Fermi National Accelerator Laboratory

Fermilab–Pub–88/192–T

Absence of Wavefunction Renormalization in Polyakov Amplitudes

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Abstract

The correct interpretation of recent coupling-constant renormalization calculations shows that there is no one-loop contribution to wavefunction renormalization for closed Polyakov string amplitudes with massless vector external states.

*Research supported in part by the Department of Energy.

12/88

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

1. Introduction

The Polyakov path integral provides a convenient means of calculating on-shell amplitudes in string theories. But it leaves many points about the results obscured. Does it yield bare or physical ('renormalized') S-matrix elements? What is the corresponding choice of space-time field variables in the low-energy effective action? We emphasize that the first question, that of wavefunction renormalization, is important not only as an abstract string theory question, but also to the practical application of strings — the efficient calculation of gauge-theory amplitudes [1,2].

In field theory, the standard technology of Feynman diagrams yields 'bare' Green's functions which must have factors of the square root of the wavefunction normalization removed from external legs in order to produce physical ('renormalized') matrix elements. In the language of effective field theories, one must adjust the Green's functions by a (finite) wavefunction renormalization when making a (finite) change in the choice of renormalization scale. Now, loop amplitudes in closed string theory include integrations over regions of punctured moduli space with loops on external legs, which look very much like field-theory diagrams with loops on external legs; it is these loops which give rise to wavefunction and mass renormalization of those legs. The gluons in string theories remain massless just as in gauge (field) theories because of gauge invariances, but for massive states, in general one expects a non-vanishing mass renormalization. Weinberg [3] has calculated this shift explicitly for the tachyon, and the result is indeed non-vanishing. The man in the street might thus expect a non-trivial wavefunction renormalization for the gluons. After all, gauge invariance does not forbid it; and a wealth of experience in field theory teaches him that whatever isn't forbidden by a symmetry will appear explicitly.

As we shall see, the man in the street would be wrong. There is in fact no wavefunction renormalization for the massless vector states of a closed string theory; and thus the Polyakov path integral yields both the 'bare' and physical S-matrices, because the two are identical. Although we shall demonstrate this statement only at one loop, for the massless vector states, we believe it is quite likely to persist at arbitrary order in perturbation theory.

Although some steps have been made towards off-shell continuations of string amplitudes [4], such formalisms are not yet ready to address the questions we have posed. We will therefore stick to on-shell Polyakov amplitudes. This will require a certain amount of care in extracting renormalizations, but there is really nothing special about string theory in this regard; a field theorist attempting to calculate on the mass shell would encounter difficulties, and with appropriate care, would also circumvent them.

In the next section, we discuss the notion of renormalization in string theories. In sect. 3, we review the β-function calculation of refs. [5,1,6] and Minahan's 'off-sheet' prescription [5] for handling the 'momentum/pole' 0/0 ambiguity that arises when calculating the renormalization of the three-point function. In sect. 4, we demonstrate the correctness of the prescription by factorizing a four-gluon amplitude. In sect. 5, we discuss the question of wavefunction renormalization. In sect. 6, we present a toy field theory exhibiting many of the same peculiarities found in the string theory. In sect. 7, we discuss our conclusions.

For our calculations, we will use a four-dimensional $N = 1$ [space-time] supersymmetric heterotic string model, using the fermionic formulation of Kawai, Lewellen, and Tye [7]. The model reduces in the infinite-tension limit to an $N = 1$ supersymmetric non-Abelian gauge theory. As the reader may verify, however, our conclusions are largely independent of the specific string model chosen, and will also apply to closed bosonic or supersymmetric strings. (We use a supersymmetric model solely to avoid technical problems with dilaton tadpoles; otherwise, and in particular in the low-energy limit, everything in this paper applies to (non-tachyonic) non-supersymmetric models as well.) The model we use, along with our conventions and normalizations, is summarized in two appendices.

2. Renormalization

One might wonder why the question of renormalization need even be discussed in a string theory. After all, according to lore, string theory is finite, and so there are no infinities to be banished. (In fact, string amplitudes are *not* finite; they contain the usual sorts of infrared divergences encountered in any field theory with massless particles. But at least there are no ultraviolet infinities.) But finiteness is beside the point; renormalization has to do with physics, not infinities. Field theory S-matrix elements are finite, too — when expressed in terms of physical couplings; and it is physical couplings and S-matrix elements that experimenters measure, not bare ones.

In field theory, one of course has the freedom to choose different renormalization points for the coupling constants, that is, different definitions of the coupling constants. For example, one may express physical S-matrix elements for a scattering experiment with center-of-mass energy of $\mathcal{O}(\mu_1)$ in terms of either $g(\mu_1)$, or say, $g(\mu_2 \gg \mu_1)$. But the sensible thing to do is to define the physical coupling for a process at energy $\mathcal{O}(\mu_1)$ using $g(\mu_1)$, not $g(\mu_2)$, because this eliminates large logarithms from the radiative corrections, and absorbs them into the definition of the coupling constant.

Although the chance of any experimenter measuring string scattering amplitudes in the visible future is vanishingly small, one nonetheless might imagine an experimenter performing hypothetical string scattering experiments. The sensible choice for the physical coupling in a string theory is then no different from that in a field theory; one should express the computed S-matrix elements in terms of couplings defined at an energy scale comparable to the energy of the scattering experiment, in order to eliminate large logarithms from the radiative corrections. What *is* different about string theory is that the 'bare' coupling is defined at a finite renormalization scale, and so one could choose to express S-matrix elements in terms of 'bare' couplings if one wished without encountering divergences. Renormalization in string theory is thus analogous to *finite* renormalizations that occur in field theory when changing renormalization scales by a finite ratio. But the finiteness of coupling-constant renormalizations in string theory should not obscure the existence of such renormalizations. (It is true that in certain supersymmetric string theories, just as in certain supersymmetric field theories — for example four-dimensional $N = 4$ supersymmetric theories — there is no coupling constant renormalization; but this is a consequence of supersymmetry, and has nothing particular to do with string theory.)

In field theory, there is always a prescription dependence in renormalizations. This comes about because (small) finite shifts in the renormalization can be absorbed into small changes in the radiative corrections. Thus finite renormalizations are always somewhat ambiguous: is the renormalization scale s or $s/2$? The situation in string theory is of course the same; all renormalizations are finite, and thus prescription-dependent. Nonetheless, one thing is clear: all terms which become UV-divergent in the infinite-tension limit should be considered renormalizations, in order to match on to a low-energy effective field theory in a sensible fashion (and in order to accomplish the stated goal of absorbing all large logarithms into renormalization of the coupling constant). With this criterion, our conclusions are general, holding for a variety of prescriptions.

From the viewpoint of a person calculating scaling in a non-Abelian gauge theory, the string tension is simply a physical cut-off (albeit unusual in that it preserves both gauge invariance and unitarity) that one eventually wishes to remove; the scaling behavior can be extracted in the manner one is familiar with from non-gauge theories, by differentiating with respect to the cut-off. From this perspective, the prescription dependence discussed earlier applies only to the finite pieces of the wavefunction renormalization.

In field theory, one studies the various renormalizations by computing various off-shell Green's functions — the two-point function to examine wavefunction and mass renormalization, three- and higher-point functions to examine coupling constant renormalization. If we attempt to carry out a similar program with our on-shell Polyakov amplitudes, we run into trouble immediately, since the two-point function for massless vectors vanishes trivially on shell, as a consequence of gauge invariance. We therefore turn our attention to the three-point function.

3. Coupling Constant Renormalization

The Polyakov path integral leads to an expression for the three-point function,

$$\mathcal{A}_3 = \frac{1}{2(16\pi^2)}\frac{1}{\lambda^2}\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}(\mathrm{Im}\,\tau)\int d^2\nu_1 d^2\nu_2\ \mathcal{Z}_B(\tau)\sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}}\mathcal{Z}_F\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau)\ \langle V^{a_1}(\nu_1)V^{a_2}(\nu_2)V^{a_3}(\nu_3)\rangle^{\vec{\alpha}}_{\vec{\beta};\,\tau} \tag{3.1}$$

where $\lambda = \pi\alpha'$ is the inverse string tension; $\mathcal{Z}_B$ the bosonic partition function; $\vec{\alpha}$ and $\vec{\beta}$ the various boundary conditions for the world-sheet fermions ('spin structures') over which one must sum to obtain a modular-invariant answer; $\mathcal{Z}_F\left[\begin{smallmatrix}\vec{\alpha}\\ \vec{\beta}\end{smallmatrix}\right]$ the fermionic partition function for a given set of boundary conditions (detailed expressions are in appendix II) and $\langle VVV\rangle$ the expectation value of the vertex operators for gauge boson states. This vertex operator [8, 9] is (in the F_1 formalism for the right-movers)

$$V^a(\varepsilon,k;\nu,\overline{\nu}) = -2g\sqrt{\lambda}T^a{}_i{}^j : \Psi^{i\dagger}(\nu)\Psi_j(\nu)\ \varepsilon\cdot\left(\partial_{\overline{\nu}}\overline{X}(\overline{\nu}) + i\sqrt{\lambda}\overline{\Psi}(\overline{\nu})\,k\cdot\overline{\Psi}(\overline{\nu})\right) e^{i\sqrt{\lambda}k\cdot\left(X(\nu)+\overline{X}(\overline{\nu})\right)} : \tag{3.2}$$

(note that the normalization conventions differ slightly from refs. [5,1]). (All calculations in this paper are in Minkowski space, with convergence provided where necessary by rotation to Euclidean space.)

