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Multicut Criticality in the Penner Model and c=1 Strings†

S. Chaudhuri, H. Dykstra, and J. Lykken

Theory Group, MS106
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, IL 60510

The steepest descent solution of the mth critical point of the Penner matrix model has an m-component eigenvalue support, consisting of symmetrically placed arcs in the complex eigenvalue plane. Criticality results when the branch points of this support coalesce in pairs to form a closed contour. We derive the string equations of these matrix models for arbitrary m, using the orthogonal polynomial method. The double-scaled continuum solutions are described by non-linear finite-difference equations. The free energy of the mth model is shown to be the Legendre transform of the free energy of the c=1 string compactified to a circle of radius equal to an integer multiple, m, of the self dual radius.

7/91

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1. Introduction

In this talk, I will give an overview of recent work done in collaboration with Hans Dykstra and Joe Lykken on multicut criticality in the Penner matrix model. I will particularly enlarge on aspects that have not been elaborated upon in our earlier work.

The Penner matrix model was first introduced as a means of computing the orbifold Euler characteristic of the moduli space of punctured Riemann surfaces [1][2]. Penner's basic observation, building on earlier work by Harer and Zagier, was to note that a triangulation of moduli space based on "fat" graphs leads very naturally to a combinatorics problem that is efficiently solved by the Feynman diagram expansion of a hermitian matrix model. (A very readable introduction to this work is given in [4][5].) Since in computing topological invariants on moduli space one must include surfaces with infinitely many punctures, the diagrammatic expansion contains arbitrarily high order vertices, and the corresponding matrix potential is *non-polynomial*. Using the techniques of [6], Penner succeeded in computing exactly the large N expansion for the sum over connected diagrams in this model.

Distler and Vafa asked the question whether this ($\frac{1}{N}$) expansion could be made ***critical***, and thereby identified with the continuum free energy of a theory of two-dimensional gravity. The surprising answer is that this is indeed possible. The double-scaled free energy coincides with the Legendre transformed free energy (the generating function of 1PI amplitudes) of the $c = 1$ string with compact target space at the self-dual radius [4][7]. Recently, we discovered that this property extends to an infinite series of matrix model solutions that are polynomial perturbations of the Penner model [8]. Their free energy coincides with the generating function of 1PI amplitudes of the c=1 string with compact target space of radius equal to some integer, m, of the self-dual radius.

The earliest analysis of the phase structure of the Penner model, and an elucidation of its critical behavior was done by C-I Tan [9]. Two puzzles had been left unanswered by Distler and Vafa's result. The first was the nature of criticality in the Penner model. It was unclear what feature of the large N behavior had been used to tune the couplings to criticality. The other difficulty was in the application of orthogonal polynomial techniques which would necessarily require the computation of matrix elements of *non-polynomial* operators, such as $\hat{\phi}^{-1}$.

In this talk, I hope to leave you with the answers to both of these puzzles. Unfortunately, I will not be able to discuss a much deeper mystery which is the meaning of this curious correspondence between the c=1 matrix model (a 1+1 dimensional field theory) and a matrix model in zero embedding dimensions (naively, a 1 dimensional field theory). But that is the subject of future work.

2. The Multicut Solutions

In general, the matrix integral

$$e^F = \int dM e^{Nt\ Tr(U(M)+log(1-M))} \tag{2.1}$$

is ill-defined in the neighbourhood of $t = -1$ both because of the branch cut of the logarithm and the possibility that the polynomial part of the potential $U(M)$ is unbounded from below. Changing variables to $\Phi = 1 - M$, and diagonalizing the matrix Φ, we get

$$e^F = \int_C \prod_{i=1}^{N} d\lambda_i \lambda_i^{Nt} e^{Nt\ U(1-\lambda_i)} det(P_j^{(Nt)}(\lambda_k)) \tag{2.2}$$

where C is a suitably defined integration path over the eigenvalues λ_i of Φ. In the Penner model, for example, one could restrict C to lie along the positive half of the real axis (to avoid the branch cut) and define the integral by an analytic continuation in the overall coupling, $t \to -t$, so that (2.2) defines an orthonormal measure for the associated Laguerre polynomials [3][4].

