Fermi National Accelerator Laboratory

FERMILAB–PUB–88/113–T
November, 1988

How to compute scattering amplitudes in hot gauge theories

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Abstract

I outline a method for the consistent perturbative calculation of scattering amplitudes in hot gauge theories. As the first step in this program, I determine the spectral representations for the renormalized gauge field and fermion propagators in the limit of high temperature. The interaction of an abelian gauge field with an external current illustrates how the collective longitudinal mode enters, and where infrared divergences arise. Massless fermions have a collective mode, for which the usual relation between chirality and helicity is flipped. The collective mode persists for light, massive fermions.

Based on talks given at "Quark Matter '88", Lenox, Mass., Sept., 1988, and the "Workshop on Thermal Field Theories", Case Western Reserve Univ., Oct., 1988.

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

I. Motivation

Consider "hot" gauge theories: for QCD, at a temperature T well into the chirally symmetric phase; for QED, at a temperature $T \gg m$, where m is the mass of the matter field. The latter is in contrast to "cold" QED, when $T \ll m$.

It is well known that at finite temperature, the gauge fields, like everything else, act as if they had a "mass" $m_g \sim gT$ (g is the coupling constant). It is direct to compute these masses, and as expected m_g^2 is positive and independent of gauge.

Now go one step beyond. At $T \neq 0$, sooner or later any state will scatter off of particles in the thermal distribution, and so acquire a non-zero imaginary part to its self-energy. The diagram for a gluon decaying into two virtual gluons is shown in fig. (1), where the dotted line represents the sum over intermediate states, *etc.*. To try and incorporate the gluon "mass", use the corresponding mass shell for the external legs, such as $p_0 = -im_g$, $\bar{p} = 0$. This is why the external legs in fig. (1) have a dot on them: they are meant to represent renormalized propagators, which include some effects of $\sim gT$.

Even for a gauge-variant field, there is physical information contained in its propagator: namely, the position of its pole(s), and the residues about them. A non-trivial example is provided by the W and Z bosons in the weak interactions.

Computing fig. (1), the imaginary part found is $Im\,\Pi \sim g(gT)^2$, with a gauge-dependent coefficient. In contradiction to general expectation, it appears as if the plasmon's pole is gauge-dependent.

This has lead several people to argue that one needs to define the plasmon by some more complicated, but gauge invariant, manner. While this may be useful in itself, I suggest that it doesn't matter *what* is used to compute the plasmon's pole, but *how* it is computed.

I assert that for scattering amplitudes involving soft quanta in hot gauge theories, there is a breakdown between powers of g, and the order of the loop expansion. To see this, consider the two-loop contribution to $Im\,\Pi$ in fig. (2). If the internal momenta flowing through the leg with the self-energy correction is K, then the ratio of fig. (2) to (1) is

$$\frac{\Pi(K)}{K^2} \sim \frac{m_g^2 + c_1 g^2 T|K| + \dots}{K^2} \underset{K\sim m_g}{\sim} 1\,. \tag{1.1}$$

I have expanded the self-energy about zero momentum; neglecting indices and the like, I take the leading order term to be given by the gluon "mass". The virtual momenta in these diagrams are $K \sim m_g$, so one concludes that eq. (1.1) is of order one, and the two loop correction in fig. (2) is of exactly the same magnitude as fig. (1), $\sim g(gT)^2$.

To solve this problem, I insist that one must first sum up *all* effects of $\sim gT$, to determine renormalized propagators and vertices that are exact to this order. Then these renormalized objects are sewn together to compute a damping rate. Thus instead of fig. (1), one should compute fig. (3): the important difference between the two is that in fig. (3), both the external *and* the internal legs are renormalized.

