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MULTI-FLUID MODELS FOR PLASMA WAKE-FIELD PHENOMENA

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ABSTRACT

In this paper we present various treatments of plasma wake-field phenomena which employ multi-fluid models. These models generalize the one-dimensional, nonlinear, relativistic single fluid model which has been used extensively in previous plasma wake-field calculations. Using a two-fluid model, we discuss the interaction of a low energy continuous electron beam with wake-field generated plasma waves. The phenomena of continuous beam modulation and wave period shortening are discussed. The relationship between these effects and the two-stream instability is also examined. Also, using a three-fluid model, effects due to plasma electron temperature in nonlinear plasma wake-fields are examined and compared to previous work. Finally, the consequences of ion motion induced by large amplitude electron plasma waves are calculated by including the fluid behavior of the ions.

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1. Introduction

The Plasma Wake-field Accelerator (PWFA), proposed by Chen, Dawson, Huff and Katsouleas^[1] in 1984, uses electrostatic fields generated in plasma waves driven by a relativistic electron beam. Charged particles can be accelerated in this electrostatic wake-field to ultra-high energies, making this scheme attractive for future high energy physics accelerators. The linear theory of the PWFA is well developed, and extensive analytical and numerical treatments exist of such issues as transformer ratio (the ratio of maximum accelerating field behind the driving beam to the maximum decelerating field inside the driving beam) and beam loading of the plasma waves.^[2,3] Experimental observation, at the ANL Advanced Accelerator Test Facility, of acceleration by the PWFA mechanism in both linear and nonlinear regimes has been recently reported.^[5]

Although the experiments performed in the nonlinear regime of the PWFA displayed interesting three-dimensional effects,^[4] only one-dimensional treatments of nonlinear plasma wake-fields currently exist. This problem has been treated by Rosenzweig,^[6] Ruth *et al.*^[7] and Amatuni *et al.*^[8] The theoretical methods employed in these works are all based on the approach originally pioneered by Akhiezer and Polovin in 1956.^[9] These general methods, in which the fully relativistic, nonlinear plasma wave motion was calculated using a cold fluid model, were simplified for the case of relativistic phase velocity plasma waves by Noble^[10] in his treatment of nonlinear plasma waves in the Plasma Beatwave Accelerator (PBWA). Much of the subsequent investigations into the nonlinear plasma dynamics in the PWFA is derived from these fundamental works. These investigations predict certain advantages of the nonlinear regime over linear PWFA

schemes. Operation in the one-dimensional limit of the nonlinear regime allows a novel, straightforward method of transformer ratio enhancement, and lowers the plasma density needed to achieve high accelerating gradients in the wake-fields. There are no theoretical predictions as yet on the effects of transverse plasma dynamics on this scheme.

The high transformer ratios obtained in the nonlinear scheme depend on driving the plasma electron density waves to extremely large amplitudes. For this reason, the maximum wave amplitude, which is limited by trapping of thermal plasma electrons, has been investigated in detail.^[11,12] For cool laboratory plasmas ($kT_e \simeq 10$ eV), the transformer ratio R is limited to approximately ten by thermal considerations. In Ref. 11 the motion of charged particles in the plasma wake-fields is treated generally, to include both the trapping of background plasma electrons as they are overtaken by the driving beam and the motion of particles externally injected into the wake-fields.

Both trapping and injection of charged particles into the plasma wave imply beam loading of the wave's accelerating field. This problem is treated in the case of rigidly positioned beam charge distributions in Ref. 3 for the linear case and Ref. 6 for the nonlinear case, and general solutions for the wake-fields in the presence of beam loading have been obtained. The self-consistent motion of the beam charge distributions is ignored in these treatments. Below we will provide a bridge between the two approaches, by calculating the self-consistent effects of beam loading and equations of motion simultaneously for a continuous, lower energy electron beam as it is overtaken by an ultra-relativistic driving beam in a plasma. This calculation requires that we generalize the previous single (plasma

electron) fluid treatment to include a second fluid, that of the continuous beam electrons. We also discuss the relationship of the two-stream instability to this result, and examine the potential use of this instability for amplification of plasma wake-field amplitudes.

Once we have generalized the cold fluid model of these waves to include a beam fluid, it is straightforward to model plasma electron thermal effects by a three-fluid model. In this case, one approximates the thermal distribution of electrons by an initially stationary component and forward and backward propagating electron ‘beams’ of initial velocity $v_{th} = \sqrt{3kT_e/m_e}$. This simplified approach is quite similar to the waterbag model employed in Refs. 11 and 12; its results are compared those previously obtained.

Finally, we can relax the assumption, made in all previous wake-field calculations, of infinitely massive, or stationary ions. The ion dynamics can also be treated self-consistently as a second fluid in the problem. We discuss below the effects of the ion motion on nonlinear plasma waves, and the implications these effects have on the nonlinear PWFA scheme.

