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# BRST Quantization of Superstring in Backgrounds

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**Abstract**

We study the superconformal algebra of Neveu–Schwarz–Ramond superstring in the background of graviton and antisymmetric tensor fields. The classical BRST charge is obtained and the gravitational Ward identities are presented using the invariance properties of $S$–matrix generating functional.

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## I. Introduction

In the past few years, we have seen a great surge of interest in string theories. These theories—at least the supersymmetric ones—have the potential to unify all the known fundamental interactions in a desirable manner[1,2]. The dynamics of the string theory, however, is quite in its infancy. That is to say, we do not yet have acceptable, second quantized, string field theories at least for the superstrings. Consequently, much of the dynamical information about strings is derived from the first quantized theories. For example, by considering the evolution of the string in the presence of its massless background excitations, it is possible to construct the generating functional for the $S$-matrix elements for the scattering of the massless states of the string[3,4] and develop a perturbation theory where we have to consider only one surface of a given genus at each order of the perturbation theory. Moreover, we can give nonzero vacuum expectation values to these background fields in order to study the mechanisms of compactification in the string theory. However, the background fields are not allowed to acquire any arbitrary configuration.

The guiding symmetry in the case of the string theories is the infinite dimensional conformal invariance. This symmetry is quite essential for the consistency of the string theories. Therefore, the background has to be compatible with the requirements of conformal invariance of the string theory. As we know, the $\beta$-function in a theory is a measure of the conformal anomaly. The compatibility of the background fields with conformal invariance can, therefore, be translated to a requirement that the allowed background field configurations give rise to vanishing $\beta$-functions of the theory[5]. The $\beta$-functions can, of course, be calculated perturbatively[6] and consequently the compatibility conditions on the background fields can be determined perturbatively in the first quantized theory and these conditions are the equations of motion satisfied by the backgrounds.

An equivalent, but more elegant and algebraic method of implementing the requirements of conformal invariance, is through the method of BRST quantization. In this approach, one constructs the BRST charge appropriate to the conformal invariance of the string theory. The BRST charge, $Q_{BRST}$, by construction is fermionic and anticommutes with itself (nilpotent) in the absence of any anomaly. Consequently, a background which is compatible with the conformal invariance can be determined to be the one which maintains the nilpotency of $Q_{BRST}$.

While the BRST approach has been worked out for a bosonic string coupled to non-trivial backgrounds[7-10], much remains to be done for the superstrings. In particular, the Hamiltonian structure of the Green-Schwarz superstring[11] is far from clear at present. Consequently, as a first step, we have chosen to study the BRST quantization of the Neveu-Schwarz-Ramond closed type II superstring propagating on the background of a gravitational and antisymmetric tensor field which are massless states of the NSR string. Although there exists a Hamiltonian formulation of the type II superstring in curved background,[12] a complete BRST formulation for graviton and antisymmetric tensor background is lacking so far. The presence of the antisymmetric field is quite interesting in that it gives rise to torsion in the propagating manifold. The consistency of the superconformal algebra has never been explicitly exhibited in the presence of torsion. Therefore, such a calculation is quite worthwhile, since we know from our experience with supergravity theories that the presence of torsion can sometimes alter the physics in our interesting way. Another attractive aspect of the present investigation is that some of the salient features of the model can be reformulated in a geometrical manner which is both economical and elegant.

The rest of the paper is organized as follows: In Section II, we describe the Lagrangian with its various symmetries. Then we discuss various identities which are appropriate to a manifold with torsion and which are useful in the study of the algebra. In Section III, We construct the generators of the super conformal algebra, the stress energy momentum tensor $T_{++}$ and $T_{--}$; and the supercharge densities $J_{++}$ and $J_{--}$ and we explicitly verify the classical algebra of the constraints which is identical to the superconformal algebra and construct the BRST charge appropriate to this theory. In Section IV, we obtain the classical Ward identities following from the symmetry properties of the $S$–matrix generating functional. The question of the nilpotency of the quantum BRST charge, $Q_{BRST}$, and the resulting equations of motion for the background fields would be discussed in a subsequent paper.

## II. The Model

The Lagrangian describing the propagation of a NSR string in a non–trivial background is given by[13]

$$
\begin{aligned}
e^{-1}L &= \frac{1}{2}g^{\mu\nu}\partial_\mu X^i\partial_\nu X^j G_{ij}(X) + \frac{ke^{-1}}{8\pi}\epsilon^{\mu\nu}B_{ij}(X)\partial_\mu X^i\partial_\nu X^j \\
&+ \frac{i}{2}\overline{\psi}\rho^\mu\left(\partial_\mu\psi^j + \Gamma^j_{k\ell}\partial_\mu X^k\psi^\ell\right)G_{ij}(X) \\
&- \frac{ik}{16\pi}T_{ijk}\overline{\psi}^i\rho^\mu\rho_5\psi^j\partial_\mu X^k \\
&- \frac{1}{12}R_{ijk\ell}\overline{\psi}^i\psi^k\overline{\psi}^j\psi^\ell \\
&- \frac{k}{64\pi}D_kT_{i\ell j}\overline{\psi}^i\psi^k\overline{\psi}^j\psi^\ell \\
&- \frac{k^2}{512\pi^2}G^{mn}T_{ikm}T_{j\ell n}\overline{\psi}^i\rho_5\psi^k\overline{\psi}^j\rho_5\psi^\ell \\
&+ \overline{\chi}_\mu\rho^\nu\rho^\mu\psi^i\partial_\nu X^j G_{ij}(X)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{ik}{48\pi} T_{ijk} \overline{\chi}_\mu \rho^\nu \rho^\mu \psi^i \overline{\psi}^j \rho_\nu \rho_5 \psi^k \\
&- \frac{1}{4} \overline{\chi}_\mu \rho^\nu \rho^\mu \chi_\nu \overline{\psi}^i \psi^j G_{ij}(X)
\end{aligned}
\tag{2.1}
$$

where the world-sheet indices $\mu, \nu = 0, 1$ and $i, j, k, \ell = 1, 2, \ldots d$. $e_{a\mu}$ denotes the Zweibein of the world sheet satisfying

$$e_{a\mu} e^a_\nu = g_{\mu\nu}$$

$$e_{a\mu} e^\mu_b = \eta_{ab}$$

and

$$e = \det e_{a\mu} \tag{2.2}$$

Our metric convention is $(+,-)$ and $(+,-,-,\ldots,-)$ for the world sheet and the target manifold respectively. The flat space antisymmetric tensor $\epsilon^{01} = 1$ and the two dimensional Dirac matrices in the flat background are chosen to be

$$
\begin{aligned}
\rho^0 &= \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \rho^{0\dagger} \\
\rho^1 &= -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\rho^{1\dagger} \\
\rho_5 &= \rho^0 \rho^1 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \rho_5^\dagger
\end{aligned}
\tag{2.3}
$$

so that

$$\rho^a \rho^b = \eta^{ab} + \epsilon^{ab} \rho_5 \tag{2.4}$$

holds.