To understand why the resummation of terms $\sim gT$ works, let me assume that the only such effects are represented by the gluon mass. I estimate a correction to fig. (3), like that of fig. (2), except that inside the loop of fig. (2), everything gets renormalized. Then the ratio of this renormalized fig. (2), to fig. (3), is

$$\frac{\Pi - m_g^2}{K^2 + m_g^2} \sim \frac{c_1 g^2 T|K| + \dots}{K^2 + m_g^2} \underset{K\sim m_g}{\sim} g \; . \qquad (1.2)$$

Because the one-loop mass term has already been included in the propagator, in the renormalized form of fig. (2), it is necessary to use $\Pi - m_g^2$ instead of just Π. With $K \sim m_g$, the renormalized fig. (2) is g times smaller than fig. (3).

For a scalar theory with only quartic self-interaction $\sim g^2$, the scalar mass $m_s \sim gT$ is the only term of $\sim gT$: there are no momentum dependent terms in the self-energy $\sim gT$, and no vertex renormalizations to this order. Thus the resummation of perturbation theory is trivial — all that is needed is to change the bare propagator, $1/K^2$, to $1/(K^2 + m_s^2)$.

This is not true for fermion or gauge fields. The self-energy of either includes terms $\sim gT$ which are involved functions of the energy and three-momentum. Also, the Ward identities instruct us that if the self-energies are renormalized, then at least some vertices will be as well: this is why the vertex in fig. (3) is dotted.

The self-energies $\sim gT$ for fermions and gauge fields were first computed, in momentum space, by Klimov and Weldon [2]. To compute efficiently with these propagators, I generalize the methods of ref. [3], and compute here the complete spectral representation for these renormalized propagators: gauge fields are the subject of sec.

II, fermions of sec. III.

Determining these renormalized propagators is actually easy. The precise definition of terms $\sim gT$ is given in ref. [1]: loosely speaking, they are things which are powers of gT, with no g's left over. The interesting thing is that not only are they given merely by diagrams at one-loop order, but by a small subset of the complete one-loop diagram, when the virtual momenta within the diagram is hard, $\sim gT$. This vastly simplifies the determination of terms $\sim gT$. What is difficult is to go beyond leading order, to determine something like a damping rate, which is g times a power of gT. For such quantities, soft momenta $\sim gT$ inevitably contribute; when this happens, it is imperative to use all of the machinery of renormalized propagators and vertices that are exact to $\sim gT$.

Without bothering with the technical details, it is apparent why using renormalized quantities, as in fig. (3), makes such a difference. Consider the decay of a transverse gluon into a virtual transverse gluon plus an unphysical longitudinal mode. In fig. (1), since the propagators inside the loop aren't renormalized, this certainly contributes, as a massive quanta can decay into two massless ones. But fig. (3) represents the decay of a massive mode into a massive plus massless mode, which is not allowed kinematically. Thus the gauge dependence of fig. (1) and fig. (3) will be very different.

This kind of argument is naive. Consider the decay of a transverse gluon into two virtual transverse gluons: since everything is massive, one expects fig. (3), and so $Im\,\Pi$, to vanish to $\sim g(gT)^2$, and only non-zero at $\sim g^2(gT)^2$. This is what I first thought [3], but it's wrong. In summing all terms $\sim gT$ for fermion or gauge fields, not only are there the (massive) quasi-particle excitations, which lie above the light cone, but as well the contribution of a cut, below the light cone. This cut is due to Landau damping by pairs of particles with hard momenta, $\sim T$. As these cuts lie below the light cone, they dominate the imaginary parts of amplitudes on mass shell. My argument in ref. [3] — that massive states don't decay until two-loop order — is correct, but by resumming terms $\sim gT$, one sees effects at leading order, as in fig. (3), that wouldn't appear with bare propagators until at least two-loop order.

Further complications arise for gauge theories, especially in QCD. At zero temperature, the gauge field contains two physical, transverse degrees of freedom, and two unphysical longitudinal modes. At $T \neq 0$, the transverse modes are still physical, but one of the longitudinal modes becomes a physical, collective mode; the other longitudinal mode remains unphysical. In Coulomb gauge it is easy to pick out the physical longitudinal mode, but in covariant gauge, the physical and unphysical longitudinal modes are all mixed up. Further, what cancels the contributions of the ghost loop in covariant gauge QCD involves both the unphysical longitudinal mode and vertex renormalization; again, Coulomb gauge is much easier, as then the ghosts don't contribute to discontinuities.