2. Modulation of Continuous Electron Beams

We begin our treatment of the interaction of a continuous electron beam with plasma wake-fields by writing, in the spirit of Refs. 6 - 12, the cold fluid equations for two fluids: the plasma electrons, and the continuous beam electrons. We assume that the plasma ions form an immobile neutralizing background of density n_0 and define the plasma electron density n , velocity $\mathbf{v} = \beta c$, and the

plasma frequency $\omega_p = \sqrt{4\pi e^2 n_0 / m_e}$. The continuous beam and velocity are symbolized by n_c and $\mathbf{v}_c = \beta_c c$, and ultra-relativistic bunched beam densities and velocities (which are taken to be constant) by n_b and $\mathbf{v}_b = \beta_b c$. The fluid equations are thus

$$\nabla \cdot \mathbf{E} = 4\pi e(n_0 - n - n_b - n_c), \quad \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (1)$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - 4\pi e(n\beta + n_b\beta_b + n_c\beta_c), \quad \nabla \cdot \mathbf{B} = 0, \quad (2)$$

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{p} = -e(\mathbf{E} + \beta \times \mathbf{B}), \quad (3)$$

where for the plasma electron fluid motion $\mathbf{p} = m_e \mathbf{v} / \sqrt{1 - \beta^2} = \gamma m_e \mathbf{v}$ is the momentum. The analogous equation of motion for the continuous beam electrons is obtained by substitution of \mathbf{v}_c for \mathbf{v} in Eq. (3). The equation of continuity for the plasma electrons is

$$\nabla \cdot (n\mathbf{v}) + \frac{\partial n}{\partial t} = 0, \quad (4)$$

with an identical expression holding for the continuous beam electrons.

For the one-dimensional treatment we take the direction of beam propagation to be z , and assume that there is no transverse motion. Additionally we assume the wave motion is a function only of the variable $\tau = \omega_p(t - z/v_b)$, and, taking the limit that the driving bunched beam velocity (and thus the wave phase velocity) $v_b \rightarrow c$, we can obtain simple nonlinear differential equations for the fluid motion.^[13] The steps we have just described, which constrain the equations to describe a one-dimensional, steady state system, are the root of both the strength

and shortcomings of our general approach. These constraints allow integrable equations which we can solve to yield physical insight into the system's behavior. On the other hand, if we try to relax the constraints to study the problems of transverse effects or instabilities, the mathematics becomes intractable. Thus we must discuss these problems in an approximate way, outside of the formalism of the exact results we obtain through our present approach.

Assuming total charge and current neutrality before the arrival of the bunched beam, and writing the initial unperturbed continuous beam density and velocity as n_{c0} and $v_{c0} = \beta_{c0}$ we have from the equations of continuity

$$\frac{n}{n_0} = \frac{1 + \epsilon(\beta_{c0} - 1)}{1 - \beta} \quad (5)$$

and

$$\frac{n_c}{n_0} = \epsilon \left[\frac{\beta_{c0} - 1}{\beta_c - 1} \right]. \quad (6)$$

where $\epsilon = n_{c0}/n_0$. In analogy with previous treatments, we now introduce the dependent variables $x = [(1 - \beta)/(1 + \beta)]^{1/2}$ and $x_c = [(1 - \beta_c)/(1 + \beta_c)]^{1/2}$, and Eqs. (5) and (6) become

$$\frac{n}{n_0} = \frac{1}{2} \left(1 + \frac{1}{x^2} \right) [1 + \epsilon(\beta_{c0} - 1)] \quad (7)$$

and

$$\frac{n_c}{n_0} = \frac{\epsilon}{2} (1 - \beta_{c0}) \left(1 + \frac{1}{x_c^2} \right) \quad (8)$$

Under this transformation, the fluid equations derived from Eqs. (1) – (3) reduce

to one expression

$$x'' = x_c'' = \frac{1}{2} \left[2\alpha + \frac{1}{x^2} - 1 + \epsilon(1 - \beta_{c0}) \left(\frac{1}{x^2} - \frac{1}{x_c^2} \right) \right] \quad (9)$$

where the $\alpha = n_b/n_0$; when α is piecewise constant, we can always find the first integral of these equations. This expression can be understood physically by noting that x and x_c are proportional to the electrostatic potential^[6] and that Eq. (9) is equivalent to the Poisson equation. That it is also the fluid equation of motion can be verified by further noting that $x = \gamma(1 - \beta) = (U - pc)/m_e c^2$, where U is the total energy of the electron. Explicitly, we can write the longitudinal electric field as

$$E = \pm(m_e c \omega_p / e) x'. \quad (10)$$

There are no magnetic or transverse electric fields allowed in this one-dimensional model. Thus $x_c = x - \lambda$, where the constant λ is evaluated from the initial conditions to be

$$\lambda = \left[\frac{1 + \epsilon(1 - \beta_{c0})}{1 - \epsilon(1 + \beta_{c0})} \right]^{1/2} - \left[\frac{1 - \beta_{c0}}{1 + \beta_{c0}} \right]^{1/2}, \quad (11)$$

and the fluid equations have been reduced to a single elliptic equation. Upon substitution of Eq. (11) we obtain the first integral of Eq. (9)

$$(x')^2 = C - \left[2\alpha + x + \frac{1}{x} + \epsilon(1 - \beta_{c0}) \left(\frac{1}{x} - \frac{1}{x - \lambda} \right) \right]. \quad (12)$$