$\chi_\mu$ represents the two-dimensional gravitino and $\psi^i$ is the super partner of the string variable $X^i$. Both $\chi_\mu$ and $\psi^i$ are majorana spinors. The gravitational and the antisymmetric tensor background fields are represented by $G_{ij}(X)$ and $B_{ij}(X)$ respectively and satisfy

$$\begin{aligned} G_{ij}(X) &= G_{ji}(X) \\ B_{ij}(X) &= -B_{ji}(X) \end{aligned} \tag{2.5}$$

In terms of these variables, then, we have

$$\begin{aligned} \Gamma^j_{k\ell} &= \frac{1}{2} G^{jm} \left( \partial_k G_{m\ell} + \partial_\ell G_{km} - \partial_m G_{k\ell} \right) \\ T_{ijk} &= \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} \\ R^i{}_{jk\ell} &= \partial_\ell \Gamma^i_{jk} - \partial_k \Gamma^i_{j\ell} + \Gamma^m_{jk} \Gamma^i_{\ell m} - \Gamma^m_{j\ell} \Gamma^i_{km} \end{aligned} \tag{2.6}$$

$D_i$ is the covariant derivative with respect to the affine connection defined above in Eq. (2.6) and $k$ is an arbitrary constant parameter required, for the consistency of the Wess–Zumino action, to take on integer values only.

The Lagrangian of Eq. (2.1) is invariant under the following symmetry transformations.

(i) Reparameterization invariance:

$$\begin{aligned} \delta e_{a\mu} &= \xi^\nu(\sigma,\tau)\, \partial_\nu e_{a\mu} + \left( \partial_\mu \xi^\nu(\sigma,\tau) \right) e_{a\nu} \\ \delta \chi_\mu &= \xi^\nu(\sigma,\tau) \partial_\nu \chi_\mu + \left( \partial_\mu \xi^\nu(\sigma,\tau) \right) \chi_\nu \\ \delta X^i &= \xi^\mu(\sigma,\tau)\, \partial_\mu X^i \\ \delta \psi^i &= \xi^\mu(\sigma,\tau) \partial_\mu \psi^i \end{aligned} \tag{2.7}$$

where $\xi^\mu(\sigma,\tau)$ is the parameter of the transformation.

(ii) Local Lorentz invariance

$$\begin{aligned} \delta e_{a\mu} &= \lambda \epsilon_a^b e_{b\mu} \\ \delta \chi_\mu &= -\frac{1}{2}\lambda \rho_5 \chi_\mu \\ \delta \psi^i &= -\frac{1}{2}\lambda \rho_5 \psi^i \end{aligned} \tag{2.8}$$

where $\lambda(\sigma,\tau)$ is the parameter of transformation.

(iii) Weyl scaling:

$$\begin{aligned} \delta e_{a\mu} &= \Lambda e_{a\mu} \\ \delta \chi_\mu &= \frac{1}{2}\Lambda \chi_\mu \\ \delta \psi^i &= -\frac{1}{2}\Lambda \psi^i \end{aligned} \tag{2.9}$$

where $\Lambda(\sigma,\tau)$ is the parameter of transformation.

(iv) Local supersymmetry:

$$\begin{aligned} \delta e_{a\mu} &= 2i\bar{\epsilon}\rho_a \chi_\mu \\ \delta \chi_\mu &= \left(\partial_\mu - \frac{1}{2}\rho_5\left(\omega_\mu - \overline{\chi}_\mu \rho_5 \rho \cdot \chi\right)\right)\epsilon \\ \delta X^i &= -\bar{\epsilon}\psi^i \\ \delta \psi^i &= i\rho^\mu\left(\partial_\mu X^i + \overline{\chi}_\mu \psi^i\right)\epsilon \\ &+ \Gamma^i_{jk}\psi^k \bar{\epsilon}\psi^j \\ &+ \frac{k}{16\pi} G^{i\ell} T_{jk\ell}\left(\overline{\psi}^j \rho_5 \psi^k\right)\epsilon \end{aligned} \tag{2.10}$$

Here $\epsilon(\sigma,\tau)$, a Majorana spinor, is the parameter of transformation and

$$\omega_\mu = -\epsilon^{ab}\left[e_a^\lambda\left(\partial_\mu e_{b\lambda} - \partial_\lambda e_{b\mu}\right) + e_a^\lambda e_b^\rho\left(\partial_\rho e_{C\lambda}\right)e_\mu^C\right]$$

(v) Conformal supersymmetry:

$$\delta\chi_\mu = i\rho_\mu\eta \tag{2.11}$$

where $\eta(\sigma,\tau)$ is the parameter of the transformation.

In addition, the Lagrangian is also invariant under the reparameterization of the target manifold. However, there is no space–time supersymmetry for this Lagrangian.

The various symmetries of the Lagrangian allow us to choose a gauge fixing to simplify the calculations. A particularly simple choice is the orthonormal gauge where we choose

$$e_a^\mu = \delta_a^\mu$$

and

$$\chi_\mu = 0. \tag{2.12}$$

The Lagrangian in this gauge becomes

$$\begin{aligned} L &= \frac{1}{2}\eta^{\mu\nu}\partial_\mu X^i\partial_\nu X^i G_{ij}(X) + \frac{k}{8\pi}\epsilon^{\mu\nu}B_{ij}(X)\partial_\mu X^i\partial_\nu X^j \\ &+ \frac{i}{2}\overline{\psi}^i\rho^\mu\left(\partial_\mu\psi^j + \Gamma^j_{k\ell}\partial_\mu X^k\psi^\ell\right)G_{ij}(X) \\ &- \frac{ik}{16\pi}T_{ijk}\overline{\psi}^i\rho^\mu\rho_5\psi^j\partial_\mu X^k \\ &- \frac{1}{12}R_{ijk\ell}\overline{\psi}^i\psi^k\overline{\psi}^j\psi^\ell \\ &- \frac{k}{64\pi}D_k T_{i\ell j}\overline{\psi}^i\psi^k\overline{\psi}^j\rho_5\psi^\ell \\ &- \frac{k^2}{512\pi^2}G^{mn}T_{ikm}T_{j\ell n}\overline{\psi}^i\rho_5\psi^k\overline{\psi}^j\rho_5\psi^\ell \end{aligned} \tag{2.13}$$

and that we can consider all the two dimensional quantities as the flat space objects in this gauge.

Let us note here that the antisymmetric tensor $B_{ij}(X)$ gives rise to torsion in the manifold even in the absence of fermions[14]. Furthermore, the torsion acts differently on different light cone vectors. This can be most easily seen by calculating the change in the bosonic Lagrangian under an arbitrary variation of the target manifold coordinates. Thus

$$\begin{aligned}
\delta L_{bosonic} &= \delta\left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu X^i\partial_\nu X^j G_{ij}(X) + \frac{k}{8\pi}\epsilon^{\mu\nu}B_{ij}(X)\partial_\mu X^i\partial_\nu X^i\right) \\
&= -\delta X^i(\eta^{\mu\nu}G_{ij}\partial_\mu\partial_\nu X^j \\
&+ \left\{\frac{1}{2}\eta^{\mu\nu}\left(\partial_k G_{ij} + \partial_j G_{ik} - \partial_i G_{jk}\right)\right. \\
&- \left.\frac{k}{8\pi}\epsilon^{\mu\nu}\left(\partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}\right)\right\}\partial_\mu X^j\partial_\nu X^k \\
&= -\delta X^i G_{ij}\Big[\eta^{\mu\nu}\partial_\mu\partial_\nu X^j + \eta^{\mu\nu}\Gamma^j_{k\ell}\partial_\mu X^k\partial_\nu X^\ell \\
&- \frac{k}{8\pi}\epsilon^{\mu\nu}T^j_{k\ell}\partial_\mu X^k\partial_\nu X^\ell\Big] \\
&= -\frac{1}{4}\delta X^i G_{ij}\left[D^{(1)}_-\partial_+ X^j + D^{(2)}_+\partial_- X^j\right]
\end{aligned} \tag{2.14}$$

where we have defined

$$\partial_\pm X^j = \left(\partial_0 \pm \partial_1\right)X^j \tag{2.15}$$

and

$$\begin{aligned}
D^{(1)}_-\partial_+ X^j &= \partial_-(\partial_+ X^j) + \Gamma^{(1)j}{}_{k\ell}(\partial_- X^k)(\partial_+ X^\ell) \\
D^{(2)}_+\partial_- X^j &= \partial_+(\partial_- X^j) + \Gamma^{(2)j}{}_{k\ell}(\partial_+ X^k)(\partial_- X^\ell)
\end{aligned} \tag{2.16}$$

with

$$\begin{aligned}\Gamma^{(1)j}{}_{k\ell} &= \Gamma^{j}{}_{k\ell} - \frac{k}{8\pi}T^{j}{}_{k\ell} \\ \Gamma^{(2)j}{}_{k\ell} &= \Gamma^{j}{}_{k\ell} + \frac{k}{8\pi}T^{j}{}_{k\ell}\end{aligned} \tag{2.17}$$