Evaluating the correlation functions for the three-point amplitude and integrating by parts to eliminate double derivatives of the bosonic Green's functions gives

$$\begin{aligned}
&-i\lambda\frac{4g^3}{(16\pi^2)}\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int_T d^2\nu_1 d^2\nu_2\,(\mathrm{Im}\,\tau)\sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}}\mathcal{Z}\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau)\\
&\qquad\Big[\varepsilon_1\cdot\varepsilon_2\dot{G}_B(\overline{\nu}_{12})\Big(\varepsilon_3\cdot k_1\dot{G}_B(\overline{\nu}_{13})\Big(k_1\cdot k_2\dot{G}_B(\overline{\nu}_{12}) - k_2\cdot k_3\dot{G}_B(\overline{\nu}_{23})\Big)\\
&\qquad\qquad + \varepsilon_3\cdot k_2\dot{G}_B(\overline{\nu}_{23})\Big(k_1\cdot k_2\dot{G}_B(\overline{\nu}_{12}) + k_1\cdot k_3\dot{G}_B(\overline{\nu}_{13})\Big)\Big)\\
&\qquad - \varepsilon_1\cdot\varepsilon_2 k_1\cdot k_2(-\dot{G}_B(\overline{\nu}_{12})\varepsilon_3\cdot k_2 - \dot{G}_B(\nu_{13})\varepsilon_3\cdot k_1)G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{12})^2\\
&\qquad - \varepsilon_1\cdot\varepsilon_2(\varepsilon_3\cdot k_1 k_2\cdot k_3 - \varepsilon_3\cdot k_2)G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{12})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\nu_{13})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\nu_{23})\\
&\qquad\qquad + \text{cyclic permutations} + (\varepsilon\cdot k)^3\text{ terms}\Big]\\
&\qquad\times\Big[\mathrm{Tr}\,(T^{a_1}T^{a_2}T^{a_3})\,G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{12})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{23})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{13})\\
&\qquad\qquad + \mathrm{Tr}\,(T^{a_1}T^{a_3}T^{a_2})\,G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{13})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{23})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{12})\Big]\\
&\qquad\times\exp\Big(\lambda k_1\cdot k_2 G_B(\nu_{12}) + \lambda k_2\cdot k_3 G_B(\nu_{23}) + \lambda k_1\cdot k_3 G_B(\nu_{13})\Big)
\end{aligned} \tag{3.3}$$

where G_B is the bosonic Green's function, G_F is a fermionic Green's function, where α_G and β_G are the boundary conditions on the torus of world sheet fermions associated with the gauge group of interest, and where $\alpha_\uparrow$ and $\beta_\uparrow$ are the boundary conditions of the right-mover world-sheet fermions carrying the space-time index. The terms of interest for our discussion are the $\varepsilon_i \cdot \varepsilon_j$ terms, which can yield terms proportional to the tree amplitude. The $(\varepsilon \cdot k)^3$ terms in the amplitude do not contribute to renormalization.

Each term is proportional to a momentum invariant $k_i \cdot k_j$, which vanishes on shell (this is stronger than the usual kinematic vanishing of the three-point amplitude). However, there are regions of punctured moduli space which produce poles in the momentum invariants; these can leave a non-vanishing contribution if we cancel the pole against the momentum invariant. (Of course, we will need a prescription to resolve this 'momentum/pole' 0/0 ambiguity). Such pole contributions arise only from regions where the Koba-Nielsen variables come together so as to isolate the loop on an external leg. Using the short distance behavior of the Green functions given in appendix II, the poles in the $k_i \cdot k_j$ channel are obtained by taking $\nu_i \to \nu_j$ and then integrating over $\nu_{ij} = \nu_i - \nu_j$. The only contributions are from the regions where $\nu_{ij} \simeq 0$, and the integral over ν_{ij} is of the form

$$\int d^2\nu_i |\nu_{ij}|^{-2-\lambda k_i \cdot k_j/\pi} = -\frac{2\pi^2}{\lambda k_i \cdot k_j} \tag{3.4}$$

For example, to extract the $k_1 \cdot k_2$ pole from the $\varepsilon_1 \cdot \varepsilon_2$ terms we perform the integral around $\nu_{12} \simeq 0$ so that the relevant terms in the amplitude (3.3) reduce to

$$\begin{aligned}
&-i\lambda\varepsilon_1 \cdot \varepsilon_2 \,\mathrm{Tr}\,(T^{a_1}[T^{a_2}, T^{a_3}]) \\
&\qquad \frac{4g^3}{(16\pi^2)} \int_{\mathcal{F}_1} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \int_T d^2\nu \,(\mathrm{Im}\,\tau) \sum_{\vec\alpha,\vec\beta} C^{\vec\alpha}_{\vec\beta} \mathcal{Z}\begin{bmatrix}\vec\alpha\\ \vec\beta\end{bmatrix}(\tau) \\
&\qquad\qquad \times \Big(\dot G_B^2(\bar\nu_{23}) - G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar\nu_{23})^2\Big) \\
&\qquad\qquad \times \frac{1}{(2\pi)^2}|\nu_{12}|^{-\lambda k_1 \cdot k_2/\pi - 2}\Big(-\varepsilon_3 \cdot k_1\, k_2 \cdot k_3 + \varepsilon_3 \cdot k_2\, k_1 \cdot k_3\Big) \\
&\qquad\qquad \times G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{23}) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{23}) \exp\Big(\lambda(k_2 \cdot k_3 + k_1 \cdot k_3) G_B(\nu_{13})\Big) \\
&= i\frac{\varepsilon_3 \cdot k_1\, \lambda k_2 \cdot k_3 - \varepsilon_3 \cdot k_2\, \lambda k_1 \cdot k_3}{2\lambda k_1 \cdot k_2}\, g^3 L(\lambda; k_2 + k_1, k_2)
\end{aligned} \tag{3.5}$$

where the loop factor L is

$$\begin{aligned}
L(\lambda; k_1, k_2) \equiv -\frac{4}{(16\pi^2)} \int_{\mathcal{F}_1} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \int_T d^2\nu \,(\mathrm{Im}\,\tau) \sum_{\vec\alpha,\vec\beta} C^{\vec\alpha}_{\vec\beta} \mathcal{Z}\begin{bmatrix}\vec\alpha\\ \vec\beta\end{bmatrix}(\tau) \exp(\lambda k_1 \cdot k_2 G_B(\nu)) \\
\Big(\dot G_B^2(\nu) - G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\nu)^2\Big)\, G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu)
\end{aligned} \tag{3.6}$$

We can now introduce a renormalization scale μ, and separate L into renormalizations and finite pieces,

$$
\begin{aligned}
L(\lambda,\mu;k_1,k_2) &= \mathcal{R}(\lambda;\mu) + L'(\lambda,\mu;k_1,k_2)\\
\mathcal{R}(\lambda;\mu) &= -\frac{4}{(16\pi^2)}\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int_T d^2\nu\,(\mathrm{Im}\,\tau)\sum_{\vec\alpha,\vec\beta} C^{\vec\alpha}_{\vec\beta}\mathcal{Z}\begin{bmatrix}\vec\alpha\\ \vec\beta\end{bmatrix}(\tau)\exp\left(\lambda\mu^2 G_B(\nu)\right)\\
&\qquad\left(\dot G_B^2(\nu) - G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\nu)^2\right)G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu)G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu)\\
L'(\lambda,\mu;k_1,k_2) &= -\frac{4}{(16\pi^2)}\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int_T d^2\nu\,(\mathrm{Im}\,\tau)\sum_{\vec\alpha,\vec\beta} C^{\vec\alpha}_{\vec\beta}\mathcal{Z}\begin{bmatrix}\vec\alpha\\ \vec\beta\end{bmatrix}(\tau)\\
&\qquad\left(\exp\left(\lambda k_1\cdot k_2 G_B(\nu)\right) - \exp\left(\lambda\mu^2 G_B(\nu)\right)\right)\\
&\qquad\left(\dot G_B^2(\nu) - G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\nu)^2\right)G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu)G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu)
\end{aligned}
\tag{3.7}
$$

The renormalization piece $\mathcal{R}$ contains all the $\ln\lambda$ contributions, and thus all the renormalization contributions in the low-energy limit, while L' contains only [on-shell] infrared divergences as $\lambda\to 0$. This split-up is thus natural for linking up with low-energy effective field theories. Alternatively, it is possible to define the renormalization to include *only* the $\ln\lambda$ terms, absorbing all the rest into the loop amplitude. These two prescriptions differ only by terms which vanish as $\lambda\to 0$, and the use of such a prescription would not alter our conclusions.

