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RENORMALIZED LATTICE PERTURBATION THEORY*

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Apparent large failures of perturbation theory in lattice gauge theory at $g^2 \leq 1.0$ result from the use of the bare coupling constant of the lattice regulator in the perturbative expansion. They can be cured by the use of a coupling constant defined by some physical process. This situation is the same as that encountered in dimensionally regularized perturbation theory, where the use of a coupling constant definition suggested by the regulator (the MS scheme) can make the perturbative expansion appear to be more poorly behaved then it really is.

1. INTRODUCTION

Although in principle lattice methods allow the calculation of any quantity in QCD, without recourse to perturbation theory, in practice perturbation theory is important to lattice QCD in several ways. Firstly, it provides the essential connection between low-energy lattice simulations and the high-energy arena of perturbative QCD phenomenology through such methods as the operator product expansion. Secondly, perturbation theory can account for effects on low-energy phenomena due to physics at distance scales shorter than the lattice spacing. Provided the lattice spacing a is small enough, errors of order a and higher can be removed from the theory by perturbatively correcting the action and operators that define the lattice theory. Finally, agreement between lattice Monte Carlo and perturbative results for short distance quantities, where both approaches are expected to be reliable, is necessary in order to have confidence in Monte Carlo calculations of nonperturbative quantities.

It is therefore at first sight disturbing to find many cases in which Monte Carlo results seem to agree poorly with perturbative calculations. To take two examples out of many, the first order perturbative result[1] for the renormalization of the critical hopping parameter κ_c is $\kappa_c = 1/(8 - .869g^2) = .140$ at $\beta \equiv 6/g^2 = 6.0$ using the bare coupling constant of the lattice regulator as the expansion parameter, while Monte Carlo results[2] give $\kappa_c = .157$. Similarly, the first order perturbative result for the expectation value of the lattice gluon field U in Landau gauge is $1 - \frac{1}{3}Tr(U) = .078g^2 = .078$ at $\beta = 6.0$, while Monte Carlo results yield $1 - \frac{1}{3}Tr(U) = .139$.[3]

In this paper we show that the above facts, while true, are misleading, and that lattice perturbation theory at $g^2 \leq 1.0$ (cutoff momenta $\pi/a \approx 6$ GeV) is in fact about as well behaved as familiar dimensionally regularized perturbation at similar momenta. The key point is that α_{lat} is not the best expansion parameter for perturbation theory, despite its natural connection with the lattice regulator. The situation of perturbation theory using lattice regularization is the same as the situation of perturbation theory using the other most common regulator for QCD, dimensional regularization, where the series obtained in terms of α_{MS}, the "natural" coupling constant for dimensional regularization, often looked like nonsense. These series were usually reasonably well behaved when expressed in terms of a coupling constant defined from some physical process, or in terms of an ad hoc definition such as $\alpha_{\overline{MS}}$ chosen to give sensible results for many physical processes. (For a discussion of the issues in dimensionally regularized QCD perturbation theory, see [4].)

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2. SYMPTOMS OF A POOR CHOICE OF EXPANSION PARAMETER

If an expansion parameter α_{good} produces a well behaved perturbation series, an alternative expansion parameter $\alpha_{bad} \equiv \alpha_{good}(1 - 10,000\alpha_{good})$ will have large, uniform second order coefficients of around 10,000. Series in terms of α_{bad}, although formally correct, will be misleading when truncated and compared with data. The symptom of a poor choice of expansion parameter is the presence of large, uniform second order coefficients.

A large second order coefficient appeared in the first second order calculation done on the lattice: the calculation of the gluonic three point function used by the Hasenfratz's to obtain the ratio of Λ parameters of the lattice and the continuum.[5] They found that $g(M)^2_{\widetilde{mom}}$, defined as the coefficient of this three point function, had the expansion

$$g(M)^2_{\widetilde{mom}} = g_0^2\,[1 + g_0^2(\beta_0 \ln\left(\frac{\pi}{aM}\right)^2 + .4312)],$$

where $g_0^2 = 4\pi\alpha_{lat}$ and $\beta_0 = 11/16\pi^2 y$. The constant .4312 in this expression is responsible for most of the large ratio of the Λ parameters.

Since long distance quantities are well behaved when expanded in terms of $g(M)^2_{mom}$, it is immediately obvious that all other long distance quantities will have a similar constant term. For example, the heavy quark potential $V(\bar{q})$ at momentum transfer $\bar{q}$ has the expansion[6]

$$V(\bar{q}^2) = -\frac{C_f g_0^2}{\bar{q}^2}\,[1 + g_0^2(\beta_0 \ln\left(\frac{\pi}{a\bar{q}}\right)^2 + .374)].$$

The crucial point is that, although it may or may not be obvious, it is also true that the short distance lattice quantities which are now known have a similar term. For example, the corrections to the heavy potential as a function of distance have the form[7]

$$V(R) = -\frac{C_f g_0^2}{4\pi R}\,[1 + g_0^2(\beta_0 \ln\left(\frac{\pi R}{a}\right)^2 + C_R)],$$

where C_R for various values of R is given in the following table.