Nevertheless, I expect that in the end that the general arguments prevail, and that the damping rates found will be gauge invariant and positive, indicating stability of the perturbative vacuum. I have completed the computation of the decay of a massive fermion [1]. This example is elementary, as corrections to the fermion propagator, and its vertex, can be neglected. At zero momentum, I find that the self-energy of the fermion is $Im\,\Sigma_F \sim +g(gT)$. Non-zero momentum is surprising: $Im\,\Sigma_F \sim +log(1/g)g(gT)$; both are independent of gauge. The $log(1/g)$ at non-zero momentum is special to gauge theories: contrary to the estimate of eq. (1.2) above, which indicates that corrections are down by at least one power of g, I can only reliably compute the coefficient of the $\sim log(1/g)$.

I believe this example is generic. The great difficulty in computing for light fields at soft momenta is the necessity of including vertex renormalization. I have computed the vertex renormalization for QED, which I used to determine the damping rate of a massless fermion [4].

After completing ref. [1], Gordon Baym and Chris Pethick informed me that they and H. Monien have also investigated a resummation of terms $\sim gT$ [5]. Although they did not resum all terms of $\sim gT$, they did include the most important — the Landing damping mentioned above. They find that 1/viscosity is $\sim log(1/g)g^4$. This $\sim log(1/g)$ has a different origin from that in the self-energy: in the viscosity it comes from $log(T/(gT))$, while in the self-energy, it arises from $log((gT)/(g^2T))$.

II. Gauge fields

In this section I determine the the gauge propagator to $\sim gT$. My results for this, and for massless fermions in the next section, build upon those of Klimov and Weldon [2]. My principal contribution lies in the computation of the spectral representation for these propagators. This requires not only the positions of the poles in the propagator [2], but the residues about these poles, as well. Further, in order to construct the complete renormalized propagator to $\sim gT$, the effects of cuts due to Landau damping must also be included. I then use the renormalized gauge propagator to compute the interaction with an external current. This standard example demonstrates some general features of the infrared limit of hot gauge theories.

Let the self-energy of the gauge field be $\Pi_{\mu\nu}$. At a temperature $T \neq 0$, the Ward identity, $P^\mu P^\nu \Pi_{\mu\nu} = 0$, implies that there are two independent components to $\Pi_{\mu\nu}$. I choose to define these as the transverse and longitudinal terms, Π_t and Π_ℓ, respectively:

$$\Pi_{00} = \Pi_\ell \; , \; \Pi_{0i} = -\frac{p_0 p^i}{p^2}\,\Pi_\ell \; , \; \Pi_{ij} = \left(\delta^{ij} - \hat{p}^i\hat{p}^j\right)\Pi_t + \hat{p}^i\hat{p}^j\,\frac{p_0^2}{p^2}\,\Pi_\ell \, . \tag{2.1}$$

I use upper case letters to denote four-vectors, with $P^\mu = (p_0, \bar{p})$, so $P^2 = p_0^2 + p^2$; $p \equiv |\bar{p}|$ and $\hat{p} = \bar{p}/p$. The transverse and longitudinal propagators are defined as

$$\Delta_t = \frac{1}{p_0^2 + p^2 - \Pi_t} \; , \; \Delta_\ell = \frac{1}{p^2 - \Pi_\ell} \, . \tag{2.2}$$

In the limit of high temperature, the leading terms in the propagator are those $\sim (gT)^2$, with the neglect of those $\sim g(gT)$, *etc.* [1]. To illustrate the results for Π to $\sim gT$, I consider the contribution of N_f flavors of massless fermions in QCD:

$$\Pi_t^f = -2g^2\, N_f \left(tr_{(-)}\frac{1}{K^2} + tr_{(-)}\frac{k^2 - (\bar{k}\cdot\hat{p})^2 + P\cdot K}{K^2(P-K)^2} \right) \; ,$$

$$\Pi_s^f = -2g^2 N_f \left(tr_{(-)}\frac{1}{K^2} + tr_{(-)}\frac{-2k^2 - k_0 p_0 + \bar{k}\cdot\bar{p}}{K^2(P-K)^2} \right) \, . \tag{2.3}$$