The constant C is evaluated from the causal boundary conditions at the bunched beam boundaries. As a simple example, for the wake-fields behind an infinitesi-

mally short δ -function beam of surface charge density σ_b we have

$$C = \left(\frac{\omega_p \sigma_b}{n_0 c}\right)^2 + x_0 + \frac{1}{x_0} + \epsilon(1 - \beta_{c0})\left(\frac{1}{x_0} - \frac{1}{x_0 - \lambda}\right), \quad (13)$$

where $x_0 = \lambda + [(1 - \beta_{c0})/(1 + \beta_{c0})]^{1/2}$. The electric field is now known explicitly as a function of x , as $eE = (m_e c \omega_p) x'$, and its maximum deceleration amplitude is $eE_m = 4\pi e^2 \sigma_b$ directly behind the driving beam. The electric field also attains this maximum amplitude in the wave's accelerating phase, when $x = x_0$ and $x' = -x'_0 = -(\omega_p \sigma_b / n_0 c)$.

The solution to Eq. (12) can be written in terms of elliptic integrals, but this does not illuminate the behavior of the plasma waves or the continuous beam distribution well. However, one can deduce the prominent characteristics of the continuous beam loaded wake-fields from inspection of Eqs. (8) and (12). If the beam is relativistic and $\epsilon \ll 1$, then $x_0 \simeq 1$ and $\lambda \simeq 1 + \epsilon - 1/2\gamma_{c0}$ where $\gamma_{c0} = (1 - \beta_{c0}^2)^{1/2}$ is the initial continuous beam Lorentz factor. If the driving beam charge is large enough then the value of $x = x_{min}$ at the density turning point in the forward motion of the plasma electrons can approach λ from above. The continuous beam electrons build up a very high density near this point, as can be seen by rewriting Eq. (8) as $n_c/n_0 = \epsilon(1 - \beta_{c0})[1 + 1/(x - \lambda)^2]$. This concentration of beam electrons is due to the wave potential's near trapping of the beam. As these electrons are accelerated to a very high energy $\gamma_c m_e c^2 = 1/2(x_c + 1/x_c)m_e c^2 \simeq m_e c^2/2(x_{min} - \lambda)$, they load down the plasma wave and become detrapped and decelerated after attaining a high energy. This observation validates the assumptions used in deriving the saturation of wave amplitude due to background trapping in Ref. 11. The continuous beam loading causes the wave

electric field to turn faster, as can be seen from Eq. (12). For $\gamma_{c0} \gg 1$, $\lambda \simeq 1$ and even small amplitude waves have nonlinear interactions with the continuous electron beam, as once $x \leq 1$ the last term in Eq. (12) becomes dominant and nonlinear. The local oscillation frequency increases dramatically in this case due to the sharp buildup of the continuous beam electrons. The wave period is shortened by this nonlinear effect, in analogy to the period shortening associated with thermal effects.^[11] The modulation of the continuous beam electron energy density corresponds to the excitation of a very nonlinear positive energy wave in the electron beam.

To illustrate the effects of beam modulation and period shortening we display a numerical example. A wave is assumed to be excited by a δ -function beam with $\sigma_b = 0.3cn_0/\omega_p$, which gives an electric field amplitude of $eE_m/mc\omega_p = 0.3$. The wave electric field and perturbed plasma density $n_1/n_0 = n/n_0 - 1 - \epsilon$, in absence of a continuous beam, shown in Fig. 1(a) are slightly nonlinear, as we have chosen a fairly large wave amplitude in which $n_1/n_0 \simeq 0.3$ is not very small compared to unity. Upon introduction of continuous beam of initial speed $\beta_{c0} = 0.99$ ($\gamma_{c0} \simeq 7.7$) and $\epsilon = 0.1$, we observe the sharp peak of continuous electron beam density which serves to nearly completely suppress the region of positive perturbed plasma electron density, as illustrated in Fig. 1(b). The maximum continuous beam density reached is approximately $n_c \simeq 7.8n_0$ (off scale), and the maximum energy $\gamma_c \simeq 62.5$. The net wave period has contracted to approximately 0.6 of the period shown in Fig. 1(a).