Of course, there are other potential contributions in the $\nu_1\to\nu_2$ region; collecting all the $\varepsilon\cdot\varepsilon$ terms which contribute in this region we obtain the kinematic factor

$$
\begin{aligned}
&\varepsilon_1\cdot\varepsilon_2\,\frac{k_1\cdot\varepsilon_3\;k_2\cdot k_3 - k_2\cdot\varepsilon_3\;k_1\cdot k_3}{k_1\cdot k_2} + \varepsilon_2\cdot\varepsilon_3\,\frac{k_2\cdot\varepsilon_1(k_1\cdot k_3 + k_2\cdot k_3) - k_3\cdot\varepsilon_1\;k_1\cdot k_2}{k_1\cdot k_2}\\
&\qquad + \varepsilon_1\cdot\varepsilon_3\,\frac{k_3\cdot\varepsilon_2\;k_1\cdot k_2 - k_1\cdot\varepsilon_2(k_1\cdot k_3 + k_2\cdot k_3)}{k_1\cdot k_2}
\end{aligned}
\tag{3.8}
$$

which is ill-defined, since both the numerator and denominator of each term vanish.

This type of ambiguity arises in general N-point gluon amplitudes, when considering a configuration of Koba-Nielsen variables where all but one are brought close together; in the case that the isolated leg is the last one, the denominator is

$$
\sum_{i<j<N} k_i\cdot k_j = k_N^2/2 = 0 \tag{3.9}
$$

while the numerator vanishes because of gauge invariance (this configuration of Koba-Nielsen variables would correspond to a mass renormalization of the gluon if the numerator did not vanish).

Minahan [5] has proposed a prescription for resolving this 'momentum/pole' 0/0 ambiguity. In this prescription, momentum conservation is relaxed, in such a way so that modular invariance is maintained. This is equivalent to requiring each external momentum, and their sum $p = \sum_i k_i$, to be a null vector, so that

$$\sum_{i<j}^{N} k_i \cdot k_j = 0 \tag{3.10}$$

This prescription is sufficiently 'off-sheet' to resolve the ambiguity discussed earlier. It may seem odd, but at least at the three-point level can be justified by examining the four-point function, as we shall do in the next section.

We can rewrite the kinematic factor (3.8) as

$$\begin{aligned} &\varepsilon_1 \cdot \varepsilon_2 \, k_1 \cdot \varepsilon_3 \, \frac{k_2 \cdot k_3 + k_1 \cdot k_3}{k_1 \cdot k_2} + \varepsilon_2 \cdot \varepsilon_3 k_2 \cdot \varepsilon_1 \, \frac{k_1 \cdot k_3 + k_2 \cdot k_3 + k_1 \cdot k_2}{k_1 \cdot k_2} \\ &\quad - \varepsilon_1 \cdot \varepsilon_3 \, k_1 \cdot \varepsilon_2 \, \frac{k_1 \cdot k_2 + k_1 \cdot k_3 + k_2 \cdot k_3}{k_1 \cdot k_2} \end{aligned} \tag{3.11}$$

where all terms proportional to $p \cdot \varepsilon$ have been dropped.

Using the Minahan prescription in equation (3.8), we find for the contribution of the ν_{12} term

$$\begin{aligned} \mathcal{A}_3\big|_{\nu_{12}} &= -\frac{1}{2} i\varepsilon_1 \cdot \varepsilon_2 k_1 \cdot \varepsilon_3 \; g^3 L(\lambda; k_1 + k_2, k_3) \\ &= -\frac{1}{2} i\varepsilon_1 \cdot \varepsilon_2 k_1 \cdot \varepsilon_3 \; g^3 \left(\mathcal{R}(\lambda;\mu) + L'(\lambda; k_1 + k_2, k_3)\right) \end{aligned} \tag{3.12}$$

Since pinches in the Koba-Nielsen variables correspond in tree-level diagrams to the appearance of an explicit pole due to gluon exchange, one is tempted to identify this configuration as a loop isolated on an external leg, and this contribution as a wavefunction renormalization [5]. But the contribution is not cyclicly symmetric in the external leg, and so fails to factorize into a product of a tree-level three-point vertex and a propagator on an external leg. This suggests that this contribution cannot be *all* wavefunction renormalization, but also includes some vertex renormalization. It is only when we add all the different pole contributions (which appear to be loops on the various external legs) that cyclic symmetry, and resulting proportionality to the three-point tree diagram, is restored. One might worry that this behavior is due to the peculiarity of the Minahan procedure, but as we shall see, a similar state of affairs arises in a pole configuration of the four-point variables that does not require any 'off-sheet' prescription to become to well-defined.

The most general form one might expect for the ν_{12}-pole contribution to the $\mathcal{R}$ terms is

$$c_1\varepsilon_1 \cdot \varepsilon_2 \, k_1 \cdot \varepsilon_3 + c_2\varepsilon_2 \cdot \varepsilon_3 \, k_2 \cdot \varepsilon_1 + c_2\varepsilon_1 \cdot \varepsilon_3 \, k_3 \cdot \varepsilon_2 \tag{3.13}$$

(the last two coefficients must be equal because of the reflection symmetry of the Polyakov amplitude). We can decompose this into a cyclicly symmetric piece,

$$c_2 \left(\varepsilon_1 \cdot \varepsilon_2 \, k_1 \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 \, k_2 \cdot \varepsilon_1 + \varepsilon_1 \cdot \varepsilon_3 \, k_3 \cdot \varepsilon_2\right) \tag{3.14}$$

and a left-over asymmetric piece. By factorization, the three-point diagrams which contain wavefunction renormalization must be proportional to the three-vertex terms which are cyclicly symmetric, so naïvely, one would interpret c_2 as a contribution to the wavefunction renormalization, and the left-over asymmetric term as a contribution to vertex renormalization. In equation (3.12), c_2 vanishes, so this implies that the renormalization contributions we are seeing are *all* vertex renormalization, and contain no wavefunction renormalization piece at all. One cannot actually draw this conclusion from the three-point function alone, because the identification of cyclicly symmetric terms as contributions to wavefunction renormalization is not a unique procedure; vertex renormalization can always be shifted into wavefunction renormalization and an appropriate rescaling of external states*. Nevertheless, as we shall see in sect. 5, the naïve conclusion turns out to be correct.

4. Renormalization of the Four-Point Amplitude

Similar to the renormalization of the three-point amplitude considered in the previous section, one may compute the renormalization of the four-point amplitude. Here, one finds that after integration by parts, there are two different sorts of configurations of the Koba-Nielsen variables that contribute: one kind of configuration where two independent pairs of νs are pinched (an 'internal-loop' piece), shown pictorially in Fig. 1a, and another where three νs are pinched together (an 'external-loop' piece), shown in Fig. 1b. Both leave a factor of an integral of the same form as in equation (3.6), which gives rise to the renormalization piece. (Terms with fewer pinches cannot give rise to $\ln\lambda$'s and thus do not yield any contributions proportional to $\mathcal{R}$; they should not be interpreted as a renormalization of the coupling with either of the renormalization prescriptions mentioned earlier. Terms with all four Koba-Nielsen variables pinched together vanish because of supersymmetry.) The first kind of contribution suffers from no 'momentum/pole' 0/0 ambiguity, and does not require any sort of 'off-sheet' prescription to make it well-defined. The second kind does suffer from the ambiguity, and a prescription, such as the Minahan modular-invariant prescription, is necessary in order to make it well defined. There are two important things the four-point amplitude can teach us. First, the contributions from the two kinds of configurations are not independently gauge-invariant. Only when the two are added together do we get a kinematic factor proportional to the tree amplitude (and thus gauge-invariant on shell). This demonstrates the validity of the Minahan prescription for the four-point amplitude, as it preserves gauge invariance as

* In ordinary field theory, this potential ambiguity is resolved by the two point function, which defines wavefunction renormalization. In string theory, we cannot use the two-point function to determine the value of the wavefunction renormalization since it vanishes because of the on-shell condition, and the off-shell extension is obscure. Thus, one must resort to more indirect methods.

well as the renormalizability of the effective low-energy theory. Furthermore, the factorization of the prescription-independent piece of the four-point amplitude verifies that the Minahan prescription is correct for the three-point amplitude.

That the 'internal-loop' piece is not by itself proportional to the tree amplitude tells us that it cannot be interpreted as a renormalization of an internal propagator. Explicitly, the contribution proportional to $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$ is

$$\begin{aligned}\mathcal{A}_4\Big|_{\text{internal loop renormalisation}} &= \\ 2g^4\mathcal{R}(\lambda;\mu)\frac{1}{st}\big(&\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4t^2+\varepsilon_1\cdot\varepsilon_3\varepsilon_2\cdot\varepsilon_4st+\varepsilon_1\cdot\varepsilon_4\varepsilon_2\cdot\varepsilon_3s^2-\varepsilon_1\cdot\varepsilon_2k_2\cdot\varepsilon_3k_3\cdot\varepsilon_4t \\ &-\varepsilon_1\cdot\varepsilon_2k_2\cdot\varepsilon_3k_2\cdot\varepsilon_4t-\varepsilon_1\cdot\varepsilon_2k_1\cdot\varepsilon_3k_2\cdot\varepsilon_4t+\varepsilon_1\cdot\varepsilon_4k_1\cdot\varepsilon_3k_3\cdot\varepsilon_2s \\ &-\varepsilon_1\cdot\varepsilon_4k_1\cdot\varepsilon_2k_2\cdot\varepsilon_3s+\varepsilon_2\cdot\varepsilon_3k_2\cdot\varepsilon_1k_3\cdot\varepsilon_4s-\varepsilon_2\cdot\varepsilon_3k_2\cdot\varepsilon_4k_3\cdot\varepsilon_1s \\ &-\varepsilon_3\cdot\varepsilon_4k_2\cdot\varepsilon_1k_3\cdot\varepsilon_2t+\varepsilon_3\cdot\varepsilon_4k_1\cdot\varepsilon_2k_3\cdot\varepsilon_1t\big)\end{aligned} \tag{4.1}$$