In what follows, however, we use an alternative regularization of the integral that allows us to approach the critical point smoothly by analytic continuation from a stable large N solution of the generic complex potential [10]. We define the matrix integral by choosing for each λ_i a complex integration path C as follows. We first rotate λ_i so that the polynomial part of the potential is bounded for large λ_i. This determines the correct asymptotes for C in the complex λ_i plane. The contour C must connect these asymptotes smoothly to the large N eigenvalue support, C_o, avoiding the branch cut of the logarithm, and passing only through regions in the complex plane where the large N solution is stable [10]-[12].

2.1 Eigenvalue Analysis. Consider the case where $U(M)$ is quadratic, which defines the KT model [8]. This has the potential

$$V(\Phi) = -\frac{1}{2}\Phi^2 - log(\Phi) \tag{2.3}$$

The model has a smooth, normalizable, and stable, large N solution for the eigenvalue distribution density, described by the generating function [6][11]

$$F(\lambda) = \frac{t}{2}(V'(\lambda) - G'(\lambda)) = \frac{t}{2}\left(-\lambda - \frac{1}{\lambda} - \frac{\sqrt{(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - a_4)}}{\lambda}\right) \quad (2.4)$$

where the a_i are solutions to $a^4 + 2(1 + 2/t)a^2 + 1 = 0$. The action for a single eigenvalue ($G'(\lambda)$ is proportional to the eigenvalue density) is given by

$$G(\lambda) = \frac{1}{2}(\lambda^2 - 1) - log(\lambda) + 2\pi i. \quad (2.5)$$

We have chosen the branch cut of the logarithm to lie along the negative real axis and fixed the integration constant so that $G(\lambda)$ runs from 0 to $2\pi i$ as we move just outside the piecewise connected square root branch cuts from $a_1 \to a_3$, $a_4 \to a_2$. The large N eigenvalue support, C_o, is the contour $Re(G(\lambda)) = 0$, composed of two cuts whose end points a_{2i}, a_{2i-1} coalesce in pairs at criticality, $t \to -1$. Introducing a cut-off δ, $t = -1 - \delta^2\mu$, in the limit $\delta \to 0$, the end-points of the two cuts approach each other in pairs parallel to the imaginary axis† and the two disconnected pieces of the support join to form a closed loop. The large N support can be smoothly extended to the contour C shown in Figure 1 [8]. It is easy to verify that the solution is stable under small shifts in the eigenvalue density ($Re(G(\lambda))$ is positive) for any smooth deformation of C in the shaded region of the figure. Moreover, the planar free energy can be calculated directly from the integral [8]

$$E_0 = \int_{C_o} d\lambda u(\lambda)V(\lambda) + \frac{1}{2}\int_{C_o} d\lambda \int_{C_o} d\mu u(\lambda)u(\mu)log(\lambda - \mu)^2 \quad (2.6)$$

and the leading order contribution can be shown to diverge logarithmically,

$$E_0 \to \frac{1}{2}\mu^2 log\mu + \cdots \quad (2.7)$$

as in the case of the Penner model [8].

† If, instead, we approach t_c from *above*, the end-points approach each other along the real axis and consistency with the saddle-point equation would introduce a pole in the generating function. The appearance of the planar logarithmic scaling violations in this case is subtle [9].

Finally, we recall a well-known expression relating the eigenvalue analysis and the orthogonal polynomial treatment [13]. The generating function $F(\lambda)$ is defined by the sum (we use the notation of [15])

$$F(\lambda) \;=\; \sum_{n=1}^{N} \left\langle n \middle| \left(\lambda - \hat{\phi}\right)^{-1} \middle| n \right\rangle \tag{2.8}$$

and can be computed in terms of the coefficients appearing in the three-term recursion relation satisfied by the orthogonal polynomials:

$$\hat{\phi}|n\rangle \;=\; (S_n + \sqrt{R}_n e^{i\theta} + e^{-i\theta}\sqrt{R}_n)|n\rangle \tag{2.9}$$