R/a	C_R
2	.43
3	.43
4	.43
6	.45
∞	.4545

The constant for $R = \infty$ can be obtained by Fourier transforming the q space expression. It can be seen that the constants at finite R vary very little from the one at $R = \infty$. The smallness of the dependence on the distance scale may be less surprising if it is remembered that the corrections are dominated by high momentum tadpole graphs which are insensitive to the momentum of the process.

As we will see later, Wilson loops also have a similar term when Creutz ratios are used to remove self-energies.

We therefore conclude that the known perturbative expansions in α_{lat} have the uniform second order coefficients which are characteristic of poor choice of expansion parameter.

3. CURE FOR A POOR CHOICE OF EXPANSION PARAMETER

To define an improved g^2, we can use any quantity which is known to second order in α_s. To keep the argument clear and simple, we prefer to use a quantity which has been calculated strictly on the lattice. This restricts our choices to Wilson loops and the heavy quark potential $V(q)$. Bare Wilson loops are inconvenient because they have large self energy corrections which have nothing to do with the definition of the coupling constant. $V(q)$ on the other hand is ideal for our purposes: it is a physical quantity which is understood both in the continuum and on the lattice. We therefore define $g_V^2(q)$ such that

$$V(q) \equiv -\frac{C_f g_V^2(\bar{q})}{\bar{q}^2}.$$

If the second order term in the series is absorbed into the Λ parameter, we obtain [6] $\Lambda_V = 46.08\Lambda_{lat}$ for $SU(3)$ with no flavors of quarks. We then take $g_V^2(\tilde{q}) \equiv g^2(\tilde{q}/\Lambda_V)$ as the approximate coupling strength of a gluon with momentum $\tilde{q}$ and use it to expand other quantities with "typical" momenta $\tilde{q}$.

The absorption of the second order correction into the Λ parameter is equivalent to taking the second order correction as the first term in a geometric series. This boosts the effect of the coupling constant redefinition from 40% to nearly a factor of two. Explicit calculations of third and higher order terms will provide corrections to the improved coupling constant.

The use of a renormalized coupling constant with a much larger Λ parameter is by far the most important practical component of the renormalized perturba-

β	κ_c			
	Perturbation Theory			M. C.
	$\alpha_{lat}(1/a)$	$\alpha_{\overline{MS}}(\tilde{q})$	$\alpha_V(\tilde{q})$	Data
5.7	.141	.154	.160	.169
6.0	.140	.151	.156	.157
6.1	.140	.151	.155	.155
6.3	.140	.149	.152	.151

Table 1: The critical hopping parameter κ_c for Wilson quarks, calculated in first order perturbation theory and by Monte Carlo. Statistical errors in all data presented are of order one in the last digit quoted or smaller.

tion theory which we are describing. However, in order to eliminate ad hoc elements of the procedure to the greatest extent possible, we will also define a specific way of estimating an appropriate $\tilde{q}$ for each process. We define $\tilde{q}$ logarithmically, since we are really interested in a typical α for each process. If we are calculating some integral

$$I = \int d^4q\, f(q),$$

we obtain an average $\ln q^2$ for the integral as

$$< \ln q^2 >= \frac{1}{I}\int d^4q\, f(q) \ln q^2,$$

and then define $\tilde{q}$ as

$$\tilde{q} \equiv \exp(\frac{1}{2} < \ln q^2 >).$$

4. COMPARISON OF IMPROVED PERTURBATION THEORY WITH MONTE CARLO DATA

The arguments in Section 3 for choosing an appropriate definition of g^2 and scale choice have been made solely on the basis of known facts about perturbation theory, and without regard to Monte Carlo data. Only now that we have examined perturbation theory carefully are we ready to determine the extent to which it agrees with Monte Carlo calculations of short distance quantities.