There are two parts to this contribution, one arising from the double pinch $\nu_1\to\nu_2;\nu_3\to\nu_4$, and the other from the double pinch $\nu_1\to\nu_4;\nu_2\to\nu_3$. The former part separates into two terms, each of which factorizes into a product of a one-pinch contribution to the one-loop-corrected three-point function we found in the previous section, and a tree-level three-point function:

$$\begin{aligned}-i\mathcal{A}_4\Big|_{\text{internal loop renormalisation}} &= \\ &\Big(-i\mathcal{A}_3^{\text{loop}}\Big|_{\text{one pinch}}(1,2,I_\mu)\Big)\left(-i\frac{g_{\mu\nu}}{s}\right)\Big(-i\mathcal{A}_3^{\text{tree}}(I_\nu,3,4)\Big) \\ &+\Big(-i\mathcal{A}_3^{\text{tree}}(1,2,I_\mu)\Big)\left(-i\frac{g_{\mu\nu}}{s}\right)\Big(-i\mathcal{A}_3^{\text{loop}}\Big|_{\text{one pinch}}(I_\nu,3,4)\Big) \\ &+\ \text{finite as } s\to 0\end{aligned} \tag{4.2}$$

where we have extracted the part proportional to $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$, after summing over the colors of the propagator on the right-hand side. (The phases are determined by factorizing a four-point tree amplitude.)

Indeed, there is *no* contribution that can be interpreted as a product of two tree-level three-point functions with a one-loop-corrected propagator connecting the two. Although it is possible, as discussed in the previous section, to change the wavefunction renormalization and vertex renormalization by shifting part of the latter into the former along with a rescaling of the external states, the requirement that both the three- and four-point amplitudes yield the same coupling constant renormalization fixes this ambiguity. In the next section, we will rephrase this argument in a form independent of the detailed algebraic structure of the three- and four-point amplitudes.

The factorization shown above also singles out the Minahan prescription for the three-point function as yielding the unique answer consistent with unitarity. We can imagine contructing a three-point amplitude with one leg off-shell by factorizing a four-point amplitude on a gluon pole, for example on the pole in $(k_3+k_4)^2$. If we then extracted the contributions proportional to $\mathcal{R}(\lambda;\mu)$ arising from the $\nu_1 \to \nu_2$ pinch, we would find exactly the three-point loop amplitude in the first term of equation (4.2), the same answer as was obtained using the Minahan prescription.

The 'external-loop' contributions proportional to $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$ have the form

$$
\begin{aligned}
&\mathcal{A}_4\Big|_{\text{external loop renormalization}} = \\
&2g^4\mathcal{R}(\lambda;\mu)\frac{1}{st}\big(\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4 st - 2\varepsilon_1\cdot\varepsilon_3\varepsilon_2\cdot\varepsilon_4 st + \varepsilon_1\cdot\varepsilon_4\varepsilon_2\cdot\varepsilon_3 st - \varepsilon_1\cdot\varepsilon_2 k_2\cdot\varepsilon_3 k_3\cdot\varepsilon_4 t \\
&\quad +2k_2\cdot\varepsilon_1\varepsilon_2\cdot\varepsilon_3 k_3\cdot\varepsilon_4 t - 2\varepsilon_1\cdot\varepsilon_3 k_1\cdot\varepsilon_2 k_3\cdot\varepsilon_4 t - \varepsilon_3\cdot\varepsilon_4 k_2\cdot\varepsilon_1 k_3\cdot\varepsilon_2 t \\
&\quad +k_1\cdot\varepsilon_2 k_3\cdot\varepsilon_1\varepsilon_3\cdot\varepsilon_4 t - \varepsilon_1\cdot\varepsilon_2 k_2\cdot\varepsilon_3 k_2\cdot\varepsilon_4 t - \varepsilon_1\cdot\varepsilon_2 k_1\cdot\varepsilon_3 k_2\cdot\varepsilon_4 t \\
&\quad +2k_2\cdot\varepsilon_1 k_2\cdot\varepsilon_3\varepsilon_2\cdot\varepsilon_4 t + 2\varepsilon_2\cdot\varepsilon_4 k_1\cdot\varepsilon_3 k_2\cdot\varepsilon_1 t - 2\varepsilon_1\cdot\varepsilon_4 k_1\cdot\varepsilon_2 k_2\cdot\varepsilon_3 t \\
&\quad -2k_1\cdot\varepsilon_2 k_1\cdot\varepsilon_3\varepsilon_1\cdot\varepsilon_4 t + 2\varepsilon_1\cdot\varepsilon_3 k_3\cdot\varepsilon_2 k_3\cdot\varepsilon_4 s - 2\varepsilon_1\cdot\varepsilon_2 k_2\cdot\varepsilon_3 k_3\cdot\varepsilon_4 s \\
&\quad +k_2\cdot\varepsilon_1\varepsilon_2\cdot\varepsilon_3 k_3\cdot\varepsilon_4 s - 2\varepsilon_3\cdot\varepsilon_4 k_3\cdot\varepsilon_1 k_3\cdot\varepsilon_2 s - 2\varepsilon_3\cdot\varepsilon_4 k_2\cdot\varepsilon_1 k_3\cdot\varepsilon_2 s \\
&\quad +2\varepsilon_1\cdot\varepsilon_3 k_2\cdot\varepsilon_4 k_3\cdot\varepsilon_2 s + \varepsilon_1\cdot\varepsilon_4 k_1\cdot\varepsilon_3 k_3\cdot\varepsilon_2 s - \varepsilon_2\cdot\varepsilon_3 k_2\cdot\varepsilon_4 k_3\cdot\varepsilon_1 s \\
&\quad +2\varepsilon_2\cdot\varepsilon_4 k_2\cdot\varepsilon_3 k_3\cdot\varepsilon_1 s - 2\varepsilon_1\cdot\varepsilon_2 k_2\cdot\varepsilon_3 k_2\cdot\varepsilon_4 s + 2\varepsilon_2\cdot\varepsilon_4 k_2\cdot\varepsilon_1 k_2\cdot\varepsilon_3 s \\
&\quad -\varepsilon_1\cdot\varepsilon_4 k_1\cdot\varepsilon_2 k_2\cdot\varepsilon_3 s\big)
\end{aligned}
\tag{4.3}
$$

The reader may verify that the sum of the contributions in equations (4.1) and (4.3) is proportional to the coefficient of $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$ in the tree string amplitude.

This does not uniquely single out the Minahan prescription as a resolution of the 0/0 ambiguity for the four- or higher-point amplitudes, but it is quite likely that by repeating the factorization arguments given earlier for an $(N+1)$-point function, that one would derive the Minahan prescription for the N-point amplitude.

5. Wavefunction Renormalization

As we have seen in previous sections, the renormalization of the three- and four-point functions suggests that the one-loop contribution to wavefunction renormalization vanishes in Polyakov amplitudes. In gauge theories, Ward identities provide powerful constraints on renormalization. In particular, the identities constrain the numerical coefficient of the β-function for the entire four-point amplitude's coupling constant, g^2, to be twice that of the β-function for the three-point coupling, g. Since wavefunction renormalization contributes differently to the three- and four-point amplitudes, its value can be determined by comparing the three- and four-point string amplitudes.

The argument can be phrased in a simple fashion, not relying on the detailed algebra and factorizations used in the preceeding discussions. As a side benefit, it will also turn out to be independent of the over-all normalization of tree- and one-loop amplitudes.

We proceed as follows. The tree and one-loop amplitudes are

$$
\begin{aligned}
A_3^{\rm tree} &= g f_3(\{k_i, \varepsilon_i, T^{a_i}\}) \\
A_4^{\rm tree} &= g^2 f_4(\{k_i, \varepsilon_i, T^{a_i}\}) \\
A_3^{\rm loop} &= K_3^s g^3 \mathcal{R}(\lambda;\mu) f_3(\{k_i, \varepsilon_i, T^{a_i}\}) + \mathcal{O}(\lambda^0) \\
A_4^{\rm loop} &= K_4^s g^4 \mathcal{R}(\lambda;\mu) f_4(\{k_i, \varepsilon_i, T^{a_i}\}) + \mathcal{O}(\lambda^0)
\end{aligned}
\tag{5.1}
$$

where the K_i^s are renormalization coefficients, analogous to the coefficients of the 'divergent' pieces of diagrams encountered in field theory, where the f_i are proportional to the kinematic and color factors of the three- and four-point functions in a non-Abelian gauge field theory, and where μ is the renormalization scale. We have picked out the gluon pole terms in $A_4^{\rm tree}$, since we are studying its wavefunction renormalization. Note that calculating renormalization in string theory is similar to calculating renormalization in field theory from the *full* amplitude, including loops on external legs.