A physical criterion for strong buildup of continuous beam electrons can be formulated by comparing the surface charge density necessary to reverse the wave

electric field to the integrated density of continuous beam electrons overtaken by the driving beam in the distance it takes a point in the continuous beam to slip one plasma period behind the driver. This gives a measure of the potential strength of the low energy beam's effect on the plasma wave. The criterion is thus

$$\frac{\epsilon}{1 - \beta_{c0}} \geq \frac{eE_m}{m_e c \omega_p}. \quad (14)$$

This expression shows explicitly the requirement that the continuous beam be relativistic for significant nonlinear interaction with the plasma wave to occur. If Eq. (14) is satisfied, then the continuous electron beam loading will cause a sudden reversal of the accelerating phase of the wave. The concomitant period shortening is pronounced in small amplitude waves, but not in very large amplitude waves, as the positive perturbed plasma electron density region is already very short and spiky in this case.^[6]

It is of interest to place these results in the context of linear plasma wave instability theory. The dispersion relation for linear waves in the beam-plasma system we have discussed is^[14]

$$\frac{\omega_p^2}{\omega^2} + \frac{\omega_c^2}{(\omega - kv_c)^2} = 1, \quad (15)$$

where $\omega_c^2 = 4\pi e^2 n_c / \gamma_{c0}^3 m_e$. If we apply the additional wake-field excitation condition $\omega/k = v_b$ then the solution to Eq. (15) must give purely real values for the frequency and wave-number. Thus no absolutely unstable solutions which grow exponentially with the distance behind a causal excitation moving at speed v_b can exist (this form of solution is allowed by our steady state assumption).

The spatial and temporal characteristics of the two-stream instability are much more subtle, showing both absolute and convective nature. The Green's function response of a cold beam-plasma system has been found to be a growing wave-packet with frequency $\sim \omega_p$, phase velocity v_c and a group velocity two-thirds of the beam velocity $v_g \simeq 2v_c/3$.^[15]

This result suggests that one might use the two-stream instability to amplify plasma wake-fields. This is problematic for several reasons. The first is that energy in the instability can only propagate at best at two-thirds the speed of light, and thus wave energy obtained from the unstable beam can interact in only a limited way with accelerating relativistic particles. Also, since we wish to have a relativistic phase velocity in the wake-fields, and we want to extract energy from the low energy beam causally, we require the wake-field excitation to have a speed $v_b \geq v_c$. Unfortunately, waves with phase velocity near v_c have a very small nonlinear saturation level,^[15] as could be anticipated from our previous results. Thus the instability is not likely to be of value for wake-field amplification.

Once the effect of modulation of low energy electron beams in plasma wake-fields has been experimentally achieved, it could be exploited as a source of high-frequency high-power fields. By using the tightly bunched beam to excite an RF cavity one could efficiently extract the beam energy in the form of electromagnetic waves. This scheme has been utilized by Friedman and coworkers at NRL^[16] to generate gigawatts of RF power. Plasma wake-field beam modulation used as a bunching system may be of interest in extending this type of scheme to high frequency; nonlinear plasma wake-fields have been driven in at greater than 40 GHz in present PWFA experiments. Alternatively, a sudden lowering of

the plasma density at the end of the interaction length would shift the peaks in the modulated beam density into a decelerating phase, and the wake-field amplitude would grow linearly with distance behind the driving beam, resulting in a potentially very large accelerating wake-field amplitude.

In order for the modulation of a continuous low energy electron beam to be experimentally observed, the low energy beam must be stable in the interaction time against plasma-induced instabilities. The two-stream instability growth rate^[15] limits the plasma length to $L < (2c/\sqrt{3})(2/\omega_c^2\omega_p)^{1/3}$. The low energy beam can also be unstable with respect to self-focusing and filamentation. Since we have used a one-dimensional model, these results apply only in the limit that the beam is very wide compared to a plasma wavelength $\lambda_p = 2\pi c/\omega_p$. Previous investigations have shown that in this limit the beam is stable against self-focusing but prone to filamentation (Weibel) instability. It should be also noted that, if one assumes as we have here that the beam pulse is both wide and long compared to a plasma wavelength, then plasma wake-field theory validates our assumption of charge and current neutralization of the beam to a good approximation.^[17] The conditions for stabilization of the Weibel instability have been studied by Su, *et al.*^[18] The Weibel instability can be stabilized in this case by Landau damping. If the transverse temperature T_\perp of the beam is such that $kT_\perp/m_e c^2 > \epsilon/2$, then Landau damping will stabilize the beam against filamentation. In terms of the invariant emittance ϵ_n and radius σ_r of the beam, this inequality becomes $\epsilon_n > \sigma_r \sqrt{\epsilon/2\gamma}$. Note that in this discussion we have retained the notation $\epsilon = n_b/n_0$; this quantity does not represent an emittance.