The subscript on the trace indicates anti-periodic boundary conditions in time. The first term is:

$$tr_{(-)}\frac{1}{K^2} = -\frac{T^2}{24} \, . \tag{2.4}$$

To understand the other terms, consider

$$tr_{(-)}\frac{k^2}{K^2(P-K)^2} = \int \frac{d^3k}{(2\pi)^3}\frac{k^2}{2E_k 2E_{p-k}}$$

$$\left\{ (1-\tilde{n}(E_k)-\tilde{n}(E_{p-k}))\left(\frac{-1}{ip_0-E_k-E_{p-k}}+\frac{1}{ip_0+E_k+E_{p-k}}\right)\right.$$
$$\left. +(\tilde{n}(E_k)-\tilde{n}(E_{p-k}))\left(\frac{-1}{ip_0-E_k+E_{p-k}}+\frac{1}{ip_0+E_k-E_{p-k}}\right)\right\}, \qquad (2.5)$$

$\tilde{n}(E)=1/(exp(E/T)+1), E_k=k$ for massless fermions. The method of ref. [3] was used to do the k_0 integral.

Let $\omega = ip_0$ and p be $\sim gT$. If k is itself $\sim gT$, then all momenta in eq. (2.5) are of the same order, with the k integral $\sim \int d^3k\,/ip_0 \sim (gT)^2$; this only contributes $g^2(gT)^2$ to Π^f. To get a term in $\Pi^f \sim (gT)^2$, eq. (2.5) must be like eq. (2.4), $\sim T^2$. This only happens if in eq. (2.5) k is $\sim T$. In this case, I can approximate

$$E_{p-k}-E_k \sim -p\cos\theta\ ,\ \tilde{n}(E_k)-\tilde{n}(E_{p-k}) \sim -\frac{p\cos\theta}{T}\,\tilde{n}(E_k)(1-\tilde{n}(E_k))\ . \qquad (2.6)$$

Then it is easy to do eq. (2.5): in the first two terms, $ip_0 \pm (E_k+E_{p-k}) \approx \pm 2k$, which produces a constant; in the second two terms, $ip_0 \pm (E_k-E_{p-k}) \approx ip_0 - p\cos\theta$, with an easy angular integral. The final result is

$$eq.(2.5) \approx -\frac{T^2}{48}-\frac{T^2}{24}\left(1-\frac{ip_0}{2p}L(ip_0,p)\right)+\dots\ , \qquad (2.7)$$

$$L(ip_0,p)\equiv \log\left(\frac{ip_0+p}{ip_0-p}\right)\ . \qquad (2.8)$$

The other terms not represented in eq. (2.7) are $\sim pT$, etc., and so contribute $\sim g^2pT$ to Π^f, which for $p \sim gT$ is $\sim g(gT)^2$, and not $\sim (gT)^2$. The function $L(ip_0,p)$ in eq. (2.8) will appear often in the following.

For the other terms in eq. (2.3), that $\sim tr_{(-)}(\bar{k}\cdot\hat{p})^2/(K^2(P-K)^2)$ generates terms $\sim T^2$ as above. All of the other terms in eq. (2.3), however, can be ignored: e.g., $tr_{(-)}\bar{k}\cdot\hat{p}/(K^2(P-K)^2) \sim pT$, and so only contributes $\sim g(gT)^2$ to Π^f.

As claimed in sec. I, terms $\sim gT$ are due entirely to a small part of the one-loop graphs, when the virtual momenta $k \sim T$. This is especially easy to compute. For example, in the full one-loop result, the angular integrals are arduous to evaluate. In contrast, due to the simplifications of eq. (2.6), the angular integrals for the terms $\sim gT$ are trivial.

