)b^\dagger b \equiv [b,b^\dagger]_q = 1$. These have been studied for their own sake, and also for constructing representations of quantum groups.

In the light of all this, a natural question arisies, whether it may be possible to use the idea of a quantum deformation in a yet more central way, directly in the dynamics and structure of quantum mechanics itself, so that one considers a quantum deformed quantum mechanics, or more compactly (and with less emphasis on the apparent pleonasm due to the misnominous *quantum* in quantum groups), a q-quantum mechanics. This is guaranteed to have immediate physical consequences, and, it appears so far, that they can be made sensible.

There are a number of apparently inequivalent ways to deform the quantum equations of motion, but we concentrate here on only two. In order to explain them, it may be useful to consider the q-deformed su(2)-algebra, $su_q(2)$ (which is really that of $sl_q(2)$). The classical algebra can be written:

$$[J_0\ , J_\pm] = \pm 2\, J_\pm \quad ; \qquad [J_+\ , J_-] = J_0 \ . \tag{1}$$

Then one way of writing $su_q(2)$ is

$$[J_0\ , J_\pm] = \pm 2\, J_\pm \ ; \qquad [J_+\ , J_-] = [J_0]_q \ , \tag{2}$$

with the q-bracket defined as

$$[x]_q = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \ . \tag{3}$$

As $q \to 1$, the classical algebra is recovered.

Another appraoch [8] is to deform the left side of the su(2)-commutator relations instead of the right, so that one uses the quommutator defined above:

$$[J_0\ , J_\pm]_{s^2} = \pm J_\pm \ ; \qquad [J_+\ , J_-]_{1/s} = J_0 \ . \tag{4}$$

We have used both methods to deform the basic dynamical laws of quantum mechanics itself.

2. Q-Quantum Mechanics

Following these approaches for writing the commutation relations of a quantum group, but now using them in the Heisenberg equations of motion, with no suggestion of a quantum group structure, we have two ways (at least) to deform. Letting $O(t)$ be an observable in the theory, we have:

$$\text{method 1)} \qquad i\,[H(t)\,,O\,(t)\,]_q = i\,[qHO - (1/q)O\,H] = \dot{O}\,(t) \quad ; \qquad (5)$$

$$\text{method 2)} \qquad i\,[H(t)\,,O\,(t)\,] = [\dot{O}\,(t)]_q \;. \qquad (6)$$

As we shall see, each method has its pros and cons. Method 1) appears to have a problem with unitarity, but one can formally integrate the equations and one can make sense of expectation values of observables. Method 2) is highly non-linear and so, difficult to integrate; however, it appears to be unitary.

Beginning with the first method, Eq. 5, we see that by using the Hamiltonian H itself in the deformed equation of motion, we find that H is not a constant of the motion:

$$\dot{H}(t) \;=\; i\,[H(t)\,,H(t)]_q \;=\; i\,r\,H^2(t)\,, \qquad (7)$$

with

$$r \equiv q - 1/q\;. \qquad (8)$$

Eq. 7 can be formally integrated, writing $H_0 = H(t{=}0)$, to give the time-development of H:

$$H(t) \;=\; \frac{H_0}{1 - i\,r\,H_0\,t}\;. \qquad (9)$$

Having obtained an expression for H(t), we can now use it to solve Eq. 5 formally for $O\,(t)$:

$$O\,(t) \;=\; [\,1 - irH_0t\,]^{-q/r}\,O\,(0)\,[\,1 - irH_0t\,]^{1/qr}\;. \qquad (10)$$

If it happens that $[H_0, O\,(0)] = 0$, then using

$$-q/r \;+\; 1/qr \;=\; -1\,, \qquad (11)$$

we find for this case,

$$O(t) = \frac{O(0)}{1 - i\, r\, H_0\, t} \quad . \tag{12}$$

It may be of interest and amusing to observe [9] that if we rather boldly let $[H, t]_q = 0$, then indeed $H(t) = H_0$, i.e., a constant. So we have the usual time-evolution and no apparent problems with unitarity. But this requires a new and different definition of time for each Hamiltonian!

Going back to Eq. 10, we note that

$$\lim_{q\to 1} [\, 1 - irH_0 t\,]^{-q/r} = \exp(iH_0 t) \tag{13a}$$

and

$$\lim_{q\to 1} [\, 1 - irH_0 t\,]^{1/qr} = \exp(-iH_0 t) \ , \tag{13b}$$

so that we recover the usual time-evolution.

Defining the operators $S = [1 - irH_0 t]^{1/qr}$ and $S' = [1 - irH_0 t]^{-q/r}$, we note that they cannot be inverse to each other (except for $q = \pm 1$), since as seen above $S'S = [1 - irH_0 t]^{-1}$. Hence time-evolution is not unitary. However, if we restrict q to the unit circle,

$$q = e^{i\theta} \ , \quad r = 2\, i \sin\theta \ , \tag{14}$$

then, so long as $[1 + 2\sin\theta \ H_0 t] > 0$, $S' = S^\dagger$, so that hermiticity is preserved under time-evolution. (Note that if one treats q as a formal variable, as is usually done with quantum groups, and so one does not compex conjugate it when taking the hermitian adjoint of an expresssion in which q appears, then one cannot preserve hermiticity. However, we do not have the formal structure of a quantum group here, so how one treats q appears to be a matter of choice.) It should be noted that even for the condition in Eq. 14, there will be times when the above inequality is violated, so that there is no general guarantee that even hermiticity is preserved for all times. But this appears to be the best one can do, so we shall limit our discussion to the case $q = e^{i\theta}$.