One must allow enough interaction time in the plasma for the modulation to

develop, however. This implies that the continuous beam slip by approximately one plasma wavelength with respect to the driver, which can be written $L > \lambda_p/(1 - \beta_{c0})$. As an example, we take a 2 kA continuous beam of energy $\gamma = 4$, emittance $\epsilon_n = 10^{-4}$ m-rad and radius $\sigma_r = 2$ cm. If we choose the plasma density to be 2.5×10^{14} cm $^{-3}$, then $\epsilon \simeq 10^{-4}$ and $\lambda_p \simeq 2$ mm. The beam is stable against filamentation for these parameters. The modulation will develop in a plasma of length $L > 6.4$ cm, while the two-stream instability e-folding distance limits the length to approximately $L < 10$ cm.

3. Three-fluid Thermal Plasma Model

The thermal limits on one-dimensional electron plasma wave amplitude are well known. Previous investigations have modelled the problem using plasma electron trapping arguments^[11] and a warm-fluid waterbag model.^[12] In this section, we generalize the results of the previous section to provide a simplified derivation of the maximum wave amplitude due to thermal effects. We consider below an initial plasma electron distribution function approximated by three equal density ($n_0/3$) components: a stationary part and forward and backward propagating streams of initial velocity $v_{th} = \sqrt{3kT_e/m_e}$. This model preserves the essential features of the waterbag treatment without the complex derivation of the nonlinear pressure term; however, it misses some subtle results of the rigorous treatment given in Ref. 12.

In analogy with the results of the previous section, we can write down the single equation of motion for the potential variables of the initially stationary,

backward and forward electron distribution components as

$$x'' = x''_- = x''_+ = \frac{1}{6} \left[\frac{1}{x_-^2} + \frac{1}{x^2} + \frac{1}{x_+^2} - 3 \right], \quad (16)$$

respectively. By the arguments similar to those leading to Eq. (12), we can rewrite this expression as

$$x'' = \frac{1}{6} \left[\frac{1}{(x - \lambda_-)^2} + \frac{1}{x^2} + \frac{1}{(x - \lambda_+)^2} - 3 \right], \quad (17)$$

where the λ_{\pm} are constants dependent on v_{th} . For electron temperatures $kT_e \ll m_e c^2$, we have $\lambda_+ \simeq \lambda_- \simeq \lambda_0 = v_{th}/c$, and the first integral of Eq. (17) is

$$(x')^2 \simeq C - \frac{1}{3} \left[\frac{1}{(x + \lambda_0)} + \frac{1}{x} + \frac{1}{(x - \lambda_0)} + 3x \right] = C - \left[\frac{1 - (\lambda_0^2/3x^2)}{x^2 - \lambda_0^2} + x \right]. \quad (18)$$

The constant of integration is evaluated, for a δ -function driving beam, to be

$$C \simeq A^2 + 2 - \lambda_0^2/3, \quad (19)$$

where again $A = \omega_p \sigma_b / n_0 c$. The minimum allowed value of x is clearly $x_{min} = \lambda_0$. In contrast to the equations of motion derived from waterbag treatments, however, Eq. (18) does not become mathematically ill-behaved ($x \rightarrow \lambda_0$) for a certain value of A . On the other hand, we know from the Hamiltonian treatment given in Ref. 11 that the forward component of the velocity distribution will be trapped when $x \simeq \lambda_0 + 1/\gamma_b$, where γ_b is the Lorentz factor of the driving beam. The maximum value of A allowed before trapping is thus $A_m \simeq \sqrt{\gamma_b/3\lambda_0} = [\gamma_b m_e c^2 / 27 k T_e]^{1/4}$, which is similar to the result of the waterbag calculation presented in Ref. 11, up to the factor of $\gamma_b^{1/4}$. The introduction of this factor reflects the difficulty of trapping thermal electrons in an

ultra-relativistic phase velocity potential. This qualitative effect is also predicted in the the more rigorous and physically complete treatment of the full thermal effects on the one-dimensional wave amplitude found in Ref. 12.

4. Effects of Ion Motion

In this section we relax the assumption of immobility of the plasma ions by treating the ions as the second cold fluid in the problem. We shall see from this treatment that ignoring the ion motion is generally a good approximation, except for certain interesting limiting cases.