The full result is [2]:

$$\Pi_\ell = -3m_g^2 \left(1 - \frac{ip_0}{2p} L\,(ip_0, p) \right) \,,$$

$$\Pi_t = +\frac{3}{2}\, m_g^2\, \frac{p_0^2}{p^2} \left(1 - \left(1 + \frac{p^2}{p_0^2} \right) \frac{ip_0}{2p} L\,(ip_0, p) \right) \,. \tag{2.9}$$

I have dropped the superscript (f) for fermions, since the only change in adding the gluons is to alter the value of m_g^2:

$$m_g^2 = \frac{e^2 N_f}{9} T^2 \;(QED),\; m_g^2 = \frac{g^2}{9} \left(N + \frac{N_f}{2} \right) T^2 \;(SU(N)) \,, \tag{2.10}$$

where m_g is the "mass" for the gauge field.

In a general covariant gauge, the self-energies of eq. (2.9) are independent of the gauge-fixing parameter ξ. To show this, one starts with the form of the diagram in momentum space, as in eq. (2.3), before any sum over k_0 is performed. In covariant gauges, terms with double poles in $1/K^2$ appear; to evaluate these, it is best to use the identity

$$\frac{1}{(K^2)^2} = lim_{\,m_\xi \to 0} \left(\frac{1}{K^2} - \frac{1}{K^2 + m_\xi^2} \right) \tag{2.11}$$

In most terms, such as in the residue of the gauge field, its statistical distribution function, *etc.*, the dependence on m_ξ produces terms that are down by $\sim 1/k^2$, which is $\sim 1/T^2$ for hard $k \sim T$. The greatest contribution from gauge dependent terms comes from terms where m_ξ appears in an energy denominator, like the last two terms in eq. (2.5); then produces contributions $\sim 1/(ip_0 k)$, which is $\sim 1/(gT^2)$ for soft p_0 and hard k.

Using this kind of power counting, it is direct to show the ξ-independence of the gauge self-energies to $\sim gT$. What I do not have is a more direct way of showing this, using merely the fact that the gauge-dependent part of the propagator is $\sim \xi\, K^\mu K^\nu/(K^2)^2$. At present, I can only establish gauge invariance after I have contracted all indices and done the sum over k_0 — but not before.

It is noteworthy that the functional form of the self-energies is constant, with the same dependence on ip_0 and p. The temperature enters solely through the mass $m_g \sim gT$; it is only through m_g that the difference between virtual fermions and gluons is felt. This is presumably because for hard momenta $k \sim T$, the differences in spin, *etc.*, don't matter.

Beyond leading order, when terms $\sim g(gT)^2$ are included, I expect that neither of these properties is preserved: then the self-energies will depend on ξ, with a different functional dependence on p_0 and p from diagrams with virtual fermion loops versus those with virtual gluon loops.

The renormalized propagators to $\sim gT$ are formed by using eqs. (2.2) and (2.9). Computing with these renormalized propagators in the usual fashion, by performing the discrete sum over k_0, is surely intractable. I then generalize the method of ref. [3], by computing the fourier transform of the propagator in the Euclidean time τ:

$$\Delta(\tau,p) = T \sum_{\substack{n=-\infty,\\ p_0=2\pi n T}}^{+\infty} e^{-ip_0\tau}\Delta(p_0,p)$$

$$= \frac{1}{4\pi i}\oint_C dz\; e^{-iz\tau} cot\left(\frac{z}{2T}\right)\Delta(z,p) = -\frac{1}{2\pi}\oint_C dz\;\frac{e^{-iz\tau}}{e^{-iz/T}-1}\Delta(z,p)\,, \qquad (2.12)$$

C is a contour encircling the real axis counter-clockwise; I assume $\Delta(z)$ has no poles within C. The final integral in eq. (2.12) is convergent at infinity in the entire z plane for $0 \le \tau \le 1/T$. Thus C is deformed from around the real axis, into two U's around the imaginary axis, plus infinity.