For a two-cut solution, the Hilbert space is composed of two sub-sectors and the operator $\hat{\phi}$ is matrix valued [14]-[18]. The recursion coefficients converge to two distinct functions, depending on whether n is even or odd. In the large N limit,

$$(\lambda - \hat{\phi}) \;=\; \begin{pmatrix} S_e(x) & e^{-i\theta}\sqrt{R}_e(x) + \sqrt{R}_o(x)e^{i\theta} \\ e^{-i\theta}\sqrt{R}_o(x) + \sqrt{R}_e(x)e^{i\theta} & S_o(x) \end{pmatrix} \tag{2.10}$$

and the generating function is

$$F(\lambda) \;=\; \int_0^1 dx \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{2\lambda - (S_e + S_o)}{\left[(\lambda - S_e)(\lambda - S_o) - (R_e + R_o) - \sqrt{R}_e\sqrt{R}_o(e^{2i\theta} + e^{-2i\theta})\right]} \tag{2.11}$$

We will shortly demonstrate that this expression does indeed reproduce (2.4)when we substitute for $R_{e[o]}$, $S_{e[o]}$ from the string equations obtained in the next section.

2.2 Orthogonal Polynomial Analysis. The string equations are simple to derive except that they involve matrix elements of the operator $\hat{\phi}^{-1}$ due to the logarithmic piece in the potential. Evaluating this non-polynomial matrix element as a formal sum, we obtain [9][8]

$$\frac{2n+1}{Nt} \;\equiv\; \left\langle n|\hat{\phi}V'(\hat{\phi})|n\right\rangle \;=\; g_2(S_n^2 + R_n + R_{n+1}) - 1 \tag{2.12}$$

$$\begin{aligned} \frac{n}{Nt} &\equiv \left\langle n-1|V'(\hat{\phi})|n\right\rangle \\ &= g_2 R_n - \left(1 + \frac{R_n}{S_n S_{n-1}} + \frac{R_n}{S_n S_{n-1}}\left[\frac{R_n}{S_n S_{n-1}} + \frac{R_{n-1}}{S_{n-1}S_{n-2}} + \frac{R_{n+1}}{S_{n+1}S_n}\right] + \cdots\right) \end{aligned} \tag{2.13}$$

where $g_2 = t = -1$ at criticality, and $S_c^2 = 1,\; R_c = 0$. To take the double scaling limit, we introduce a cut-off δ, and define the renormalized variables [20]-[22]

$$\nu^{-1} = N\delta^2, \quad z = (x_c - x)N, \quad \mu = (t_c - t)N \tag{2.14}$$

Note that the potential, and eigenvalue distribution, are symmetric under rotations $\lambda \to e^{\pi i}\lambda$, but only up to shifts of the log branch cut. We will assume a scaling ansatz consistent with the symmetries of our two-cut solution

$$\begin{gathered} S_n = 1 + \delta\sigma(z) + \delta^2\sigma_1(z), \quad S_{n-1} = \omega(1 + \delta\sigma(z+1) + \delta^2\sigma_1(z+1)) \\ R_n = \delta\rho(z) + \cdots, \quad R_{n-1} = \delta\rho(z+1) + \cdots \end{gathered} \tag{2.15}$$

where $\omega = e^{\pi i}$. We note, then, that the string equations are identical for even or odd n, so that we have only two distinct equations to solve. To $O(\delta)$, (2.12) gives the constraint

$$\sigma(z) \;=\; -\frac{1}{2}\left(\rho(z) + \rho(z-1)\right) \tag{2.16}$$

while the $O(\delta^2)$ condition simply allows us to solve for the function $\sigma_1(z)$ in terms of $\sigma(z)$. (2.13) is trivially solved to $O(\delta)$ with no new constraints, and at $O(\delta^2)$ yields an equation for $\rho(z)$:

$$\rho(z)\left[\rho(z+1) + \rho(z-1)\right] \;=\; 2\nu(\mu + z) \tag{2.17}$$

Now, we note that any solution to the equation

$$\rho(z)\rho(z-1) \;=\; \nu(\mu - \frac{1}{2} + z) \tag{2.18}$$

is a solution to (2.17). But a solution to (2.18) is easy to construct since this is nothing but a well-known gamma function identity. Shifting $\mu \to \mu + \frac{1}{2}$ we find that