Assuming the plasma electrons and ions to be initially at rest, with the plasma and ion equilibrium density both equal to n_0 (ion charge state assumed $Z = 1$), the equation of continuity for the plasma gives, in direct analogy to Eq. (7), an expression for the plasma electron density

$$\frac{n}{n_0} = \frac{1}{1 - \beta} = \frac{1}{2} \left[1 + \frac{1}{x^2} \right], \quad (20)$$

where x is as defined above. Likewise, if we define the variable $x_i = [(1 - \beta_i)/(1 + \beta_i)]^{1/2}$, where $\beta_i = v_i/c$ is the normalized ion velocity, we obtain the ion density

$$\frac{n_i}{n_0} = \frac{1}{1 - \beta_i} = \frac{1}{2} \left[1 + \frac{1}{x_i^2} \right]. \quad (21)$$

Thus the equations of motion for both fluids can be written as two coupled equations; for the electrons

$$x'' = \frac{1}{2} \left[2\alpha + \frac{1}{x^2} - \frac{1}{x_i^2} \right], \quad (22)$$

and for the ions

$$x_i'' = -\frac{\mu}{2}\left[2\alpha + \frac{1}{x^2} - \frac{1}{x_i^2}\right], \quad (23)$$

where $\mu = m_e/m_i$ is the electron-to-ion mass ratio. As these equations both have the same initial conditions, we can parameterize the solution of Eq. (23) in terms of the solution to Eq. (22) by

$$x_i - 1 = \mu(1 - x). \quad (24)$$

Eq. (24) can be understood by noting that the quantities $x_i - 1$ and $1 - x$ are the electrostatic potential energy of the ions and electrons, respectively, normalized to their rest masses. Substituting Eq. (24) into Eq. (22), we obtain

$$x'' = \frac{1}{2}\left[2\alpha + \frac{1}{x^2} - \frac{1}{[1 + \mu(1 - x)]^2}\right], \quad (25)$$

This equation shows that the effects of ion motion are ignorable unless μx is not small compared to unity. This means that the plasma electron motion must be extremely nonlinear – the electrons must have relativistic negative velocities for this condition to hold. This condition is approached in the nonlinear PWFA scheme described in Ref. 6. The first integral of Eq. (25) is

$$(x')^2 = C - \left[\frac{1}{x} + \frac{1}{\mu[1 + \mu(1 - x)]} - 2\alpha x\right], \quad (26)$$

where the constant C is again determined by the causal boundary conditions in the problem.

As an example, for the case where the driving beam is chosen to give $\alpha = n_b/n_0 = 1/2$, nonlinear theory with stationary ions predicts^[6] that arbitrarily large transformer ratios can be achieved. The value of x increases monotonically within the beam, indicating an approach to total charge neutralization, and thus a constant decelerating field within the driving beam. The transformer ratio obtained is related to the final value of $x = x_f$; it is simply $R = \sqrt{x}$. If we take $\alpha = 1/2$ in Eq. (26) to examine the effects of ion motion on this scheme, we have

$$(x')^2 = \frac{1}{\mu} - \left[\frac{1}{x} + \frac{1}{\mu[1 + \mu(1 - x)]} - x \right], \quad (27)$$

inside the driving beam and

$$(x')^2 = x_f + \frac{1}{\mu} - \left[\frac{1}{x} + \frac{1}{\mu[1 + \mu(1 - x)]} \right] \quad (28)$$

in the free oscillations behind the driver. The maximum accelerating field behind the beam occurs at $x = 1$, which gives

$$E_+ = (m_e c \omega_p / e) \sqrt{x_f - 1}. \quad (29)$$

Eqs. (27) and (28) have an ill-behaved point at $x = (1 + \mu)/\mu$, which corresponds to trapping of the ions in the driving electron beam's potential well. This point represents a strict upper bound on the length l_b of the driving beam,^[6] i.e. $l_b < \lambda_p/\mu$. From the point of view of maximizing the transformer ratio, this restriction is much more severe. If we expand Eq. (27) to first order in μ , we find

$$(x')^2 = 1 - \frac{1}{x} + \frac{\mu}{2}(1 - x)^2, \quad (30)$$

and note that the maximum in x' occurs at approximately $x = \mu^{-1/3}$, which optimizes the beam length at approximately $l_b \leq 2\pi c/\mu^{1/3}\omega_p$. Taking $x_f = \mu^{-1/3}$

in Eqs. (29) and (30) we have $R \leq \mu^{-1/6}$, which for hydrogen ions gives $R \leq 3.5$, which is a tighter constraint than that arising from the trapping of thermal electrons. Use of heavier singly ionized species ions raises this limit only slightly, by a factor the one-sixth power of the atomic weight.

This limit on transformer ratio can be theoretically circumvented by allowing the driving electron beam's density to rise slightly towards the end of the bunch. This would negate the effects of the rise in ion density associated with forward ion motion. Such a scheme would be very sensitive to the exact profile of the driving beam, and is thus problematic. Concerns of this sort are also present in examining the dynamics of the free oscillations behind the driver. The presence of nontrivial ion motion indicates the possible onset of the modulational instability. Previous investigations have speculated that this instability would destroy a very nonlinear plasma wave ($eE_m > m_e c \omega_p$) in less than one oscillation.^[12] Our result here helps quantify this concern; a small amount of ion motion during the passing of the driving beam causes large changes in the subsequent wave motion.