The coupling-constant renormalization coefficient can be expressed in terms of the K_i^s and the wavefunction renormalization coefficient as follows,

$$
\begin{aligned}
K_g &= K_3^s + \frac{3}{2} K_A \\
K_{g^2} &= 2K_g = K_4^s + 2K_A
\end{aligned}
\tag{5.2}
$$

so that

$$
K_A = K_3^s \left(\frac{K_4^s}{K_3^s} - 2 \right)
\tag{5.3}
$$

We can write the ratio of the K_i^s directly as

$$
\frac{K_4^s}{K_3^s} = \frac{\text{Coefficient of } \mathcal{R}(\lambda;\mu) \text{ in } A_4^{\rm loop}}{A_4^{\rm tree}} \, \frac{A_3^{\rm tree}}{\text{Coefficient of } \mathcal{R}(\lambda;\mu) \text{ in } A_3^{\rm loop}}
\tag{5.4}
$$

which form makes it clear, as advertised, that it is independent of the over-all normalization of tree or one-loop amplitudes. Explicit calculation gives

$$
\begin{aligned}
A_3^{\rm tree} &= g f_3(\{k_i, \varepsilon_i, T^{a_i}\}) \\
A_4^{\rm tree} &= g^2 f_4(\{k_i, \varepsilon_i, T^{a_i}\}) \\
\text{Coefficient of } \mathcal{R}(\lambda;\mu) \text{ in } A_3^{\rm loop} &= g^3 f_3(\{k_i, \varepsilon_i, T^{a_i}\}) \\
\text{Coefficient of } \mathcal{R}(\lambda;\mu) \text{ in } A_4^{\rm loop} &= 2g^4 f_4(\{k_i, \varepsilon_i, T^{a_i}\})
\end{aligned}
\tag{5.5}
$$

We thus find that $K_4^s/K_3^s = 2$; this in turn implies that $K_A = 0$. Now, the one-loop β-function of the string model is given by $g^3 \partial\mathcal{R}(\lambda;\mu)/\partial \ln\sqrt{\lambda}$; in the low-energy limit, $\lambda \to 0$, this yields exactly the β-function of the low-energy non-Abelian gauge theory [5]; if the one-loop contribution to wavefunction renormalization did not vanish, we would have to peel away three-halves of the latter from the coefficient of $A_3^{\rm loop}$ in order to obtain the correct β-function. The previous calculations [5,1,6] are thus consistent with the result obtained here (although they did not give the correct interpretation). Kaplunovsky [10] has also calculated the β-function of the low-energy field theory limit of the string theory, using a background-field method, in agreement with the other results. The background-field method [11] does not, however, tell us anything about Polyakov amplitudes. Were the Polyakov amplitude generating background-field S-matrix elements, we would have found that $K_4^s/K_3^s = 12/7$ as required by the background-field Ward identity $Z_3 = Z_A^{-1/2}$, where Z_A and Z_3 are the usual two- and three-point renormalization constants.

One remaining puzzle involves the two-point function. It has the form

$$k_1 \cdot k_2\, \varepsilon_1 \cdot \varepsilon_2 g^2 \mathcal{R}(\lambda;\mu) + \mathcal{O}(\lambda^0) + (\varepsilon \cdot k)^2 \text{ terms} \tag{5.6}$$

As noted earlier, it vanishes on-shell, because of gauge invariance. It also vanishes when using Minahan's prescription. One might nonetheless be tempted to simply remove the $k_1 \cdot k_2$ in front, and interpret the remaining coefficient of the $\varepsilon \cdot \varepsilon$ term as a one-loop contribution to wavefunction renormalization. Our earlier discussion shows that this naïve off-shell continuation is simply not correct.

6. A Field Theory Toy Model

In the infinite tension limit, the string amplitudes should reduce to the amplitudes of the low energy effective field theory; but as discussed in the previous sections the amplitudes contain features which are quite puzzling from a field theory point of view.

In this section, we construct a simple scalar field theory toy model whose amplitudes display many of the features we found in the string amplitudes. We will present a scalar field theory with the following properties: (a) its amplitudes possess an on-shell 'momentum/pole' 0/0 ambiguity; (b) there is no one-loop contribution to wavefunction renormalization, although the theory has non-trivial scattering amplitudes; (c) a naïve analysis of the three-point function yields a fake wavefunction renormalization; (d) the theory contains fake propagators.

Our model field theory has the action

$$\begin{aligned} S[\phi] = \int d^D x\, \Big[&\frac{1}{2}\phi(-\partial^2 - m^2)\phi - \frac{\lambda}{3!}\phi^3 + \alpha\lambda\phi^2(-\partial^2 - m^2)\phi - \frac{\alpha\lambda^2}{2}\phi^4 \\ &+ \frac{\alpha^2\lambda^2}{2}\phi^2(-\partial^2 - m^2)\phi^2 - \frac{\alpha^2\lambda^3}{2}\phi^5 - \frac{\alpha^3\lambda^4}{3!}\phi^6 \Big] \end{aligned} \tag{6.1}$$

where α is to be chosen appropriately.

Actually, this theory has two other properties: it is renormalizable on shell in six dimensions and super-renormalizable in four dimensions; and its perturbative S-matrix elements are unitary. At first sight, these last claims may appear to be manifestly wrong, but as we shall see they are in fact rather trivially true.

The three-point vertices generated by this Lagrangian are displayed in Fig. 2. From the vertices, it is not difficult to construct the two-point diagrams given in Fig. 3, which determine the mass and wavefunction renormalization. (For convenience, we have neglected the tadpole and cactus diagrams which only contribute to mass renormalization.) The truncated two-point function is

$$\Pi(p^2) = \frac{\lambda^2}{2}\Big[\delta m^2 + (p^2 - m^2)K_\phi + 4\alpha(p^2 - m^2)\delta m^2 + O((p^2 - m^2)^2)\Big] \tag{6.2}$$

where

$$\delta m^2 \equiv \int \frac{d^D p}{(2\pi)^4}\,\frac{1}{(p^2 - m^2)^2} \tag{6.3}$$

$$K_\phi \equiv \frac{d}{dk^2}\int \frac{d^D p}{(2\pi)^4}\,\frac{1}{(p^2 - m^2)((p-k)^2 - m^2)}\Big|_{k^2=m^2}$$

From the form of the vacuum polarization it is clear that for the choice

$$\alpha = -\frac{K_\phi}{4\delta m^2} \tag{6.4}$$

the wavefunction renormalization vanishes. For $D < 4$ the integrals converge and α is some finite constant. For $D \geq 4$ where the integrals diverge α will depend on the dimensional regularization parameter $\epsilon = 4 - D$.

The three-point amplitude contains an on-shell ambiguity, analogous to the one encountered in the on-shell Polyakov string amplitude. For example, when the diagram in Fig. 4 is placed on shell we find an on-shell internal propagator multiplied by an vanishing vertex — this is the 'momentum/pole' 0/0 ambiguity. Of course, in the field theory case we can work entirely off shell with a well defined two-point function, an alternative not available in the standard Polyakov approach. This allows for an unambiguous factorization of the three-point function in field theory and the resolution of the ambiguity is clear.

One feature of striking similarity to the string amplitude is the incorrect conclusion of non-trivial wavefunction renormalization as determined from a naïve analysis of the three-point amplitude. Consider the truncated three-point one-loop amplitude with momenta k_2 and k_3 on shell, but with momentum k_1 off shell. In the string, the analogous amplitude is obtained by factorizing the four-point amplitude, as discussed in the previous section. A naive approach to determining the wavefunction renormalization on the first leg would be to collect all terms proportional to $k_1^2 - m^2$

in the truncated amplitude (before dividing by the propagator pole) and calling that the wavefunction renormalization. The Feynman diagrams contributing to this are given in Fig. 5. Summing over the four contributions yields a non-vanishing result for the "wavefunction renormalization" in contradication to the two-point computation. The problem is that we have implicitly assumed an incorrect factorization of the three-point amplitude; the diagram of Fig. 5d should not be included in the wavefunction renormalization part of the $\lambda\phi^3$ coupling-constant renormalization.

The unitarity and renormalizability properties mentioned earlier follow from the fact that the action (6.1) generates the physical S-matrix elements of ordinary ϕ^3 theory; the reader may convince himself of this by explicitly working out Feynman diagrams to arbitrarily high orders. Alternatively, it is clear that the action (6.1) is related the standard ϕ^3 action

$$S[\phi] = \int d^D x \, \left[\frac{1}{2}\phi(-\partial^2 - m^2)\phi - \frac{\lambda}{3!}\phi^3\right] \tag{6.5}$$

by the change of variables

$$\phi \to \phi + \alpha\lambda\phi^2 \, . \tag{6.6}$$

Since the logarithm of the jacobian of this transformation is proportional to $\delta^D(0)$ (which we can drop in dimensional regularization), the field redefinition theorem [12] guarantees that the physical S-matrix elements from either action (6.1) or (6.5) are identical. However, the renormalization constants differ for the two choices of field variables, as we have explicitly seen.

The diagrammatic analysis of the correspondence between the action (6.1) and ordinary an ϕ^3 theory teaches us another important lesson. The pole in the diagram of Fig. 4 does not appear in the S-matrix of the action (6.1); the correct interpretation of this diagram in the on-shell limit is given in Fig. 6. Poles in Feynman diagrams do not necessarily imply the existence of those poles in the S-matrix, since momentum factors associated with vertices may cancel them. (With the standard choice of field variables, as in the action (6.5), the poles are in Feynman diagrams are in one-to-one correspondence with the poles in the S-matrix.) As the Polyakov amplitude does not specify the field variables of the low-energy field theory, poles which arise from the pinching of Koba-Nielsen variables need not correspond to poles in the S-matrix. Our analysis in previous sections gives explicit examples of pinches which do not give rise to physical poles.