$$\rho(z) = \sqrt{2\nu}\frac{\Gamma\left(\frac{z+\mu+2}{2}\right)}{\Gamma\left(\frac{z+\mu+1}{2}\right)} \tag{2.19}$$

is a solution to (2.17), which allows us to reconstruct

$$R_n \sim \sqrt{\frac{2}{N}}\frac{\Gamma\left(\frac{N-n+\mu+2}{2}\right)}{\Gamma\left(\frac{N-n+\mu+1}{2}\right)} \tag{2.20}$$

A little algebra [8] then yields the complete free energy, in the double scaling limit, and in discrete form:

$$E = log\prod_{k=1}^{N-1} R_{N-k}^k \;\to\; \sum_{k=1}^{N/2-1} klog[(2k+\mu+1)(2k+\mu-1)] \tag{2.21}$$

Finally, as promised, we will take the large N limit of our string equations and compute the generating function (2.11) in order to compare with what we obtained directly from the eigenvalue analysis. In the large N limit, $S_e + S_o = 0$, $S_e S_o \equiv -S^2$, and $R_e = R_o \equiv R$. Substituting in (2.11), and performing the θ integration, gives

$$F(\lambda) \;=\; \int_0^1 dx \frac{\lambda}{[(\lambda^2 - (S^2 + 2R))^2 - 4R^2]^{1/2}} \tag{2.22}$$

Now, in the large N limit, (2.12) and (2.13) reduce to

$$S^2 + 2R = -(\frac{2x}{t} + 1), \quad 2R + \frac{S}{\sqrt{S^2 + 4R}} = -(\frac{2x}{t} + 1) \quad \rightarrow \quad R^2 = \frac{x(x+t)}{t^2} \tag{2.23}$$

Substitution in (2.22) yields (2.4) after a trivial integration, exactly as expected.

Our presentation of the KT model here differs slightly from that given in Ref.[8]. The derivation is more straightforward, and obviates the concerns raised in Ref.[23].

2.3 The General Case. It is easy to generalize these results to the case when $U(\lambda)$ is a higher order polynomial. In general, we have an m-cut solution for the planar eigenvalue density with the arcs symmetrically placed about the origin, and coalescing at their end-points to form a closed loop at criticality. The locations of the pairs of end-points at criticality are given by the m distinct roots of unity, $S_c^m = -1$, $R_c = 0$, and the large N support is described by the transcendental equation $Re(G(\lambda)) = 0$, where $G(\lambda)$ is obtained from the generating function. Corresponding to the potential $V = (-1)^{m-1}\lambda^m/m - log(\lambda)$, the generating function $F(\lambda)$ is given by

$$F(\lambda) = \frac{t}{2}\left((-\lambda)^{m-1} - \frac{1}{\lambda} - \frac{1}{\lambda}\sqrt{(\lambda - a_1)\cdots(\lambda - a_{2m})}\right) \tag{2.24}$$

or, equivalently, as will follow from the large N limit of the string equations,

$$F \;=\; \int_0^1 dx \frac{\lambda^{m-1}}{\{[(-1)^{m-1}\lambda^m - (1 + 2x/t)]^2 - 4x(x+t)/t^2\}^{1/2}} \tag{2.25}$$

The recursion coefficients converge to $2m$ different functions in the continuum limit. However, due to the special symmetry of these solutions they are related by phase rotations, and through translations of the argument:

$$S_{n-l} = \omega^l(1 + \delta^{2/m}\sigma(z+l) + \delta^{4/m}\sigma_1(z+l) + \cdots), \quad l = 0, \cdots, m-1 \tag{2.26}$$

$$R_{n-l} = \omega^{2l}(\delta^{2/m}\rho(z+l) + \delta^{4/m}\rho_1(z+l) + \cdots) \tag{2.27}$$

where $\omega = e^{2\pi i/m}$.