Despite these problems, we have found that it is possible to have quite sensible and finite expectation values of operators. To discuss this, let us go to the Schrödinger representation [1] for an operator $O(t)$ obeying Eq. 10.

Then the operator $O_S(t)$ is time-independent, $O_S(t) = O$, and time-evolution is given by

$$\overline{O}(t) = \frac{\langle \Psi(t) | O | \Psi(t) \rangle}{\langle \Psi(t) | \Psi(t) \rangle} . \tag{15}$$

We expand the numerator and denominator in eigenstates of H_0,

$$H_0 |n\rangle = \lambda_n |n\rangle \quad , \tag{16}$$

(including a continuous spectrum, if there, despite the notation) to obtain for the dominant term at $t = t_k = -(2\sin\theta\ \lambda_k)^{-1}$,

$$\overline{O}(t_k) = \langle k | O | k \rangle \ , \tag{17}$$

which is, in general, finite. Hence even though $\langle \Psi(t_k) | \Psi(t_k) \rangle$ may be infinite, the expectation value of an operator remains finite.

It also can be demonstrated that in the limit $|t| \to \infty$, $\overline{O}(t)$ is well-defined, despite $\langle \Psi | \Psi \rangle \to 0$.

So eigenstates of H_0 are indeed still "stationary states" since $\langle k | O | k \rangle$ is time-independent.

At this point it may be instructive to look at an example from classical mechanics, in order to get a clearer notion of the unusual dynamics present here. Hence, we consider a q-deformed Poisson bracket,

$$\{A, B\}_q \equiv q \frac{\partial A}{\partial x} \frac{\partial B}{\partial p} - (1/q) \frac{\partial B}{\partial x} \frac{\partial A}{\partial p} \ . \tag{18}$$

Using this we deform Hamilton's equations of motion to:

$$\dot{x} = \{x, H\}_q \ ; \qquad \dot{p} = \{p, H\}_q \ . \tag{19}$$

Then for a conventional Hamiltonian of the form $H = (p^2/2m) + V(x)$, one finds

$$\dot{x} = q\,(p/m)\ ; \qquad \dot{p} = -\,(1/q)\,V'(x)\,, \tag{20}$$

and so one obtains the changed equation of motion for x: $m\ddot{x} = q\dot{p} = -V'(x)$. This is despite the fact that

$$\dot{H} = \{H, H\}_q \neq 0\,, \tag{21}$$

so the Hamiltonian is not conserved. Nevertheless,

$$\tilde{H} = q(p^2/2m) + (1/q)\,V(x) \tag{22}$$

is conserved, $\{\tilde{H}, \tilde{H}\}_q = 0$. Hence, just because H is not conserved, is not necessarily fatal: there are other constants of the motion, and the dynamics may be quite sensible. In the case of quantum mechanics, one may use the constant operator H_0 , whose eigenstates form a useful basis, and whose eigenvalues can be thought of as energy.

In considering quantum mechanics examples, we restrict our discussion to one-dimensional cases, and, furthermore, we assume usual commutation relations for X and P, so [X, P] = i. One could have, of course, also used variables X' and P' which quommute, $[X', P']_q = i$. We chose the simpler commuting case, but it should also be noted there exits an admittedly non-linear map between the two kinds of variables:

$$X' = f(G)\,X\ ; \quad P' = h(G)\,P\,, \tag{23}$$

where the anti-hermitian operator G = (XP + PX) / 2i , and so [G, P] = P , and [G, X] = -X, and

$$h(G+1) = \frac{1}{f(G)\,(G+1/2)}\left[\frac{C}{q^{2G}} + \frac{q}{q^2-1}\right], \tag{24}$$

with C an arbitrary constant.

One may also consider whether it may be possible to find mapping functions similar to f(G) which transform H and any operator O so that they obey the usual commutator Heisenberg equations of motion rather than the quommutator Eq. 6. We have searched for such maps, but due to the right-hand-side of Eq. 6 being another operator, and the time derivative of O itself to boot, rather than simply i as in the X, P case, our attempts to find such maps have been unsuccessful so far. So to the extent that one may assume such maps do not exist, it seems that the quommutator deformation Eq. 6 is not trivially equivalent to the undeformed case.

We now go to the case of a "free" particle: $H = P^2/2m$. Since $[H, P] = 0$, $P(t)$ follows Eq. 12,

$$P(t) = P_0 \,[1 - (irP_0{}^2/2m)t]^{-1} \cdot \tag{25}$$

To find the solution for X(t) requires a bit more work, with the result that

$$X(t) \;=\; X_0\,[1 - (irP_0{}^2/2m)t]^{-1} \;+\; qP_0 t\,[1 - (irP_0{}^2/2m)t]^{-2} . \tag{26}$$

Also in this simple case, $X = X^\dagger$ for all t. Furthermore, one can determine that the initial commutation relation, $[X, P] = i$, is preserved in time in this case.