In fact, this analysis suggests that in such a case, we can write down two generalized curvatures for the manifold following from Eqs. (2.6) and (2.17), namely,

$$\begin{aligned}R^{(1)i}{}_{jk\ell} &= \partial_\ell\Gamma^{(1)i}{}_{jk} - \partial_k\Gamma^{(1)i}{}_{j\ell} \\ &+ \Gamma^{(1)m}{}_{jk}\Gamma^{(1)i}{}_{\ell m} - \Gamma^{(1)m}{}_{j\ell}\Gamma^{(1)i}{}_{jm} \\ R^{(2)i}{}_{jk\ell} &= \partial_\ell\Gamma^{(2)i}{}_{jk} - \partial_k\Gamma^{(2)i}{}_{j\ell} \\ &+ \Gamma^{(2)m}{}_{jk}\Gamma^{(2)i}{}_{\ell m} - \Gamma^{(2)m}{}_{j\ell}\Gamma^{(2)i}{}_{km}\end{aligned} \tag{2.18}$$

Written explicitly, the two generalized curvatures of Eq. (2.18) take the form

$$\begin{aligned}R^{(1)}_{ijk\ell} &= R_{ijk\ell} + \frac{k}{8\pi}\left(D_k T_{\ell ij} - D_\ell T_{kij}\right) \\ &+ \frac{k^2}{64\pi^2}G^{mn}\left(T_{ikm}T_{j\ell n} - T_{i\ell m}T_{jkn}\right) \\ R^{(2)}_{ijk\ell} &= R_{ijk\ell} - \frac{k}{8\pi}\left(D_k T_{\ell ij} - D_\ell T_{kij}\right) \\ &+ \frac{k^2}{64\pi^2}G^{mn}\left(T_{ikm}T_{j\ell n} - T_{i\ell m}T_{jkn}\right)\end{aligned} \tag{2.19}$$

Let us note further from the relation

$$D_k T_{\ell ij} - D_\ell T_{ijk} + D_i T_{jk\ell} - D_j T_{k\ell i} = 0 \tag{2.20}$$

that

$$R^{(1)}{}_{ijk\ell} = R^{(2)}{}_{k\ell ij} \tag{2.21}$$

Moreover, $R^{(1)}_{ijk\ell}$ and $R^{(2)}_{ijk\ell}$ satisfy the following Bianchi identities which are useful in proving the superconformal algebras, given in Section III.

$$\begin{aligned} R^{(1)}_{ijk\ell} &+ \frac{k}{4\pi} D^{(1)}_j T_{ik\ell} + \frac{k^2}{16\pi^2} T_{ijm} T^m{}_{k\ell} + cyclic\ \ perm\ \ jk\ell = 0 \\ R^{(2)}_{ijk\ell} &- \frac{k}{4\pi} D^{(2)}_j T_{ik\ell} + \frac{k^2}{16\pi^2} T_{ijm} T^m{}_{k\ell} + cyclic\ \ perm\ \ jk\ell = 0 \end{aligned} \tag{2.22}$$

$$\begin{aligned} D_j R^{(1)}_{imk\ell} &+ \frac{k}{8\pi} G^{pq} \left( T_{ijp} R^{(1)}_{qmk\ell} - T_{pjm} R^{(1)}_{iqk\ell} \right) \\ &+ cyclic\ \ perm\ \ jk\ell = 0 \\ D_j R^{(2)}_{imk\ell} &- \frac{k}{8\pi} G^{pq} \left( T_{ijp} R^{(2)}_{qmk\ell} - T_{pjm} R^{(2)}_{iqk\ell} \right) \\ &+ cyclic\ \ perm\ \ jk\ell = 0 \end{aligned} \tag{2.23}$$

The analysis of constraints and the algebra take a particularly simple form if we go to the local coordinates of the target manifold. Let us define the vielbeins, $E_{Ai}$, which satisfy

$$\begin{aligned} E_{Ai}(X) E^A_j(X) &= G_{ij}(X) \\ E_{Ai}(X) E^i_B(X) &= \eta_{AB} \end{aligned} \tag{2.24}$$

where $i, j$ and $A, B$ take values $1, 2, \ldots d$. We can also define $X$–independent spinors through the relation

$$\psi^i(X) = E^i_A(X) \psi^A(\sigma, \tau) \tag{2.25}$$

as well as the one component spinors, $\psi^A_\pm$, as

$$\frac{1}{2}(1 \pm \rho_5)\psi^A = \psi^A_\mp \tag{2.26}$$

In terms of these variables, the Lagrangian of (2.13) becomes

$$
\begin{aligned}
L &= \frac{1}{2}\left(\eta^{\mu\nu}G_{ij}(X) + \frac{k}{4\pi}\epsilon^{\mu\nu}B_{ij}(X)\right)\partial_\mu X^i \partial_\nu X^j \\
&- \frac{i}{2}\psi_-^A\left(\eta_{AB}\partial_+\psi_-^B - w^{(2)}_{i,AB}\partial_+ X^i \psi_-^B\right) \\
&+ \frac{i}{2}\psi_+^A\left(\eta_{AB}\partial_-\psi_+^B - w^{(1)}_{i,AB}\partial_- X^i \psi_+^B\right) \\
&+ \frac{1}{4}R^{(1)}{}_{ijk\ell}\psi_-^i\psi_-^j\psi_+^k\psi_+^\ell
\end{aligned}
\tag{2.27}
$$

The two spin connection terms are defined with the two different torsions as

$$
\begin{aligned}
w^{(1)}_{i,AB} &= -E_{Aj}\partial_i E_B^j - E_{Aj}\Gamma^{(1)j}{}_{i\ell}E_B^\ell \\
&= w_{i,AB} + \frac{k}{8\pi}E_{Aj}T^j{}_{i\ell}E_B^\ell \\
w^{(2)}_{i,AB} &= -E_{Aj}\partial_i E_B^j - E_{Aj}\Gamma^{(2)j}{}_{i\ell}E_B^\ell \\
&= w_{i,AB} - \frac{k}{8\pi}E_{Aj}T^j{}_{i\ell}E_B^\ell
\end{aligned}
\tag{2.28}
$$