5. Discussion

This work has introduced multi-fluid models to examine various physical phenomena in plasma wake-fields. In particular, introduction of more than one electron fluid species allows a self-consistent analysis of the modulation of low energy, continuous electron beams in plasma wake-fields, as well the re-examination of the thermal limit on plasma wave amplitude. The modulation of intense beams through this effect could be employed in a source of very high power, high frequency fields. Addition of ion fluid motion also yields interesting results which

have direct impact on the nonlinear enhancement of the transformer ratio in the PWFA.

These models are constrained to describe one-dimensional, steady state systems, but are of significant usefulness because they allow exact, tractable analysis which gives insight into nonlinear wave dynamics in plasma wake-fields. It is hoped that further theoretical work will allow multi-dimensional description of these phenomena. This is important because the experimental results obtained thus far show significant, multi-dimensional nonlinear behavior. Also, all experiments are performed on bounded systems, which implies the need for a transient, non-steady state analysis. It would be of considerable value to have a more complete theoretical description of these effects to guide simulation and experimental investigation of plasma wake-fields.

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FIGURE CAPTIONS

Figure 1. Perturbed plasma electron density $n_1 = n - n_0 + n_{c0}$, continuous beam density n_c and electric field E , in wake-field plasma wave driven by δ -function beam with $\sigma_b = 0.3cn_0/\omega_p$ ($eE_m/mc\omega_p = 0.3$). Densities normalized to n_0 , electric field normalized to $mc\omega_p/e$. (a) No continuous beam. (b) Continuous beam with $\epsilon=0.1$ and $\beta_{c0} = 0.99$. Maximum beam density $n_c \simeq 7.8n_0$ (off scale).