7. Conclusions

In this paper, we have demonstrated the validity of the modular-invariant Minahan prescription [5] for resolving an on-shell ambiguity in Polyakov amplitudes, and we have used this prescription to show that the one-loop correction to the wavefunction renormalization vanishes. This is a general feature of Polyakov amplitudes, holding for a variety of renormalization prescriptions which match on to the usual low-energy effective field theory limit. It is also independent of the particular gauge group of the string theory.

In gauge field theory, one can of course make loop contributions to wavefunction renormalization vanish by choosing an appropriate gauge parameter, $\xi = 3/13$, for pure Yang-Mills. Here, however, the situation is different. The propagator, $1/L_0$, has trivial tensor structure, analogous to Feynman gauge ($\xi = 1$), so that is presumably not the explanation.

At tree level, the ambiguity in the low energy effective action of the string has been extensively studied [13] and as expected the S-matrix of the effective field theory is *completely* unaffected by any local field redefinition. However, different choices of field variables can lead to different values for the wavefunction renormalization at loop level, as we have seen in the toy field theory example. While we do not know what field variables A_μ (of the effective low-energy field theory) are chosen by the Polyakov path integral, the absence of loop corrections to the wavefunction renormalization gives a clue.

While the field redefinition theorem [12] guarantees that the perturbative physical S-matrix is unchanged under a wide class of field redefinitions, there is no such guarantee for non-perturbative physics, and so understanding the choice of field variables is in principle important for non-perturbative string physics, as well as for the use of string technology in calculating loop corrections to gauge theory amplitudes.

The research of both authors was supported in part by the Department of Energy. We would like to thank the Niels Bohr Institute for its hospitality during the commencement of this research, H. B. Nielsen for helpful discussions, and J. Polchinski and S. Weinberg for asking the right questions.

Appendix I. String Model

The string model used as a reference in this paper has an $N = 1$ supersymmetry, and contains an $SU(18)$ supersymmetric non-Abelian gauge theory in its low-energy limit.

In the KLT formalism, the boundary conditions for the complex Bardakci-Halpern [14] world-sheet fermions are represented by vectors $W_i = (l_1 \ldots l_{22} | r_1 \ldots r_{10})$, where the l_i component signifies that the ith left-mover fermion picks up an $\exp(-2\pi l_i)$ phase when going around the appropriate (world-sheet space or time) closed loop. A model specified by a set of basis vectors W_i is a consistent string theory if it satisfies certain constraint equations, eqs. (3.33–35) of ref. [7]. The space boundary-condition vectors specify the sectors, while the time boundary-condition vectors determine the generalized GSO projections that constrain the spectrum. The mass squared of a given state is determined by adding the quanta of the fermionic world-sheet oscillators to the vacuum energy with the usual left-right level-matching. Modular invariance requires that in calculating the partition function (or scattering amplitudes) we sum over time- and space-boundary conditions, with coefficients given in eq. (3.32) of ref. [7].

The model at hand is specified by the five "basis" vectors,

$$\begin{aligned}
W_0 &= \left(\frac{1}{2}^{22}\middle|\,\frac{1}{2}^{10}\right),\\
W_1 &= \left(0^4\frac{1}{3}^{18}\middle|\,0^{10}\right),\\
W_2 &= \left(\frac{1}{2}\frac{1}{2}00^{19}\middle|\,\frac{1}{2}\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)\left(\frac{1}{2}00\right)\left(\frac{1}{2}00\right)\right),\\
W_3 &= \left(0\frac{1}{2}^2 0^{19}\middle|\,\frac{1}{2}\left(0\frac{1}{2}0\right)\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)\left(0\frac{1}{2}0\right)\right),\\
W_4 &= \left(\frac{1}{2}0\frac{1}{2}0^{19}\middle|\,\frac{1}{2}\left(00\frac{1}{2}\right)\left(00\frac{1}{2}\right)\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)\right)
\end{aligned} \tag{I.1}$$

with "structure constants"

$$k_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}. \tag{I.2}$$

This model has the nice feature that the gauge bosons are found only in the W_0 sector while the gluinos are only in the $W_0 + W_2 + W_3 + W_4$ sector.

Appendix II. Notation and Normalizations

We define theta functions for general twisted boundary conditions by

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu\,|\,\tau) = \sum_{n\in Z} e^{\pi i(n+\alpha-1/2)^2\tau} e^{2\pi i(n+\alpha-1/2)(\nu-\beta-1/2)} \tag{II.1}$$

Then

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu\,|\,\tau) = e^{\pi i\alpha^2\tau} e^{2\pi i\alpha(\nu-\beta-1/2)}\vartheta_1(\nu+\alpha\tau-\beta|\tau) \tag{II.2}$$

where $\vartheta_1 = \vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ is the conventional first Jacobi theta function.

We remind the reader of the definition of the Dedekind η function,

$$\begin{aligned}
\eta(\tau) &= e^{\pi i\tau/12}\prod_{n=1}^{\infty}\left(1-e^{2\pi i n\tau}\right)\\
&= \sqrt[3]{\vartheta'\begin{bmatrix}0\\0\end{bmatrix}(0\,|\,\tau)\Big/2\pi}
\end{aligned} \tag{II.3}$$

where the prime indicates differentiation with respect to the first argument.

The bosonic partition function is

$$\mathcal{Z}_B(\tau) = \left(\eta(\tau)\overline{\eta}(\tau)\sqrt{\operatorname{Im}\tau}\right)^{2-D} \tag{II.4}$$

where D is the number of spacetime dimensions.

We define $\mathcal{Z}_1\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\tau)$ to be the partition function for a single left-moving complex fermion with $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ boundary conditions,

$$\mathcal{Z}_1\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) = \operatorname{Tr}\left[e^{2\pi i \hat{H}_\alpha \tau} e^{2\pi i(\frac{1}{2}-\beta)\hat{N}_\alpha}\right] = \frac{e^{-2\pi i(1/2-\alpha)(1/2+\beta)}\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](0\,|\,\tau)}{\eta(\tau)} \tag{II.5}$$

where the phase is present in order to be consistent with the KLT definition [7]. It is really irrelevant, and could be absorbed into the definitions of the summation coefficients $C^{\vec{\alpha}}_{\vec{\beta}}$.

Putting the pieces together, the complete partition function for the set of fermions with $\left[\begin{smallmatrix}\vec{\alpha}\\ \vec{\beta}\end{smallmatrix}\right]$ boundary conditions is

$$\begin{aligned}
\mathcal{Z}\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau) &= \mathcal{Z}_B(\tau)\prod_{i=1}^{\operatorname{len}\alpha_L}\mathcal{Z}_1\begin{bmatrix}\alpha_{Li}\\ \beta_{Li}\end{bmatrix}(\tau)\prod_{i=1}^{\operatorname{len}\alpha_R}\overline{\mathcal{Z}_1}\begin{bmatrix}\alpha_{Ri}\\ \beta_{Ri}\end{bmatrix}(\tau)\\
&= \left(\eta(\tau)\overline{\eta}(\tau)\sqrt{\operatorname{Im}\tau}\right)^{2-D}\\
&\quad \prod_{i=1}^{\operatorname{len}\alpha_L}\frac{e^{-2\pi i(1/2-\alpha_{Li})(1/2+\beta_{Li})}\vartheta\left[\begin{smallmatrix}\alpha_{Li}\\ \beta_{Li}\end{smallmatrix}\right](0\,|\,\tau)}{\eta(\tau)}\prod_{i=1}^{\operatorname{len}\alpha_R}\frac{e^{2\pi i(1/2-\alpha_{Ri})(1/2+\beta_{Ri})}\overline{\vartheta}\left[\begin{smallmatrix}\alpha_{Ri}\\ \beta_{Ri}\end{smallmatrix}\right](0\,|\,\tau)}{\overline{\eta}(\tau)}
\end{aligned} \tag{II.6}$$

The bosonic correlation function, $G_B(\nu)$, is defined via

$$\langle X^\mu(\nu_1,\overline{\nu}_1)\,X^\nu(\nu_2,\overline{\nu}_2)\rangle_\tau = \eta^{\mu\nu}G_B(\nu=\nu_1-\nu_2) \tag{II.7}$$

It can be expressed in terms of theta functions,

$$G_B(\nu) = -\frac{1}{\pi}\ln\left|2\pi e^{-\pi(\operatorname{Im}\nu)^2/\operatorname{Im}\tau}\frac{\vartheta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](\nu\,|\,\tau)}{\vartheta'\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](0\,|\,\tau)}\right| \tag{II.8}$$

A dotted variable, for our purposes, will always be taken to signify differentiation with respect to $\overline{\nu}$,

$$\dot{X} = \partial_{\overline{\nu}}X. \tag{II.9}$$

In a slight abuse of notation, we write the correlation function for right-movers as $\dot{G}_B(\overline{\nu})$ although in fact it is equal to $\overline{\dot{G}_B(\nu)}$.