In writing eq. (2.27) in terms of the one component spinors $\psi_\pm^A$, we have used the Majorana property and the charge conjugation matrix in our convention is

$$
C = \rho^1 = -C^T = -C^{-1} \tag{2.29}
$$

Let us also note here for completeness that we can write the generalized curvatures in terms of the spin connections as

$$
\begin{aligned}
R^{(1)}_{ABk\ell} &= E_A^i E_B^j R^{(1)}_{ijk\ell} \\
&= \partial_k w^{(1)}_{\ell,AB} - \partial_\ell w^{(1)}_{k,\,AB} + w^{(1)}_{\ell,AC} w^{(1)C}_{k,B} \\
&- w^{(1)}_{k,AC} w^{(1)C}_{\ell,\,B}
\end{aligned}
\tag{2.30}
$$

and

$$
\begin{aligned}
R^{(2)}_{ABk\ell} &= E^i_A E^j_B R^{(2)}_{ijk\ell} \\
&= \partial_k w^{(2)}_{\ell,AB} - \partial_\ell W^{(2)}_{k,AB} + W^{(2)}_{\ell,AC} w^{(2)C}_{k,B} \\
&- w^{(2)}_{k,AC} w^{(2)C}_{\ell,B} \qquad (2.31)
\end{aligned}
$$

## III. The Algebra of Constraints and BRST Quantization

It is clear from the structure of the Lagrangian in Eq. (2.1) that $e_{a\mu}$ and $\chi_\mu$ are not dynamical variables. Consequently, their equations of motion would give rise to constraints in the theory. Namely,

$$
\begin{aligned}
\frac{\delta L}{\delta e^{a\mu}} &= eT_{a\mu} \simeq 0 \\
\frac{\delta L}{\delta \overline{\chi}_\mu} &= eJ^\mu \simeq 0 \qquad (3.1)
\end{aligned}
$$

Let us note that $T\mu\nu$ and $J^\mu$ are the stress tensor and the supersymmetry current respectively. In the orthonormal gauge of Eq. (2.12), the components of $T_{\mu\nu}$ take the form

$$
\begin{aligned}
T_{+-} &= T_{01} - T_{10} = 0 \\
T_{++} &= \frac{1}{2}\left[(T_{00} + T_{11}) + (T_{01} + T_{10})\right] \\
&= \frac{1}{2}\Big[G_{ij}(X)\partial_+ X^i \partial_+ X^j + 2i\psi^A_+ \left(\eta_{AB}\partial_1\psi^B_+ - w^{(1)}_{i,AB}\partial_1 X^i \psi^B_+\right) \\
&- \frac{1}{2} R^{(1)}_{ijk\ell}\psi^i_-\psi^j_-\psi^k_+\psi^\ell_+\Big] \qquad (3.2)
\end{aligned}
$$

$$
T_{--} = \frac{1}{2}\left[(T_{00} + T_{11}) - (T_{01} + T_{10})\right]
$$

$$= \frac{1}{2}\Big[G_{ij}(X)\partial_- X^i \partial_- X^j + 2i\psi_-^A \left(\eta_{AB}\partial_1\psi_-^B - w_{i,AB}^{(2)}\partial_1 X^i \psi_-^B\right)$$
$$- \frac{1}{2}R^{(1)}_{ijk\ell}\psi_-^i\psi_-^j\psi_+^k\psi_+^\ell\Big] \tag{3.3}$$

In writing Eqs. (3.2), we have used the fermion equations following from the Lagrangian in Eq. (2.27).

In the one component notation of Eq. (2.26), we can also calculate explicitly the components of the supersymmetry current in the orthonormal gauge of Eq. (2.12), and they take the form

$$\begin{aligned}
J_+^0 + J_+^1 &= J_+^+ = J_{-+} = 0 \\
J_-^0 - J_-^1 &= J_-^- = J_{+-} = 0 \\
J_+^0 - J_+^1 &= J_+^- = J_{++} \\
&= 2G_{ij}\partial_+ X^i \psi_+^j \frac{ik}{12\pi} T_{ijk}\psi_+^i\psi_+^j\psi_+^k \\
J_-^0 + J_-^1 &= J_-^+ = J_{--} \\
&= 2G_{ij}(X)\partial_- X^i\psi_-^j + \frac{ik}{12}T_{ijk}\psi_-^i\psi_-^j\psi_-^k
\end{aligned} \tag{3.4}$$

For later convenience let us define a differently normalized supercurrent, namely,

$$\begin{aligned}
J_+ &= \frac{1}{2}J_{++} = G_{ij}(X)\partial_+ X^i\psi_+^j + \frac{ik}{24\pi}T_{ijk}\psi_+^i\psi_+^j\psi_+^k \\
J_- &= \frac{1}{2}J_{--} = G_{ij}(X)\partial_- X^i\psi_-^j + \frac{ik}{24\pi}T_{ijk}\psi_-^i\psi_-^j\psi_-^k
\end{aligned} \tag{3.5}$$

Thus the constraints of the theory take the form

$$T_{++} \simeq 0$$

$$J_+ \simeq 0$$

$$
\begin{aligned}
T_{--} &\simeq 0 \\
J_{-} &\simeq 0
\end{aligned}
\tag{3.6}
$$

We recognize these as the generators of the superconformal algebra. Therefore, we expect the constraints to be all first class and the algebra of the constraints to be the superconformal algebra.

In order to check the algebra, let us first note from Eq. (2.27) that

$$
\begin{aligned}
P_i \equiv \frac{\partial L}{\partial \dot{X}^i} &= G_{ij}\dot{X}^j + \frac{k}{4\pi}B_{ij}X'^j - \frac{i}{2}\psi_+^A\psi_+^B w_{i,AB}^{(1)} \\
&+ \frac{i}{2}\psi_-^A\psi_-^B w_{i,AB}^{(2)}
\end{aligned}
\tag{3.7}
$$

where $\dot{X}^i = \frac{\partial}{\partial\tau}X^i$ and $X'^i = \frac{\partial}{\partial\sigma}X^i$. We note that the nontrivial fundamental Poisson brackets of the theory are given by

$$
\begin{aligned}
\left\{X^i(\sigma,\tau), P_j(\sigma',\tau)\right\} &= \delta^i_j\delta(\sigma-\sigma') \\
\left\{\psi_+^A(\sigma,\tau), \psi_+^B(\sigma',\tau)\right\} &= -i\eta^{AB}\delta(\sigma-\sigma') \\
\left\{\psi_-^A(\sigma,\tau), \psi_-^B(\sigma,\tau)\right\} &= i\eta^{AB}\delta(\sigma-\sigma')
\end{aligned}
\tag{3.8}
$$

It is worthwhile to mention here that the fermionic Poisson brackets of (3.8) are to be understood as Dirac brackets due to the presence of second class constraints as follows: The canonical momenta associated with $\psi_\pm^A$ are defined as $\pi_\pm^A = \frac{\partial L}{\partial \dot{\psi}_\pm^A} = \pm\frac{i}{2}\psi_\pm^A$; and therefore, give rise to the second class constraints $\pi_\pm^A \mp \frac{i}{2}\psi_\pm^A \simeq 0$. These constraints can be eliminated by standard procedure of introducing Dirac Brackets and last two equations in (3.8) are to be interpreted in that sense. We adopt this prescription in the computation of all Poisson bracket relations involving fermions. Furthermore, we have already taken into account the Majorana nature of the fermions while obtaining

(3.8). It follows from Eqs. (2.25) and (3.8) that

$$\begin{aligned}\left\{\psi^i_\pm(\sigma,\tau),\psi^j_\pm(\sigma',\tau)\right\} &= \mp iG^{ij}(X)\delta(\sigma-\sigma')\\ \left\{P_i(\sigma,\tau),\psi^j_\pm(\sigma',\tau)\right\} &= -\partial_i\psi^j_\pm\delta(\sigma-\sigma')\end{aligned}\tag{3.9}$$