The string equations (2.12) and (2.13), with additional contributions from the polynomial part of the potential, can be solved in the double scaling limit as before. (2.12) allows us to eliminate $\sigma(z)$, $\sigma_1(z),\cdots$ in terms of $\rho(z)$, while (2.13) reduces to the equation

$$\sum_{l=0}^{m-1} \rho(z+l)\rho(z+l-1)\cdots\rho(z+l-m+1) \;=\; \nu m(z+\mu) \tag{2.28}$$

In the large N limit, $S_n, R_n \rightarrow S, R$, the string equations will simply reduce to $R^m = x(x+t)/t^2, \quad 1+2x/t = \langle n|(-\hat{\phi})^{m-1}|n\rangle$. The second of these equations is a polynomial equation relating $S(x)$ to $R(x)$.

It is easy to check (after a shift of $\mu \rightarrow \mu + \frac{1}{2}$ for even m) that the following expression is a solution to (2.28)

$$\begin{aligned} \rho(z)_{even} &= \sqrt[m]{m\nu}\,\frac{\Gamma\left(\frac{z+\mu+1+m/2}{m}\right)}{\Gamma\left(\frac{z+\mu+m/2}{m}\right)} \\ \rho(z)_{odd} &= \sqrt[m]{m\nu}\,\frac{\Gamma\left(\frac{z+\mu+1+(m-1)/2}{m}\right)}{\Gamma\left(\frac{z+\mu+(m-1)/2}{m}\right)} \end{aligned} \tag{2.29}$$

which, on solving for the corresponding R_n, yields the double-scaled free energy in discrete form

$$\begin{aligned} E_{even} &\sim \sum_{k=1}^{N/m} k \left[\sum_{l=-m/2}^{m/2} log(mk+\mu+l) \;-\; log(mk+\mu) \right] \\ E_{odd} &\sim \sum_{k=1}^{N/m} k \sum_{l=-(m-1)/2}^{(m-1)/2} log\left[mk+\mu+l\right] \end{aligned} \tag{2.30}$$

where we have dropped irrelevant divergent constants, and used similar manipulations as in [8]. The free energy is equivalent to the Legendre transform of the free energy of the $c=1$ string with radius equal to m times the self dual radius.

For completeness, let us apply this analysis to the first, non-trivial, odd member of this series, $m=3$. The potential is invariant under rotations $\Phi \rightarrow e^{2\pi i/3}\Phi$. This leads to the large N eigenvalue distribution shown in Figure 2, consisting of three symmetrically placed cuts, which join at the ends at criticality to form a closed ring. The generating function can be expressed in the form:

$$F(\lambda) = \frac{t}{2}\left(\lambda^2 - \frac{1}{\lambda} - \frac{\sqrt{(\lambda-a_1)\cdots(\lambda-a_6)}}{\lambda}\right) \tag{2.31}$$

where the end-points are the solutions to $a^6 - 2(1+2/t)a^3 + 1 = 0$. For $t = -1$, these branch points coalesce in pairs at $a = -\omega^n$, $n = 0,1,2$ where $\omega = e^{2\pi i/3}$.

As is appropriate for this three-cut solution, we assume that in the continuum limit, R_{3n}, R_{3n+1}, and R_{3n+2} approach different continuous functions, related by phase rotations and by shifts of the argument, and similarly for the S recursion coefficients. A straightforward expansion of the matrix elements yields the string equations

$$\frac{2n+1}{Nt} = g_3(S_n^3 + 2S_n(R_n + R_{n+1}) + R_{n+1}S_{n+1} + R_nS_{n-1}) - 1 \tag{2.32}$$

$$\begin{aligned}\frac{n}{Nt} &= g_3R_n(S_n + S_{n-1}) - \\ &\Big(1 + \frac{R_n}{S_nS_{n-1}} + \frac{R_n}{S_nS_{n-1}}\Big[\frac{R_n}{S_nS_{n-1}} + \frac{R_{n-1}}{S_{n-1}S_{n-2}} + \frac{R_{n+1}}{S_{n+1}S_n}\Big] + \\ &\frac{R_n}{S_nS_{n-1}}\Big\{\Big[\frac{R_n}{S_nS_{n-1}}\Big(\frac{R_n}{S_nS_{n-1}} + \frac{R_{n-1}}{S_{n-1}S_{n-2}} + \frac{R_{n+1}}{S_{n+1}S_n}\Big)\Big] + \\ &[n \to n-1] \; + \; [n \to n+1]\Big\} \; + \; \cdots\Big)\end{aligned} \tag{2.33}$$