The second example we studied was, of course, the harmonic oscillator: $H = (P^2 + X^2)/2$. The expressions for X(t) and P(t) are rather complicated now, but they are explicitly given in ref. 1. From them it can be seen that the initial relation, $[X, P] = i$, is now not obviously preserved in time, and this is most likely the generic behaviour.

Let us turn to our second method of deforming the Heisenberg equations of motion, Eq. 6. Thus we assume the equation of motion of an operator *O(t)* to be given by:

$$i\,[\,H(t), O\,(t)\,] = [\dot{O}\,(t)]_q = \frac{q^{\dot{O}/2} - q^{-\dot{O}/2}}{q^{1/2} - q^{-1/2}} \;. \tag{27}$$

Again, as $q \to 1$ we recover the usual equations of motion. Unlike the quommutator case of Eq. 5, this deformation has the obvious advantage that, as usual, operators commute with themselves, so that, in particular, the Hamiltonian itself is a constant of the motion and energy in conserved. Furthermore, hermiticity appears to be preserved in the cases when q is treated as a formal variable, or when $q = e^{i\theta}$.

In order formally to integrate Eq. 27, let us write $q = e^h$. Then $[x]_q = \sinh(hx/2) / \sinh(h/2)$, and, writing $\tilde{\hbar} \equiv \sinh(h/2)$, we find for O (t):

$$\frac{2t}{h} = \int_{O(0)}^{O(t)} \frac{dO}{\sinh^{-1}(i\tilde{\hbar}\,[H,O\,])} . \tag{28}$$

Due to the non-linearities, it is difficult to proceed further in the general case. If one wishes to go to the Schrödinger representation, then one probably has to use a non-linear Schrödinger equation [10].

But for the free particle the differential equations separate simply, and p is constant, so that we find for X(t):

$$X(t) - X(0) = t\,(2/h)\,\sinh^{-1}(\tilde{\hbar}\;p)\;, \tag{29}$$

which is a well-behaved, single-valued function. This result is certainly different from the usual quantum mechanical result, X(t) = pt, but not only is it quite sensible, it also reduces to this usual behaviour as $q \rightarrow 1$. It is also apparent that the behaviour given by Eq. 29 is very different from that obtained from the quommutator deformation, Eq. 26. Time evolution is obviously unitary here in this free case. Although we have not yet been able to establish unitarity in general, we suspect it to be there. These are yet more indications that the two methods are not equivalent.

When we seek to go beyond the free particle, the non-linearity of this deformed quantum mechanics confronts us squarely. Even for the harmonic oscillator, the coupled non-linear differential equations,

$$\dot{X}(t) = (2/h)\,\sinh^{-1}[\tilde{\hbar}\;P(t)]\;,\quad \dot{P}(t) = (2/h)\,\sinh^{-1}[\tilde{\hbar}\,X(t)]\;, \tag{30}$$

do not lend themselves to a closed-form solution, although a numerical solution is, of course, possible.

3. Concluding Remarks

The two methods we have studied are not the only possible deformations one could employ, as there are others suggested from a consideration of quantum groups. For example, there exists the so-called quantum derivative [11]:

$$D_u f(u) \equiv \frac{f(q^{1/2}u) - f(q^{-1/2}u)}{u(q^{1/2} - q^{-1/2})} , \tag{31}$$

which becomes the usual derivative when $q \rightarrow 1$. So one could also consider as deformations of the Heisenberg equations: $D_t\mathcal{O} = i[H, \mathcal{O}]$, which is similar to our second method; or $D_t\mathcal{O} = i[H, \mathcal{O}]_q$, which is similar to a combination of methods 1) and 2). However, both of these suggestions are even harder to integrate than Eq. 6. Hence, we have avoided a detailed study of these choices simply on pragmatic grounds.

Besides its intrinsic interest and its value as a tool for probing the essentials of mechanics, one of the possible practical implications of a q-quantum mechanics lies in the suggestion that the quantum deformation parameter acts as a cutoff which discretizes the time-evolution of a system. This comes about from looking at the quantum time-derivative above, which clearly appears as a discretization. From another viewpoint, the time-evolution operator S(t) introduced earlier can also be considered a discretization, now of the exponential e^{-iH_0t}.

One of the many open questions is whether and how either or both of our methods of deforming the Heisenberg equations, can be generalized to many degrees of freedom and finally field theory. This will be of interest, since it is clear much can be learned from these deformations. We saw this, for example, in studying Eq. 5, where there is a lack of unitarity. However, what the lack of unitarity seems to imply, is that just as in the passage from classical to usual quantum mechanics one gives up determinism in favour of probability amplitudes which are defined as matrix elements in a Hilbert space, here one is abandoning the physical significance of the Hilbert space inner product but, it appears, retaining the meaning of an expectation value, the appropriate ratio of such inner products. All in all, q-quantum mechanics is an excellent laboratory for exploring what is truly essential for a sensible mechanics.

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