We are now in a position to calculate the algebra of the constraints. Let us note that

$$\begin{aligned}\Big\{J_+(\sigma,\tau) \quad &, \quad J_+(\sigma',\tau)\Big\}\\ &= \left\{G_{ij}\partial_+X^j\psi^i_+(\sigma,\tau),G_{pq}\partial_+X^q\psi^p_+(\sigma',\tau)\right\}\\ &+ \frac{ik}{24\pi}\left\{G_{ij}\partial_+X^j\psi^i_+(\sigma,\tau),T_{pqr}\psi^p_+\psi^q_+\psi^r_+(\sigma',\tau)\right\}\\ &+ \frac{ik}{24\pi}\left\{T_{ijk}\psi^i_+\psi^j_+\psi^k_+(\sigma,\tau),G_{pq}\partial_+X^q\psi^p_+(\sigma',\tau)\right\}\\ &- \frac{k^2}{576\pi^2}\left\{T_{ijk}\psi^i_+\psi^j_+\psi^k_+(\sigma,\tau),T_{pqr}\psi^p_+\psi^q_+\psi^r_+(\sigma',\tau)\right\}\end{aligned}\tag{3.10}$$

Each of these terms can be evaluated using the relations in Eqs. (3.7), (3.8) and (3.9). They take the forms

$$\begin{aligned}\Big\{G_{ij}\partial_+X^j\psi^i_+(\sigma,\tau) \quad &, \quad G_{pq}\partial_+X^q\psi^p_+(\sigma',\tau)\Big\}\\ &= [\,-iG_{ij}\partial_+X^i\partial_+X^j-\frac{k}{4\pi}T_{ijk}\partial_+X^k\psi^i_+\psi^j_+\\ &+ 2\psi^i_+\left(\partial_1\psi^j_+ +\Gamma^{(2)j}_{k\ell}\partial_1X^k\psi^\ell_+\right)G_{ij}\\ &+ \frac{i}{2}R^{(2)}_{ijk\ell}\psi^i_+\psi^j_+\psi^k_-\psi^\ell_-\\ &+ \frac{ik^2}{64\pi^4}G^{mn}T_{ijm}T_{k\ell n}\psi^i_+\psi^j_+\psi^k_+\psi^\ell_+\,]\,\delta(\sigma-\sigma')\end{aligned}\tag{3.11}$$

$$\begin{aligned}\frac{ik}{24\pi}\quad &\left\{G_{ij}\partial_+X^j\psi^i_+(\sigma,\tau),T_{pqr}\psi^p_+\psi^q_+\psi^r_+(\sigma',\tau)\right\}\\ = [\,&\frac{k}{8\pi}T_{ijk}\partial_+X^k\psi^i_+\psi^j_+ +\frac{ik}{24\pi}\partial_p\left(T_{ijk}\psi^i_+\psi^j_+\psi^k_+\right)\psi^p_+\end{aligned}$$

$$+ \quad \frac{ik}{8\pi} E^{Ak} T_{ijk} w^{(1)}_{P,AB} \psi^i_+ \psi^j_+ \psi^B_+ \psi^p_+ \,]\, \delta(\sigma - \sigma') \tag{3.12}$$

$$\begin{aligned} \frac{ik}{24\pi} \quad & \left\{ T_{ijk} \psi^i_+ \psi^j_+ \psi^k_+ (\sigma, \tau) \,,\, G_{pq} \partial_+ X^q \psi^p_+ (\sigma', \tau) \right\} \\ = \quad & \frac{ik}{24\pi} \left\{ G_{ij} \partial_+ X^j \psi^i_+ (\sigma, \tau) \,,\, T_{pqr} \psi^p_+ \psi^q_+ \psi^r_+ (\sigma', \tau) \right\} \end{aligned} \tag{3.13}$$

$$\begin{aligned} & \frac{-k^2}{576\pi^2} \left\{ T_{ijk} \psi^i_+ \psi^j_+ \psi^k_+ (\sigma, \tau) \,,\, T_{pqr} \psi^9_+ \psi^q_+ \psi^r_+ (\sigma', \tau) \right\} \\ = \quad & \frac{ik^2}{64\pi^2} G^{mn} T_{ijm} T_{k\ell n} \psi^i_+ \psi^j_+ \psi^k_+ \psi^\ell_+ \delta(\sigma - \sigma') \end{aligned} \tag{3.14}$$

Combining equations (3.11)–(3.14) and using the identities of Section II (2.22)–(2.23), we obtain

$$\{J_+(\sigma, \tau) J_+(\sigma', \tau)\} = 2iT_{++} \delta(\sigma - \sigma') \tag{3.15}$$

Similarly, we can show that

$$\{J_-(\sigma, \tau) \,,\, J_-(\sigma', \tau)\} = +2iT_{--} \delta(\sigma - \sigma') \tag{3.16}$$

The poisson bracket $\{T_{++}(\sigma,\tau), J_+(\sigma',\tau)\}$ can be explicitly evaluated by noting that

$$\begin{aligned} & \left\{ T_{++}(\sigma, \tau) \quad , \quad J_+(\sigma', \tau) \right\} \\ & \quad = \quad \left\{ \frac{1}{2} G_{ij} \partial_+ X^i \partial_+ X^j (\sigma, \tau) \,,\, J_+(\sigma', \tau) \right\} \\ & \quad + \quad \left\{ i\psi^A_+ \left( \eta_{AB} \partial_1 \psi^B_+ - w^{(1)}_{i,AB} \partial_1 X^i \psi^B_+ \right) (\sigma, \tau) \, J_+(\sigma', \tau) \right\} \\ & \quad + \quad \left\{ -\frac{1}{4} R^{(1)}_{ijk\ell} \psi^i_- \psi^j_- \psi^k_+ \psi^\ell_+ (\sigma, \tau) \,,\, J_+(\sigma, \tau) \right\} \end{aligned} \tag{3.17}$$

Each of these terms can be evaluated to have the form

$$\frac{1}{2} \left\{ G_{ij} \partial_+ X^i \partial_+ X^j \quad , \quad J_+(\sigma', \tau) \right\}$$

$$
\begin{aligned}
= & \ [ - \partial_+ X^i \partial_+ X^j \partial_i \psi^p G_{jp} - \frac{1}{2} \partial_+ X^i \psi_+^p \partial_p G_{ij} \\
+ & \ \partial_+ X^i \left( \partial_p \left( G_{ij} - \frac{k}{4\pi} B_{ij} \right) X^{ij} - \partial_i \left( G_{pq} - \frac{k}{4\pi} B_{pq} \right) X^{1q} \right) \psi_+^p \\
+ & \ E_j^B \partial_+ X^i \partial_+ X^j w_{i,AB}^{(1)} \psi_+^A \\
- & \ \frac{i}{2} R_{ijk\ell}^{(2)} \psi_+^i \psi_-^k \psi_-^\ell \partial_+ X^j \ ] \, \delta(\sigma - \sigma') \\
+ & \ \partial_+ X^i (\sigma, \tau) \left( \left( G_{ip} - \frac{k}{4\pi} B_{ip} \right) (\sigma, \tau) + \left( G_{pi} - \frac{k}{4\pi} B_{pi} \right) (\sigma', \tau) \right) \\
& \ \psi_+^p (\sigma', \tau) \, \partial_1 \delta (\sigma - \sigma')
\end{aligned} \tag{3.18}
$$