We put in the scaling ansatz ($g_3 = 1$ at criticality)

$$\begin{aligned} S_{n-l} &= \omega^l\Big(-1 - \delta^{2/3}\sigma(z+l) - \delta^{4/3}\sigma_1(z+l) + \cdots\Big) \\ R_{n-l} &= \omega^{2l}\Big(-\delta^{2/3}\rho(z+l) - \delta^{4/3}\rho_1(z+l) + \cdots\Big), \quad l = 0,1,2 \end{aligned} \tag{2.34}$$

Equation (2.32) gives equations relating σ, σ_1 and ρ_1 to ρ. Using these relations, (2.33) gives the string equation for ρ at order δ^2:

$$\rho(z+2)\rho(z+1)\rho(z) + \rho(z+1)\rho(z)\rho(z-1) + \rho(z)\rho(z-1)\rho(z-2) = 3\nu(z+\mu). \tag{2.35}$$

Note that all complex phases have cancelled out, and $\rho(z)$ is real. This equation is a simple generalization of the string equations for m=1 and m=2, and is solved by the following form:

$$\rho(z) = \sqrt[3]{3}\,\frac{\Gamma(\frac{z+\mu+2}{3})}{\Gamma(\frac{z+\mu+1}{3})}. \tag{2.36}$$

The pattern of string equations and their solutions for higher values of m is clear. Solving for the R_n as before, we can determine the free energy. Manipulations similar to those performed for the KT model give:

$$F \sim \sum_{k=1}^{N/3} k \log[(3k+\mu+1)(3k+\mu)(3k+\mu-1)] \tag{2.37}$$

which is equivalent to the Legendre transform of the free energy of the c=1 string with radius equal to thrice the self-dual radius. Again, we can cross-check with our eigenvalue analysis by computing the generating function for this three cut solution. We have

$$F(\lambda) = \int dx \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{\sum_{(1,2,3)}(\lambda - S_i)(\lambda - S_{i+1}) - (R_1 + R_2 + R_3)}{\left\{\prod_i(\lambda - S_i) - \sum_{(1,2,3)}(\lambda - S_i)R_{i+1} + \sqrt{R_1 R_2 R_3}(e^{3i\theta} + e^{-3i\theta})\right\}^{1/2}} \tag{2.38}$$

where the sums are taken over cyclic permutations of the indices. The large N limit of the string equations, and our scaling ansatz, $S_1 + S_2 + S_3 = 0$, $S_1 S_2 S_3 = 1$, and similarly for the R_i, gives

$$\frac{2n+1}{Nt} = S_n^3 - 3\omega R_n S_n - 1 \tag{2.39}$$

$$\frac{n}{Nt} = -\tfrac{1}{2}\left(1 - S_n/\sqrt{S_n^2 - 4\omega R_n}\right) - \omega R_n S_n \tag{2.40}$$

which reduce to a cubic equation $S(S^2 - 3\omega R) = 1 + 2x/t$ relating $S_1 \equiv S$ to $R_1 \equiv R$, and the condition $R^3 = x(x+t)/t^2$. Inserting these back in (2.38) and integrating gives the expression (2.31).

3. Conclusions

The Penner matrix model with polynomial perturbations can support a large variety of multicritical behaviors. In this paper, we have focussed on a one-parameter sequence of multicut solutions, singled out by their evident connection to certain $c = 1$ strings. Criticality is achieved when the end-points of the planar eigenvalue support coalesce in pairs to form a closed loop. An additional one-parameter sequence of one-cut multicritical behaviors is obtained when extra zeroes on the real axis, collect on to one, or both, coalescing square root branch points of the eigenvalue density at criticality [8] [23]. It is tempting to speculate that these solutions may be related to the special states of the $c = 1$ matrix model [24]. Finally, it is clear that one can construct solutions in the same universality class as the (p, q) minimal models obtained from the one-matrix model in Douglas' classification. For example, to obtain the kth model with susceptibility exponent $\gamma_0 = -1/k$, one tunes k couplings in the potential such that the eigenvalue density has $(k-1)$ extra zeroes coalescing with the branch point [25]. Of course, since $c < 1$, R_c is non-zero and the branch points no longer coalesce in pairs at criticality.