We thus have

$$\dot{G}_B(\overline{\nu}) = \frac{i\operatorname{Im}\nu}{\operatorname{Im}\tau} - \frac{1}{2\pi}\frac{\vartheta'\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](\overline{\nu}\,|\,-\overline{\tau})}{\vartheta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](\overline{\nu}\,|\,-\overline{\tau})} \tag{II.10}$$

The fermionic particle correlation function $G_F[{\alpha \atop \beta}](\nu)$ and anti-particle correlation function $\hat{G}_F[{\alpha \atop \beta}](\nu)$ are defined as follows (excluding the case $\alpha = \beta = 0$):

$$\begin{aligned}
\left\langle \Psi^{i\dagger}(\nu_1)\, \Psi^j(\nu_2) \right\rangle^{\alpha}_{\beta;\tau} &= \delta^{ij} G_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu = \nu_1 - \nu_2) \\
\left\langle \Psi^{i}(\nu_1)\, \Psi^{j\dagger}(\nu_2) \right\rangle^{\alpha}_{\beta;\tau} &= \delta^{ij} \hat{G}_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu = \nu_1 - \nu_2)
\end{aligned} \tag{II.11}$$

where here the expectation value is understood to exclude a factor of the partition function.

These correlation functions can also be expressed in terms of theta functions,

$$\begin{aligned}
G_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu) &= \frac{\vartheta\left[{\alpha \atop \beta}\right](\nu\,|\,\tau)\vartheta'\left[{0 \atop 0}\right](0\,|\,\tau)}{2\pi\vartheta\left[{0 \atop 0}\right](\nu\,|\,\tau)\vartheta\left[{\alpha \atop \beta}\right](0\,|\,\tau)} \\
\hat{G}_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu) &= G_F\begin{bmatrix}1-\alpha\\ 1-\beta\end{bmatrix}(\nu) = -G_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-\nu)
\end{aligned} \tag{II.12}$$

where the last equality derives from a theta function identity.

As $\nu \to 0$ the various Green's functions have the following behavior:

$$\begin{aligned}
\exp\left(G_B(\nu)\right) &\to |\nu|^{-1/\pi} \times \text{constant} \\
\dot{G}_B(\overline{\nu}) &\to -\frac{1}{2\pi\overline{\nu}} \\
G_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu) &\to \frac{1}{2\pi\nu}
\end{aligned} \tag{II.13}$$

The vertex operator for emission of a gauge boson, in the $\mathcal{F}_1$ picture for the right-movers, is

$$V^a(\varepsilon, k; \nu, \overline{\nu}) = \; -2g\sqrt{\lambda}T^a{}_i{}^j : \Psi^{i\dagger}(\nu)\Psi_j(\nu)\, \varepsilon \cdot \left(\partial_{\overline{\nu}}\overline{X}(\overline{\nu}) + i\sqrt{\lambda}\overline{\Psi}(\overline{\nu})\, k \cdot \overline{\Psi}(\overline{\nu})\right) e^{i\sqrt{\lambda}k\cdot\left(X(\nu)+\overline{X}(\overline{\nu})\right)} : \tag{II.14}$$

or, using Grassman variables to put it into an exponential form,

$$\begin{aligned}
V^a(\varepsilon, k; \nu, \overline{\nu}) = \; & 2g\sqrt{\lambda}T^a{}_i{}^j : \int d\theta_1\, d\theta_2\, d\theta_3\, d\theta_4 \\
& \exp\Big(i\sqrt{\lambda}k \cdot X(\nu,\overline{\nu}) + \theta_1 \Psi^{i\dagger}(\nu) + \theta_2 \Psi_j(\nu) \\
& \quad + \theta_3\theta_4 \varepsilon \cdot \dot{\overline{X}}(\overline{\nu}) + i\sqrt{\lambda}\theta_3 k \cdot \overline{\Psi}(\overline{\nu}) + \theta_4 \varepsilon \cdot \overline{\Psi}(\overline{\nu})\Big) :
\end{aligned} \tag{II.15}$$

The N-point amplitude is then given by

$$\begin{aligned}
\mathcal{A}_N = & \frac{1}{2(16\pi^2)}\frac{1}{\lambda^2}\int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\,(\mathrm{Im}\,\tau) \int d^2\nu_1 \cdots \int d^2\nu_{N-1} \\
& \mathcal{Z}_B(\tau) \sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}} \mathcal{Z}_F\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau)\; \langle V^{a_1}(\nu_1)\cdots V^{a_N}(\nu_N)\rangle^{\vec{\alpha}}_{\vec{\beta};\tau}
\end{aligned} \tag{II.16}$$

Evaluating the correlation functions gives (in Minkowski space)

$$
\begin{aligned}
&\frac{1}{2(16\pi^2)}\lambda^{N/2-2}(2g)^N\, T^{a_1}{}_{m_1}{}^{n_1}\cdots T^{a_N}{}_{m_N}{}^{n_N} \\
&\int \frac{d^2\tau}{(\operatorname{Im}\tau)^2}\int\left(\prod_{i=1}^{N} d\theta_{i1}\, d\theta_{i2}\, d\theta_{i3}\, d\theta_{i4}\right)\operatorname{Im}\tau\int\left(\prod_{i=1}^{N-1} d^2\nu_i\right)\sum_{\vec\alpha,\vec\beta} C^{\vec\alpha}_{\vec\beta} Z^{\vec\alpha}_{\vec\beta}(\tau) \\
&\prod_{i<j}\exp\Bigg[\lambda k_i\cdot k_j\, G_B(\nu_i-\nu_j) \\
&\qquad -\theta_{i1}\theta_{j2}\delta^{m_i}{}_{n_j}\, G_F\begin{bmatrix}\alpha_{m_i}\\ \beta_{n_j}\end{bmatrix}(\nu_i-\nu_j)-\theta_{i2}\theta_{j1}\delta^{m_j}{}_{n_i}\, \hat G_F\begin{bmatrix}\alpha_{m_j}\\ \beta_{n_i}\end{bmatrix}(\nu_i-\nu_j) \\
&\qquad -\theta_{i3}\theta_{j3}\lambda k_i\cdot k_j\, G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline\nu_i-\overline\nu_j) \\
&\qquad +i\sqrt{\lambda}(\theta_{i3}\theta_{j4}k_i\cdot\varepsilon_j+\theta_{i4}\theta_{j3}k_j\cdot\varepsilon_i)\, G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline\nu_i-\overline\nu_j) \\
&\qquad -i\sqrt{\lambda}(\theta_{i3}\theta_{i4}k_j\cdot\varepsilon_i-\theta_{j3}\theta_{j4}k_i\cdot\varepsilon_j)\,\dot G_B(\overline\nu_i-\overline\nu_j) \\
&\qquad +\theta_{i4}\theta_{j4}\varepsilon_i\cdot\varepsilon_j\, G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline\nu_i-\overline\nu_j) \\
&\qquad +\theta_{i3}\theta_{i4}\theta_{j3}\theta_{j4}\varepsilon_i\cdot\varepsilon_j\,\ddot G_B(\overline\nu_i-\overline\nu_j)\Bigg]
\end{aligned}
\tag{II.17}
$$

This formula is valid in all sectors except Ramond–Ramond, where the fermionic zero mode demands special treatment. However, that sector enters only into parity-violating amplitudes, and so is not relevant to any of the calculations in this paper. The normalization of the amplitude has been calculated by Polchinski [15] and Sakai and Tanii [16].

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Figures

Fig. 1. (a) An 'internal-loop' contribution to $\mathcal{A}_4$. (b) An 'external-loop' contribution to $\mathcal{A}_4$. The internal lines merely represent pinches of Koba-Nielsen variables, and not necessarily propagators.

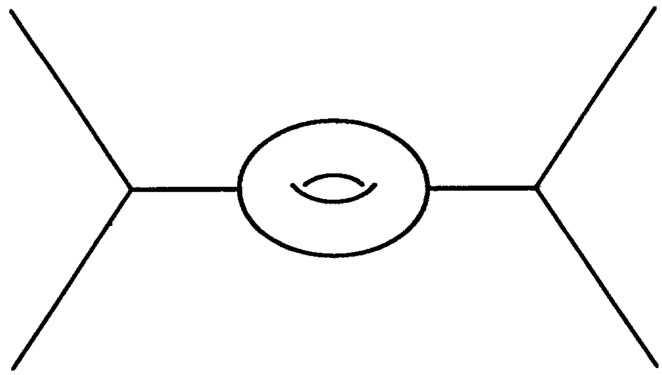
Fig. 2. (a) The three-point vertex generated by $\lambda\phi^3/3!$. (b) The three-point vertex generated by $\alpha\lambda\phi^2(-\partial^2 - m^2)\phi$ has value $-2i\alpha\lambda(k_1^2 + k_2^2 + k_3^2 - 3m^2)$.

Fig. 3. The two-point diagrams generated by the action (6.1).

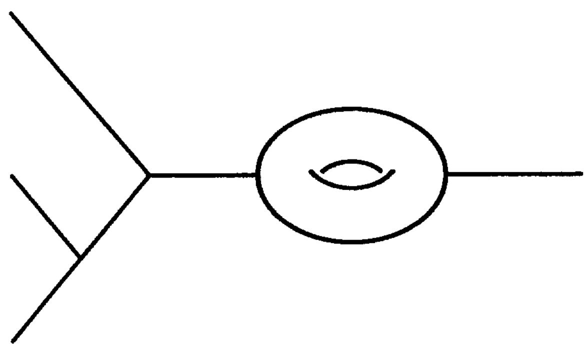
Fig. 4. A diagram containing an on-shell 'momentum/pole' 0/0 ambiguity analogous to the one found in the Polyakov amplitude.