$$
\begin{aligned}
& i \Big\{ \psi_+^A \left( \eta_{AB} \partial_1 \psi_+^B - w_{i,AB}^{(1)} \partial_1 X^i \psi_+^B \right) (\sigma, \tau), J_+ (\sigma', \tau) \Big\} \\
= & \ \Big[ - \left( \partial_1 \psi_+^A - 2 w_{i,B}^A \partial_1 X^i \psi_+^B \right) E_{Aq} \partial_+ X^q \\
- & \ i \partial_1 X^i \partial_p w_{i,AB}^{(1)} \psi_+^A \psi_+^B \psi_+^p - i w_{p,AB}^{(1)} \partial_1 \psi_+^A \psi_+^B \psi_+^p \\
- & \ 2i \partial_1 X^i w_{i,A}^{(1)} w_{p,CB}^{(1)C} \psi_+^A \psi_+^B \psi_+^p \\
- & \ \frac{ik}{8\pi} E_A^p T_{pqr} \partial_1 \psi_+^A \psi_+^q \psi_+^r \\
- & \ \frac{ik}{4\pi} \partial_1 X^i E^{Bp} T_{pqr} w_{i,AB}^{(1)} \psi_+^A \psi_+^q \psi_+^r \Big] \, \delta (\sigma - \sigma') \\
+ & \ \Big[ \psi_+^A (\sigma) E_{Aq} (\sigma') \partial_+ X^q (\sigma') - i w_{p,AB}^{(1)} (\sigma) \psi_+^A(\sigma) \psi_+^B(\sigma) \psi_+^p (\sigma') \\
+ & \ i \psi_+^A(\tau) w_{p,AB}^{(1)} (\sigma') \psi_+^B (\sigma') \psi_+^p (\sigma') \\
+ & \ \frac{ik}{8\pi} E_A^p (\sigma') T_{pqr} (\sigma') \psi_+^q (\sigma') \psi_+^r (\sigma') \psi_+^A(\sigma) \Big] \partial_1 \delta (\sigma - \sigma')
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
& \Big\{ - \frac{1}{4} R_{ijk\ell}^{(1)} \psi_-^i \psi_-^j \psi_+^k \psi_+^\ell (\sigma, \tau), J_+ (\sigma', \tau) \Big\} \\
= & \ \frac{i}{4} \Big[ R_{ijk\ell}^{(1)} \psi_+^i \psi_-^k \psi_-^\ell \partial_+ X^j - \frac{1}{8} D_p R_{ijk\ell}^{(1)} \psi_+^i \psi_+^j \psi_-^k \psi_-^\ell \psi_+^p \\
+ & \ \frac{k}{32\pi} G^{\ell r} R_{ijk\ell}^{(1)} T_{pqr} \psi_+^i \psi_+^j \psi_-^k \psi_-^q \psi_+^p \Big] \, \delta (\sigma - \sigma')
\end{aligned} \tag{3.20}
$$

Combining Eqs. (3.18) - (3.20) and using various identities of Section II, Eqs. (2.22) - (2.23), we obtain the following algebra

$$\left\{T_{++}(\sigma,\tau), J_{+}(\sigma',\tau)\right\} = \left(2J_{+}(\sigma,\tau) + J_{+}(\sigma',\tau)\right)\partial_1\delta(\sigma-\sigma') \tag{3.21}$$

Similarly,

$$\left\{T_{--}(\sigma,\tau), J_{-}(\sigma',\tau)\right\} = -\left(2J_{-}(\sigma,\tau) + J_{-}(\sigma',\tau)\right)\partial_1\delta(\sigma-\sigma') \tag{3.22}$$

A much simpler and straightforward calculation gives

$$\left\{J_{+}(\sigma,\tau), J_{-}(\sigma',\tau)\right\} = 0 \tag{3.23}$$

It then follows from the Jacobi identity

$$\left\{\left\{J_{+},J_{+}\right\},J_{-}\right\} + \left\{\left\{J_{-},J_{+}\right\},J_{+}\right\} + \left\{\left\{J_{+},J_{-}\right\},J_{+}\right\} = 0 \tag{3.24}$$

that

$$\left\{T_{++}(\sigma,\tau), J_{-}(\sigma',\tau)\right\} = 0 \tag{3.25}$$

Similarly, it follows from another Jacobi identity

$$\left\{\left\{J_{-},J_{-}\right\},J_{+}\right\} + \left\{\left\{J_{+},J_{-}\right\},J_{-}\right\} + \left\{\left\{J_{-},J_{+}\right\},J_{-}\right\} = 0 \tag{3.26}$$

that

$$\left\{T_{--}(\sigma,\tau), J_{+}(\sigma',\tau)\right\} = 0 \tag{3.27}$$

In order to derive the algebras involving $T_{++}$, we use the following Jacobi identity

$$\left\{\left\{J_{\pm},J_{\pm}\right\},T_{\pm\pm}\right\}\left\{\left\{T_{\pm\pm},J_{\pm}\right\},J_{\pm}\right\} + \left\{\left\{J_{\pm},T_{\pm\pm}\right\},J_{\pm}\right\} = 0$$

and arrive at

$$\left\{T_{\pm\pm}(\sigma,\tau), T_{\pm\pm}(\sigma',\tau)\right\} = \pm\left(T_{\pm\pm}(\sigma,\tau) + T_{\pm\pm}(\sigma',\tau)\right)\partial_1\delta(\sigma-\sigma') \tag{3.28}$$

Furthermore, the relation

$$\left\{T_{\pm\pm}(\sigma,\tau), T_{\mp\mp}(\sigma',\tau)\right\} = 0 \tag{3.29}$$

from the Jacobi identity involving $J_-, J_-$ and $T_{++}$.

This completes the demonstration that the classical algebra of the constraints is indeed the superconformal algebra.

Following the method of Batalin, Fradkin and Vilkovisky[15], we can now write down the BRST charge, $Q_{BRST}$, for the theory.

$$\begin{aligned} Q_{BRST} &= \int d\sigma\Big[T_{++}\eta_+ + T_{--}\eta_- + J_+\lambda_+ \\ &+ J_-\lambda_- + \mathcal{P}_+\partial_\sigma\eta_+\eta_+ - \mathcal{P}_-\partial_\sigma\eta_-\eta_- \\ &+ i\mathcal{P}_+\lambda_+\lambda_+ - iP_-\lambda_-\lambda_- \\ &+ \zeta_+\partial_\sigma\lambda_+\eta_+ + \frac{1}{2}\zeta_+\lambda_+\partial_\sigma\eta_+ \\ &- \zeta_-\partial_\sigma\lambda_-\eta_- - \frac{1}{2}\zeta_-\lambda_-\partial_\sigma\eta_-\Big] \end{aligned} \tag{3.30}$$

where $\eta_\pm$ are anticommuting ghosts and $\mathcal{P}_\pm$ their canonical momenta so that they satisfy following Poisson bracket relations.

$$\begin{aligned} \left\{\mathcal{P}_\pm(\sigma,\tau), \eta_\pm(\sigma',\tau)\right\} &= \delta(\sigma-\sigma'), \\ \left\{\mathcal{P}_\mp(\sigma,\tau), \eta_\pm(\sigma',\tau)\right\} &= 0 \end{aligned} \tag{3.31}$$

Similarly, the even Grassmann type ghosts $\lambda_\pm$ and their corresponding cannical momenta, $\zeta_\pm$, satisfy the canonical Poisson brackets

$$\begin{aligned} \left\{\lambda_\pm(\sigma,\tau),\zeta_\pm(\sigma',\tau)\right\} &= \delta(\sigma-\sigma') \\ \left\{\lambda_\pm(\sigma,\tau),\zeta_\mp(\sigma',\tau)\right\} &= 0 \end{aligned} \tag{3.32}$$

The BRST charge as constructed in Eq. (3.30) is nilpotent at the classical level as a result of the classical algebra. The quantum BRST operator has to be defined with proper normal ordering prescription[9] for the fields appearing in Eq. (3.30). The quantum nilpotency condition of $Q_{BRST}$ restricts the space–time dimension of the target manifold to be 10 as is well known. In the presence of nontrivial background fields $Q^2_{BRST}=0$, at the quantum level imposes strong constraints on the background field configurations. The implications of the nilpotency of the BRST charge for the present model will be reported in a subsequent publication[16].