Figure Captions

Fig. 1. Contours of $Re(G(\lambda)) = 0$ for the KT model, plotted in the complex λ plane. The two arcs (a_1, a_3), (a_4, a_2) form the piece-wise continuous planar eigenvalue support C_o. Stability of the solution is ensured under any continuous deformations of the full integration contour, C, into the shaded regions of the diagram $(Re(G(\lambda)) > 0)$.

Fig. 2. Contours of $Re(G(\lambda)) = 0$ for the three-cut solution.

References

[1] J. Harer and D. Zagier, *Invent. Math.* **85** (1986) 457.

[2] R.C. Penner, *Bull. Am. Math. Soc.* **15** (1986) 73. *J. Diff. Geom.* **27** (1988) 35.

[3] C. Itzykson and J. Zuber, "Matrix Integration and Combinatorics of the Modular Group", Saclay preprint SPhT/90/004.

[4] J. Distler and C. Vafa, *Mod. Phys. Lett.* **A6** (1991) 259.

[5] J. Distler and C. Vafa, "The Penner Model and d=1 String Theory", lecture at the Cargese Workshop on Random Surfaces, Quantum Gravity and Strings, PUPT-1212, May 1990.

[6] E. Brézin, C. Itzykson, G. Parisi, and J.B. Zuber, *Comm. Math. Phys.* **59** (1978) 35.

[7] D. Gross and I. Klebanov, *Nucl. Phys.* **B344** (1990) 475; *Nucl. Phys.* **B354** (1991) 459.

[8] S. Chaudhuri, H. Dykstra and J. Lykken, *Mod. Phys. Lett.* **A6** (1991) 1665.

[9] C-I Tan, *Mod. Phys. Lett.* **A6** (1991) 1373.

[10] F. David, *Mod. Phys. Lett.* **A5** (1990) 1019.

[11] F. David, *Nucl. Phys.* **B348** (1991) 507.

[12] S. Chaudhuri and J. Lykken, "Analyzing the Solutions of Hermitian Matrix Models", FERMILAB-PUB-90/267-T, Nucl. Phys. B, in press.

[13] D. Bessis, *Comm. Math. Phys.* **69** (1979) 147;
C. Itzykson and J. B. Zuber, *J. Math. Phys.* **21** (1980) 411;
D. Bessis, C. Itzykson, and J. B. Zuber, *Adv. Appl. Math.* **1** (1980) 109.

[14] G. Cicuta, L. Molinari and E. Montaldi, J. Phys. **A23** (1990) L421.

[15] D. Gross and A.A. Migdal, *Nucl. Phys.* **B340** (1990) 333.

[16] J. Jurkiewicz, *Phys. Lett.* **245B** (1990) 178.

[17] G. Bhanot, G. Mandal, and O. Narayan, *Phys. Lett.* **252B** (1990) 388.

[18] K. Demeterfi, N. Deo, S. Jain, and C.-I. Tan, *Phys. Rev.* **D42** (1990) 4105.

[19] C. Crnkovic and G. Moore, *Phys. Lett.* **257B** (1991) 322.

[20] M. Douglas and S. Shenker, *Nucl. Phys.* **B335** (1990) 635.