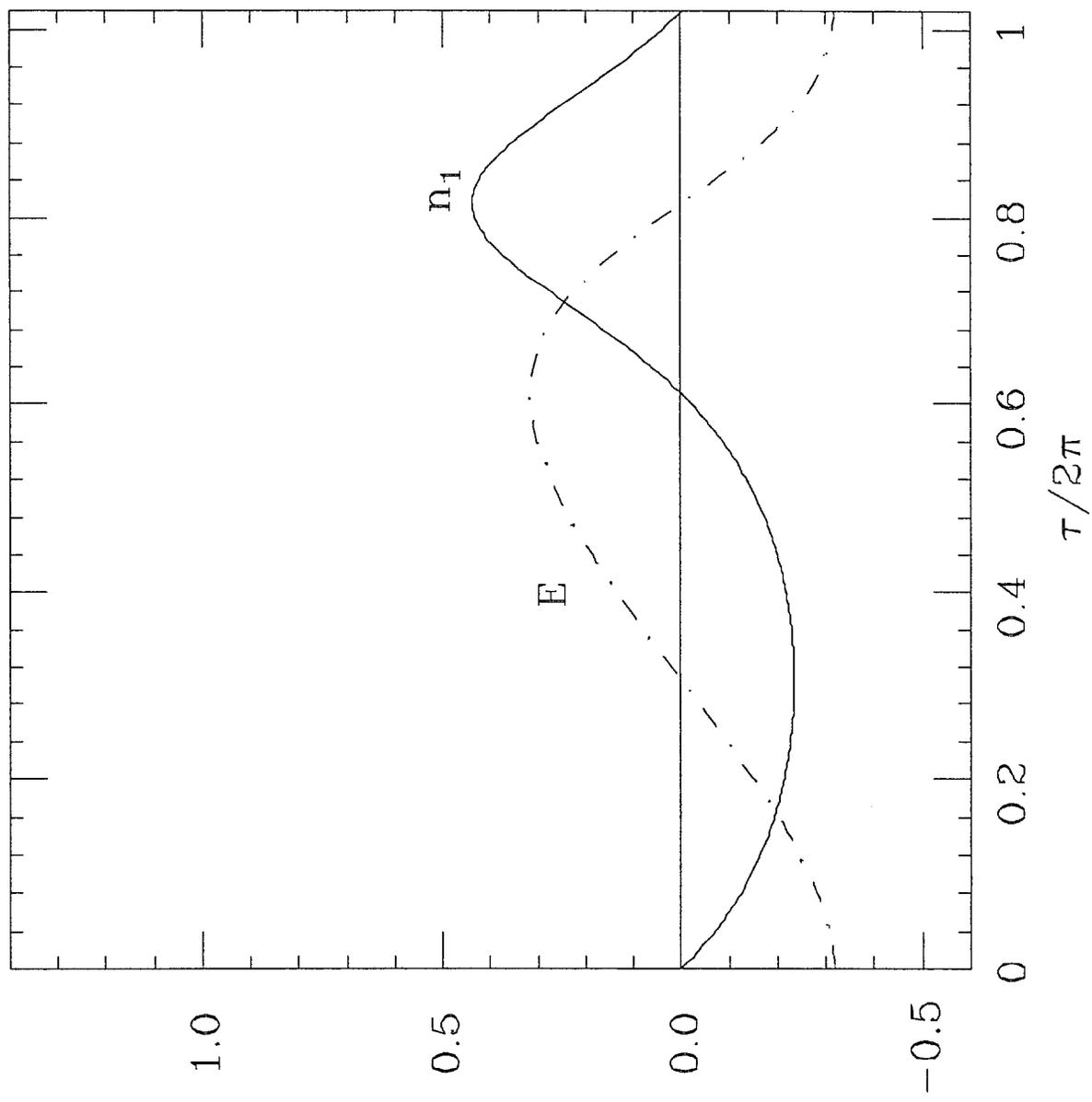


Figure 1(a)

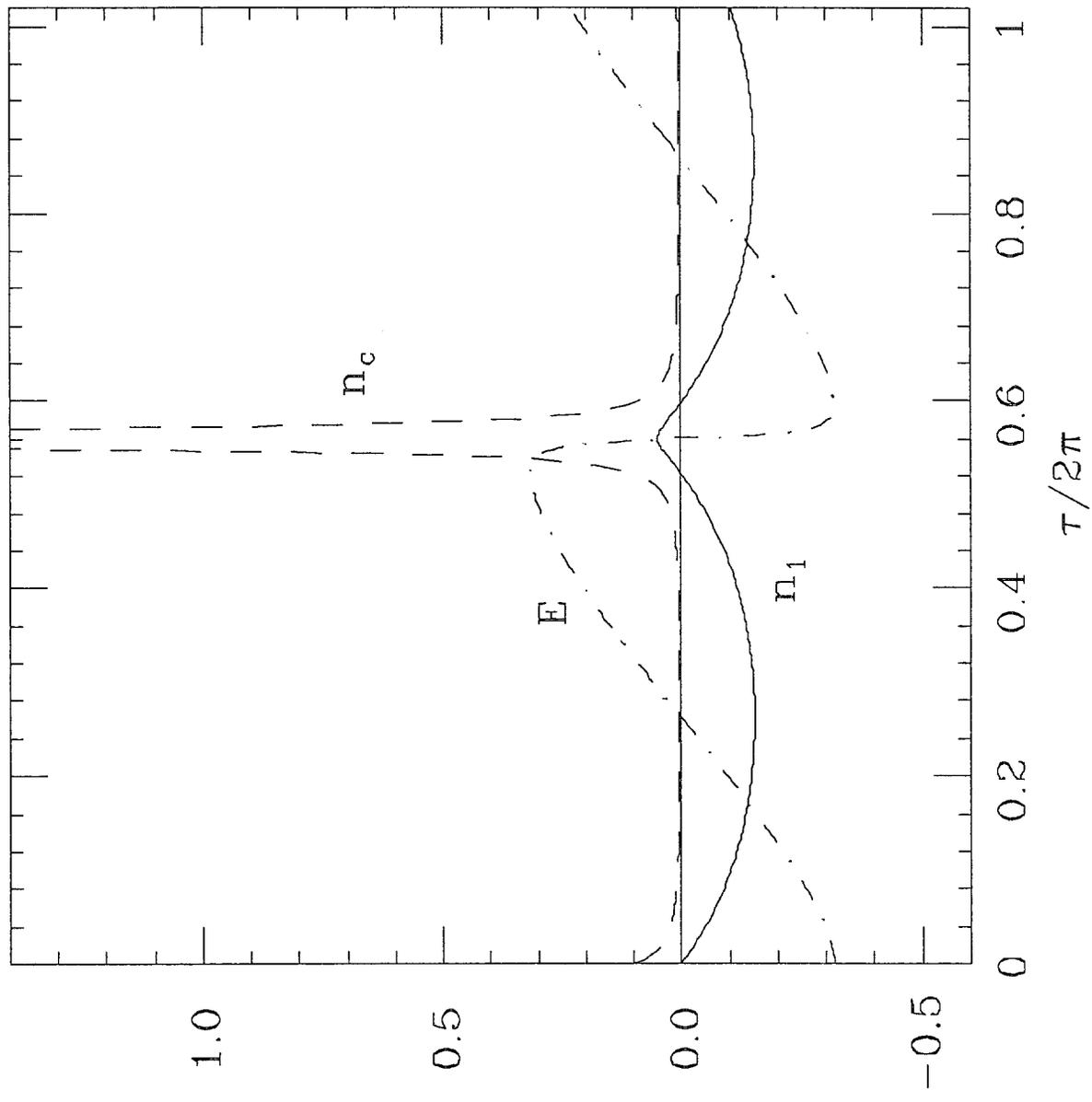


Figure 1(b)