## IV. The Ward Identities

In this section we shall discuss the invariance properties of the Fradkin–Tseytlin $S$–matrix generating functional[3], $\Sigma$, under the general coordinate transformations in the target manifold. Eventually, we shall derive the gravitational Ward identities (WI), for the $S$–matrix elements, as a consequence of the invariance properties of $\Sigma$. We define the $S$–matrix generating functional in the presence of graviton and the antisymmetric tensor background to be

$$\begin{aligned} \Sigma[G,B] &= \int\sqrt{-G}\mathcal{D}X^i\mathcal{D}\overline{\psi}^i\mathcal{D}\psi^i\mathcal{D}(ghosts) \\ &\qquad exp\Big(iS\left[G,B,X,\overline{\psi},\psi,ghosts\right]\Big) \end{aligned} \tag{4.1}$$

where $G = detG_{ij}$ and $S$ is the full gauge fixed action including the ghost fields. As was emphasized earlier, the generating functional is well defined for those background field configurations which are compatible with conformal invariance of the theory. In other words, $\Sigma$ is required to be BRST invariant. The BRST invariance of $\Sigma$, at the classical level, follows from the Fradkin–Vilkovisky theorem[15]. Indeed, the quantum BRST invariance gives rise to anomalies in general and the absence of these anomalies lead to the equations of motion of the background fields.

We shall focus our attention to study the invariance properties of $\Sigma$ under the general coordinate transformations

$$X^i(\sigma,\tau) \to X^i(\sigma,\tau) + \xi^i(X) \tag{4.2}$$

where $\xi^i(\sigma,\tau)$ is an arbitrary infinitesimal function of $X^i(\sigma,\tau)$. Under the transformation

$$\begin{aligned}
\delta_\xi X^i &= \xi^i(X) \\
\delta_\xi \psi^i &= \frac{\partial \xi^i}{\partial X^m}\psi^m \\
\delta_\xi \overline{\psi}^i &= \frac{\partial \xi^i}{\partial X^m}\overline{\psi}^m
\end{aligned} \tag{4.3}$$

Furthermore, under the shift of coordinates (4.2), $X^i \to X^i + \xi^i(X)$, the background metric $G_{ij}(X)$ and the antisymmetric tensor $B_{ij}(X)$ are shifted to

$$\begin{aligned}
\delta_\xi G_{ij}(X) &= \frac{\partial}{\partial X^m} G_{ij}(X)\xi^m(X) \\
\delta_\xi B_{ij}(X) &= \frac{\partial}{\partial X^m} B_{ij}(X)\xi^m(X)
\end{aligned} \tag{4.4}$$

Note that the variations of all the connections and generalized curvatures can be obtained in a straightforward manner using (4.4). The ghost fields remain invariant under the transformations (4.2). We can compute the variation induced on the full action, $S$, under (4.2) using eqs. (4.3) together with (4.4). It is a straightforward computation to check that the simple relation

$$\delta_\xi S = \mathcal{L}_\xi S \tag{4.5}$$

is satisfied, where $\mathcal{L}_\xi S$ is the Lie derivative with respect to the parameter $\xi$. The Lie derivative of the background fields are defined as follows

$$\begin{aligned}
\mathcal{L}_\xi G_{ij}(X) &= G_{im}(X)\partial_j\xi^m(X) + G_{jm}(X)\partial_i\xi^m(X) + \partial_m G_{ij}(X)\xi^m(X), \\
\mathcal{L}_\xi G^{ij}(X) &= -G^{im}(X)\partial_m\xi^j(X) - G^{jm}(X)\partial_m\xi^i(X) + \partial_m G^{ij}(X)\xi^m(X), \\
\mathcal{L}_\xi B_{ij}(X) &= B_{im}(X)\partial_j\xi^m(X) + B_{mj}(X)\partial_i\xi^m(X) + \partial_m B_{ij}(X)\xi^m(X),
\end{aligned} \tag{4.6}$$

Similarly we compute the Lie derivatives of all other tensors in arriving at the relation (4.5). Notice that the path integral measure remains invariant, at least classically, under the general coordinate transformations. We are aware, however, that quantum effects might destroy the invariance properties of the measure and give rise to anomalies. In fact, we have investigated[17], in the recent past, the effects of such anomalies in the case of compactified closed bosonic string coupled to the gauge background fields. Therefore, the invariance properties of $\Sigma$, presented here, are to be considered as the classical result. Now we argue that the generating functional satisfies the relation

$$\Sigma\,[G,B] = \Sigma\,[G - \mathcal{L}_\xi G, B - \mathcal{L}_\xi B] \tag{4.7}$$

due to the property (4.5) and the invariance of the path integral measure (at least classically) under (4.2). We immediately arrive at

$$0 = \int \mathcal{D}Z \left( \frac{\delta \Sigma}{\delta G_{ij}(Z)} \mathcal{L}_\xi G_{ij}(Z) + \frac{\delta \Sigma}{\delta B_{ij}(Z)} \mathcal{L}_\xi B_{ij}(Z) \right) \tag{4.8}$$

where $\mathcal{L}_\xi G_{ij}$ and $\mathcal{L}_\xi B_{ij}$ are given by (4.6). We can suitably modify the arguments of ref. 7 to deduce the desired WI. The main sequence of steps are as follows. It follows from the definition of $\Sigma$, eq. (4.1), and the relation (4.8) that

$$\begin{aligned} 0 &= < \int \mathcal{D}Z \delta(X(\sigma) - Z) \Big( \int d^2\sigma \frac{\delta L}{\delta G_{ij}} \mathcal{L}_\xi G_{ij} \\ &+ \int d^2\sigma \frac{\delta L}{\delta B_{ij}} \mathcal{L}_\xi B_{ij} \Big) > \end{aligned} \tag{4.9}$$

where $< .... >$ means the averaging with the measure of the path integral together with $\exp(iS)$. We can rewrite the above equation (4.9) as

$$< \int d^2\sigma \left( V_G^{ij} \mathcal{L}_\xi G_{ij} + V_B^{ij} \mathcal{L}_\xi B_{ij} \right) >= 0 \tag{4.10}$$

where the vertex function $V_G^{ij}$ and $V_B^{ij}$ are defined as

$$V_G^{ij} \equiv \frac{\delta L}{\delta G_{ij}} \quad \text{and} \quad V_B^{ij} \equiv \frac{\delta L}{\delta B_{ij}} \tag{4.11}$$

Using eq. (4.6) in (4.10), we get

$$\begin{aligned} &< \int d^2\sigma \Big[ V_G^{ij} \left( G_{im} \partial_j \xi^m + G_{mj} \partial_i \xi^m + \partial_m G_{ij} \xi^m \right) \\ &+ V_B^{ij} \left( B_{im} \partial_j \xi^m + B_{mj} \partial_i \xi^m + \partial_m B_{ij} \xi^m \right) \Big] >= 0 \end{aligned} \tag{4.12}$$

We recall that $\xi^m(X)$ are arbitrary infinitesimal functions. Therefore, eq. (4.12) should continue to hold good if differentiate the left hand side of the equation with respect to $\xi^m(y)$ and then set $\xi^m = 0$; and we obtain

$$< \int d^2\sigma \Big[ V_G^{ij} (G_{im}(X) \partial_j \partial (X(\sigma) - y) + G_{mj}(X) \partial_i \delta (X(\sigma) - y)$$

$$+\partial_m G_{ij}(X)\delta\left(X(\sigma)-y\right)\Big) + V_B^{ij}\Big(B_{im}(X)\partial_i\delta\left(X(\sigma)-y\right)$$

$$+B_{mj}(X)\partial_i\delta\left(X(\sigma)-y\right) + \partial_m B_{ij}(X)\delta\left(X(\sigma)-y\right)\Big)\Big] >= 0 \tag{4.13}$$

We can generate all the WI for the $S$-matrix elements, involving scattering of gravitons and the antisymmetric tensors, starting from the fundamental relations (4.13) in the following manner.