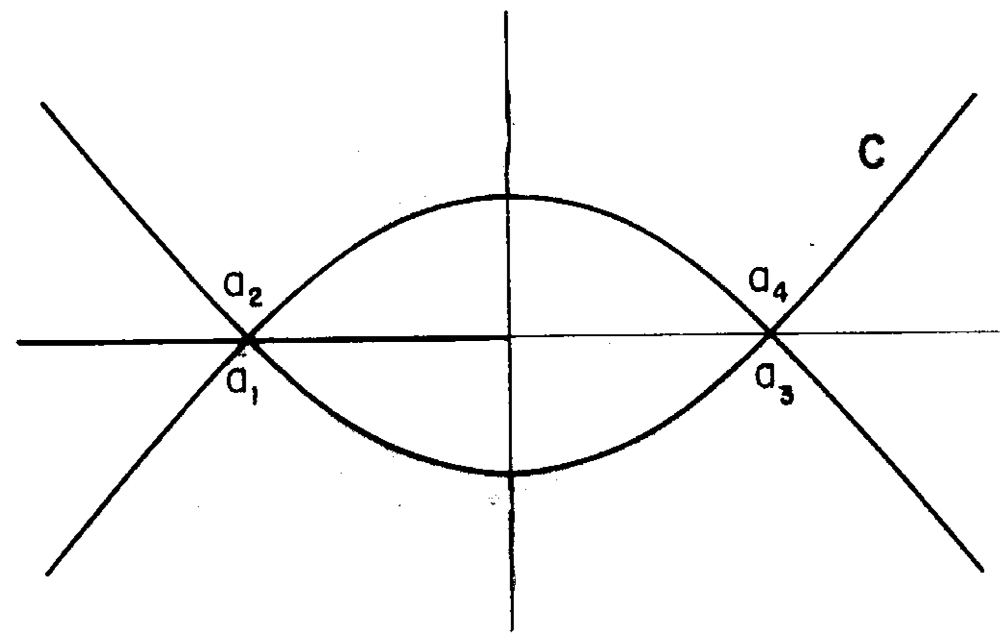
[21] E. Brézin and V. Kazakov, *Phys. Lett.* **236B** (1990) 144.

[22] D. Gross and A. A. Migdal, *Phys. Rev. Lett.* **64** (1990) 127.

[23] C-I Tan, "Generalized Penner Models and Multicritical Behavior", Brown Univ. preprint BROWN-HET-810, May 1991.

[24] U. Danielsson and D. Gross, "On the Correlation Functions of the Special Operators in c=1 Quantum Gravity", Princeton preprint PUPT-1258, May 1991.

[25] E. Alvarez and J. L. F. Barbon, *Phys. Lett.* **252B** (1990) 431.



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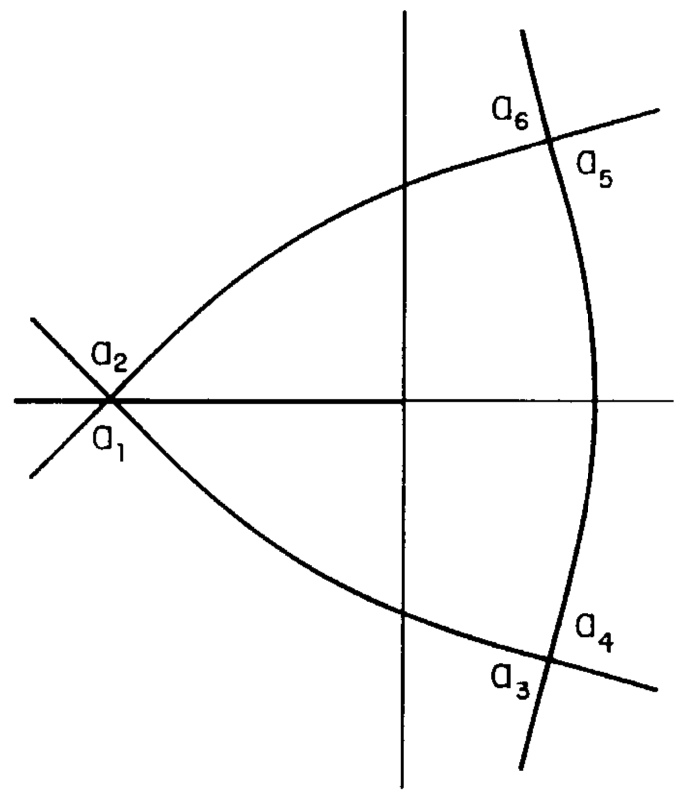

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