Fig. 5. Three-point diagrams containing factors proportional to $k_1^2 - m^2$.

Fig. 6. The correct on-shell interpretation of the diagram of Fig. 4.

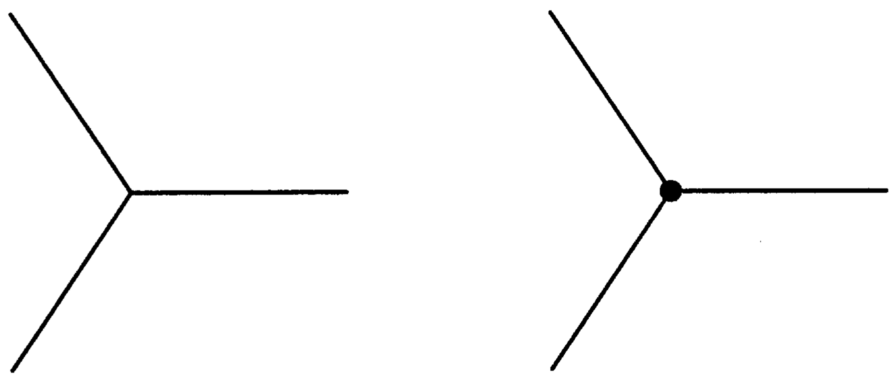


(a)



(b)

Fig. 1



(a) (b)

Fig. 2

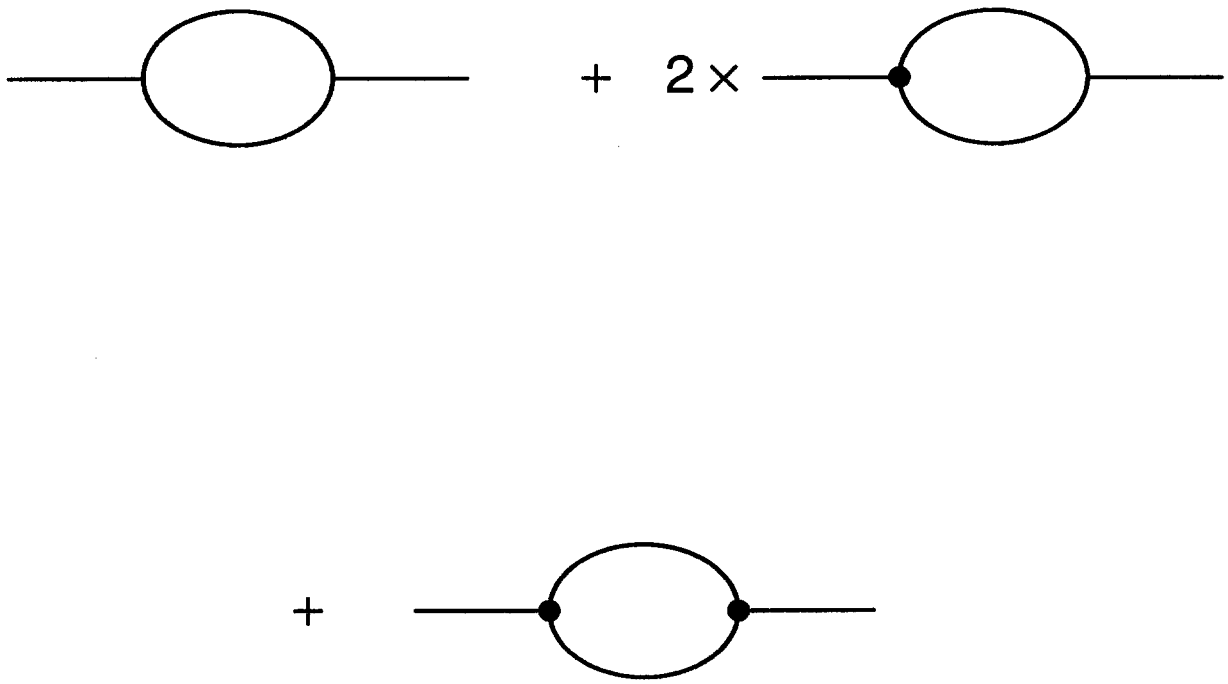


Fig. 3

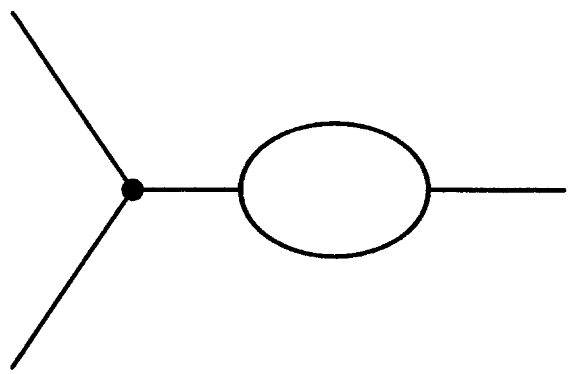


Fig. 4

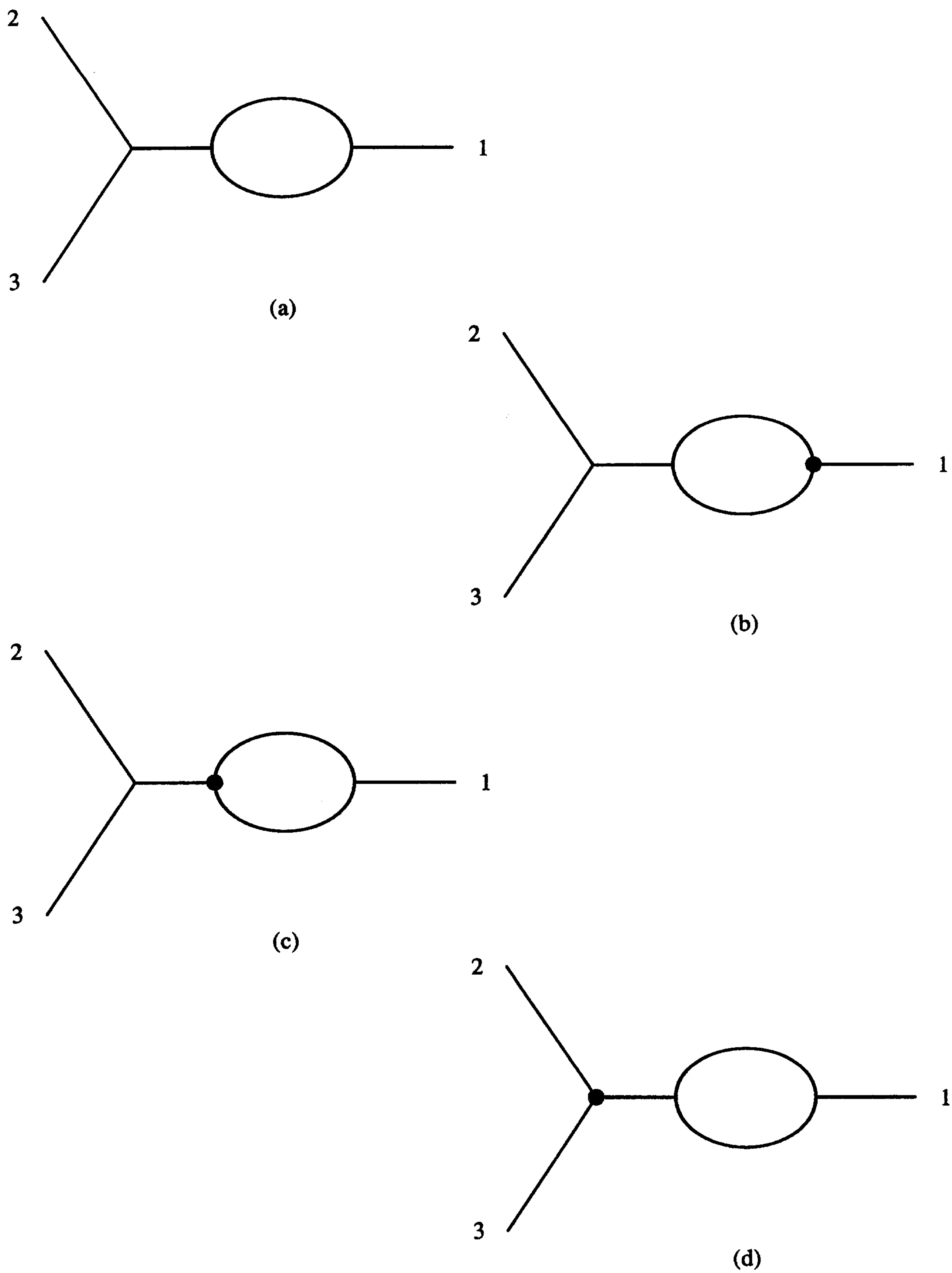


Fig. 5

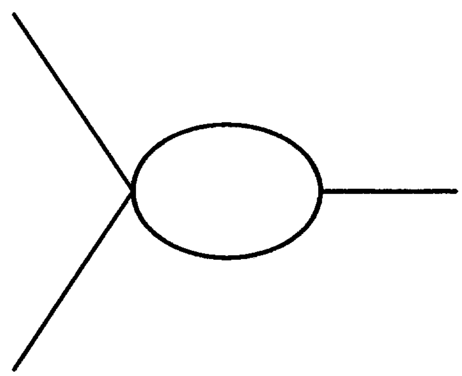


Fig. 6