Let us take functional derivatives of the LHS of (4.13) with respect to the background fields $G_{i_I j_I}(X(\sigma_I))$ and $B_{k_J \ell_J}(X(\sigma_J))$

$$\prod_{I=1}^{N}\prod_{J=1}^{M}\frac{\delta}{\delta G_{i_I j_I}\left(X\left(\sigma_I\right)\right)}\frac{\delta}{B_{k_J \ell_J}\left(X\left(\sigma_J\right)\right)} < d^2\sigma\Big[V_G^{ij}\left(G_{im}(X)\partial_i\delta\left(X(\sigma)-y\right)\right)$$

$$+G_{mj}(X)\partial_i\delta\left(X(\sigma)-y\right) + B_{mj}(X)\partial_i\delta\left(X(\sigma)-y\right)$$

$$+\partial_m B_{ij}(X)\delta\left(X(\sigma)-y\right)\Big] > \Big|_{G=G_{bg},B=B_{bg}} = 0 \tag{4.14}$$

It is understood the $G_{ij}$ and $B_{ij}$ are set to their background configurations denoted by $G_{bg}$ and $B_{bg}$ respectively after we take the functional differentiations. The background configurations are compatible with conformal (BRST) invariance of the theory.

Notice that the functional derivatives $\delta/\delta G_{i_I j_I}$ and $\delta/\delta B_{i_I j_I}$ act in three ways.

(i) The Derivative acting on the vertex function $V_G^{ij}$ and/or $V_B^{ij}$ removes $G_{ij}$ and/or $B_{ij}$ present in the vertex function and introduces a $\delta(X(\sigma)-X(\sigma_I))$.

(ii) Next, the background field functional derivatives acting on the terms involving $G_{im}$ and/or $B_{im}$ eliminate it and give rise to a $\delta(X(\sigma)-X(\sigma_I))$ again.

(iii) Finally, each $\delta/\delta G$ and/or $\delta/\delta B$ acting on the action functional, implicitly present in the definition of $< \cdots >$ bring down a factor of vertex function

$$\int d^2\sigma\delta\left(X\left(\sigma\right)-X\left(\sigma_I\right)\right)V_G \text{or} \int d^2\sigma\delta\left(X\left(\sigma\right)-X\left(\sigma_I\right)\right)V_B$$

Therefore, when a string of $N$ functional derivatives of the background fields act on $< \cdots >$ they bring down $N$–vertex functions.

As a consequence of the above arguments, we find that the divergence of $(N+1)$–point amplitude is given by a linear combination of $N$–point amplitudes. It is easy to see from eq. (4.14) that, if we take the Fourier transform then the $(N+1)$–point amplitudes get contracted with momenta $k_i$ (coming from the derivatives of the $\delta$–functions), and this is related to $N$–point amplitudes and some contact terms (appearing due to the presence of $\delta$–functions see (i) and (ii) above). This is the gravitational Ward identity[18] arising due to the invariance properties of the generating functional $\Sigma$.

The vertex functions $V_G^{ij}$ and $V_B^{ij}$ defined through (4.11) take very complicated form when we evaluate them for arbitrary nontrivial background values of $G_{ij}$ and $B_{ij}$. We present below the graviton and antisymmetric vertices for simple background field configurations such as $G_{ij} \neq \eta_{ij}$, the flat metric and $B_{ij} \neq 0$; however, torsion $T_{ijk}=0$.

$$\begin{aligned}
V_G^{pq} &\equiv \left.\frac{\delta L}{\delta G_{pq}}\right|_{G_{pq}=\eta_{pq},T_{ijk}=0} \\
&= \frac{1}{2}\eta^{\mu\nu}\partial_\mu X^p\partial_\nu X^q\delta(X-y)+\frac{i}{2}\psi_+^p\partial_-\psi_+^q\delta(X-y) \\
&- \frac{i}{2}\psi_-^p\partial_+\psi_-^q\delta(X-y) \\
&+ \frac{i}{2}\Big(\psi_+^p\psi_+^q\partial_-X^i\partial_i\delta(X-y)+\psi_+^p\psi_+^i\partial_-X^q\partial_i\delta(X-y)
\end{aligned}$$

$$
\begin{aligned}
&- \quad \psi_+^i \psi_+^q \partial_- X^p \partial_i \delta(X-y)\Big) \\
&- \quad \Big(\psi_-^p \psi_-^q \partial_+ X^i \partial_i \delta(X-y)\Big) + \psi_-^p \psi_-^i \partial_+ X^q \partial_i \delta(X-y) \\
&- \quad \psi_-^i \psi_-^q \partial_+ X^p \partial_i \delta(X-y)\Big) + \frac{1}{2} \partial_\ell \partial_j \delta(X-y)\, \psi_+^p \psi_+^j \psi_-^q \psi_-^\ell \qquad (4.15)
\end{aligned}
$$

and

$$
\begin{aligned}
V_B^{pq} &\equiv \left.\frac{\delta L}{\delta B_{pq}}\right|_{G_{pq}=\eta_{pq}, T_{ijk}=0} \\
&= \frac{k}{8\pi}\Big[\epsilon^{\mu\nu}\partial_\mu X^p \partial_\nu X^q + \frac{i}{2}\psi_+^i \psi_+^p \partial_- X^q \partial_i \delta(X-y) \\
&+ \quad \psi_+^q \psi_+^i \partial_- X^p \partial_i \delta(X-y) + \psi_+^p \psi_+^q \partial_- X^i \partial_i \delta(X-y) \\
&+ \quad \psi_-^i \psi_-^p \partial_+ X^q \partial_i \delta(X-y) + \psi_-^q \psi_-^i \partial_+ X^p \partial_i \delta(X-y) \\
&+ \quad \partial_-^p \psi_-^q \partial_+ X^i \partial_i \delta(X-y)\Big) \\
&+ \quad \frac{1}{2}\partial_i \partial_k \delta(X-y) \psi_-^i \psi_+^k \left(\psi_-^q \psi_+^p + \psi_+^q \psi_-^p\right)\Big] \qquad (4.16)
\end{aligned}
$$

## V. Summary and Discussions:

We have studied the evolution of a type II closed superstring in the background of its massless condensates, namely, the graviton and the antisymmetric Kalb–Ramond fields, We have not considered the dilaton coupling, because our study has been restricted mainly to the classical level and it breaks the superconformal invariance at this level. In this paper we have shown explicitly that the classical superconformal algebra holds even in the presence of torsion. The $Q_{BRST}$ is constructed and the classical Ward identities have been derived. We will describe the quantum nilpotency of the BRST change and the resulting background equations in a subsequent publication.[16]

## VI. Acknowledgements

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