4.1. κ_c

The integral for the renormalization of κ_c is dominated by a tadpole. We therefore expect it to be dominated by momenta of order the lattice cutoff, π/a. Using the procedure of Section 3 yields $\tilde{q} = 2.58/a$. In Table 1 we compare perturbative results[1] for $\kappa_c = 1/(8-.869g^2)$ with Monte Carlo data[2] at several values of β. We present the perturbative predictions using α_{lat}, our favorite scheme α_V, and also using $\alpha_{\overline{MS}}$, which serves to

β	$1-Tr(U)/3$ (Landau gauge)			
	Perturbation Theory			M. C.
	$\alpha_{lat}(1/a)$	$\alpha_{\overline{MS}}(\tilde{q})$	$\alpha_V(\tilde{q})$	Data
5.7	.083	.133	.152	.176
6.0	.078	.122	.138	.139
6.4	.074	.110	.122	.117

Table 2: The expectation value of the trace of a link in Landau gauge, calculated in first order perturbation theory and by Monte Carlo.

illustrate the effects of different choices of improved coupling constants. For $\beta \geq 6.0$, the deviation between the data and the free field value ($\kappa_c = .125$) agrees with renormalized perturbation theory to within 20%, but disagrees with perturbation theory using α_{lat} by around a factor of two.

4.2. $< Tr(U) >$

The integral for $1-\frac{1}{3}Tr(U) = .078g^2$ is pure tadpole, and we therefore expect it to have slightly higher average momentum than the integral for κ_c. We find $\tilde{q} = 2.80/a$. In Table 2 we show results for $1-\frac{1}{3}Tr(U)$.[3] Just as in the previous case, for $\beta \geq 6.0$, the deviation between the data and the free field value ($1-\frac{1}{3}Tr(U) = 0$) agrees with renormalized perturbation theory to within 20%, but disagrees with perturbation theory using α_{lat} by almost a factor of two.

4.3. Wilson Loops and Creutz Ratios

Large Wilson loops have badly behaved perturbative expansions for a trivial reason: they contain a self-energy contribution proportional to the length of the loop. For large loops, contributions to this self-energy approximately exponentiate, so one expects the logarithm of the Wilson loop to be better behaved in perturbation theory. Taking Creutz ratios of loops also has the effect of reducing both linear self-energies and also corner terms from loop expectation values. We therefore examine the logarithms of Creutz ratios, as the quantities having the best chance of being well behaved in perturbation theory. Since the tadpole terms present in the self-energies are removed by taking Creutz ratios, we expect to find smaller momentum scales for these objects than we did for the previous two examples. Indeed, we find for the log of the smallest Creutz ratio, $-\log(\chi_{2,2})$, the scale $\tilde{q} = 1.10/a$. We find increasingly smaller scales for increasingly large loops, also as expected. In Table 3 we

β	$-\log(\chi_{2,2})$						
	Perturbation Theory						M. C. Data
	First Order			Second Order			
	α_{lat}	$\alpha_{\overline{MS}}$	α_V	α_{lat}	$\alpha_{\overline{MS}}$	α_V	
5.7	.103	.222	.270	.156	.241	.249	.372
5.8	.101	.213	.257	.152	.232	.237	.319
5.9	.099	.205	.245	.149	.221	.227	.286
6.0	.097	.197	.234	.146	.212	.217	.265
6.1	.095	.190	.224	.142	.204	.209	.244
6.2	.094	.184	.215	.139	.196	.201	.229
6.3	.092	.178	.206	.136	.189	.193	.220

Table 3: The negative of the logarithm of the expectation value of the 2,2 Creutz ratio, $-\log(\chi_{2,2})$, calculated in first and second order perturbation theory and by Monte Carlo.

show results for $-\log(\chi_{2,2})$.[8] For $\beta \geq 6.0$, the one loop results show a pattern similar to the previous two examples. The two loop results for α_{lat} are much larger than the one loop results, a symptom of a bad expansion. The two loop results for $\alpha_{\overline{MS}}$ and α_V are quite close to the one loop results, showing that this Creutz ratio does indeed have about the same second order coefficient as $V(\bar{q})$ which we used to define the expansion parameter. They are also extremely close to each other, showing that both the renormalized α's are reasonable expansion parameters.

A feature of the data not present in the previous examples is that they lie systematically above the perturbative predictions of either of the renormalized expansions. At small β this is not surprising: a nonperturbative area law exists. What is surprising is that the observed deviations at $\beta \geq 6.0$, although they are small and perhaps cannot be taken too seriously, continue to fit the area law. Such behavior, if it were real, would signal a breakdown of the perturbative calculability of the coefficient functions in the operator product expansion for the Creutz ratios. (The relevant operators are 1 and $F^2_{\mu\nu}$. There is no dimension two operator expected to produce an area law at short distances.) This would have serious implications for the applications of the operator expansion to sum rules, and therefore clearly deserves more careful study.

5. SUMMARY

We have shown that perturbative expansions using the bare coupling constant of the lattice regulator exhibit the same symptom of a poor choice of expansion parameter as do those using the MS coupling constant: large uniform second order corrections. This is true both of long distance and short distance quantities. When a renormalized coupling constant is used for lattice perturbative calculations, many Monte Carlo calculations which seemed to be in bad disagreement with perturbation theory are seen to be in reasonably good agreement after all.

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