This produces a spectral representation for Δ:

$$\Delta_{\ell,t}(\tau,p) = \int_0^\infty d\omega\; \rho_{\ell,t}(\omega,p)\left((1+n(\omega))\,e^{-\omega\tau} + n(\omega)e^{-\omega\tau}\right)\,, \qquad (2.13)$$

$n(\omega) = 1/(exp(\omega/T)-1)$. The spectral densities $\rho_{\ell,t}(\omega,p)$ equal

$$\rho_{\ell,t}(\omega,p) = \rho^{res}_{\ell,t}(\omega_{\ell,t},p)\,\delta\left(\omega-\omega_{\ell,t}(p)\right) + \rho^{disc}_{\ell,t}(\omega,p)\theta(p-\omega)\,, \qquad (2.14)$$

$\theta(x) = 0, 1$ for $x < 0, > 0$. $\omega_{\ell,t}(p)$ are the mass shells for the longitudinal and transverse modes, and always lie above the light cone, $\omega_{\ell,t}(p) > p$; $\rho^{res}_{\ell,t}$ is the residue of the pole. $\rho^{disc}_{\ell,t}$ represents the contribution of a cut below the light cone, from $\omega = p$ down to $\omega = 0$. This represents Landau damping by pairs of particles — either virtual fermions or virtual gluons — with hard momenta $k \sim T$.

Since the self-energies computed to $\sim gT$ are independent of gauge, the transverse part of the renormalized propagator satisfies properties expected for a physical field.

For instance, the transverse density is the probability for finding a physical state, and so is never negative, $\rho_t(\omega,p) \geq 0$. It is not apparent how the longitudinal density should behave. It turns out that to $\sim gT$, the longitudinal density is never positive, $\rho_\ell(\omega,p) \leq 0$. The example of interaction with an external source, discussed later, helps to explain this.

Further, following Källen and Lehman, the canonical commutation relations determine the commutator between A_i and $\dot{A}_j$; the spectral density that appears there is the same ρ_t as in the propagator, and so the transverse spectral density satisfies a sum rule,

$$\int_0^\infty d\omega \; (2\omega) \; \rho_t(\omega,p) = 1 \; , \tag{2.15}$$

for all p. This was checked numerically. The equal-time commutation relations between A_0 and $\dot{A}_0$ are gauge dependent, and thus it is not clear whether there is any such sum rule for the longitudinal density. Numerically, I could not find any moment of ρ_ℓ, in ω, whose integral was independent of p. I note that by analyticity in the complex z plane, it is always possible to write a dispersion relation relating the real and imaginary parts of anything, as in eq. (25) of ref. [3]; what makes eq. (2.15) special is that its form is independent of p.

In examining the detailed properties of the spectral densities, I start with the more familiar example of the transverse mode.

$\omega_t(p)$ is the solution of the transcendental equation

$$L(\omega_t,p) = \frac{4p^3}{3m_g^2\omega_t(\omega_t^2-p^2)}\left(-\omega_t^2+p^2+\frac{3}{2}\,m_g^2\,\frac{\omega_t^2}{p^2}\right) \tag{2.16}$$

Its limiting forms are

$$\omega_t(p) \underset{p\to 0}{\simeq} m_g + \frac{3}{5}\frac{p^2}{m_g} - \frac{9}{35}\frac{p^4}{m_g^3} + \dots \; , \tag{2.17}$$

$$\omega_t(p) \underset{p\to\infty}{\simeq} p + \frac{3}{4}\frac{m_g^2}{p} - \frac{9}{32}\frac{m_g^4}{p^3}\left(2\,log\left(\frac{8}{3}\frac{p^2}{m_g^2}\right) - 3\right) + \dots \; . \tag{2.18}$$

The residue at this pole is

$$\rho_t^{res}(\omega,p) = \frac{\omega(\omega^2-p^2)}{3m_g^2\omega^2-(\omega^2-p^2)^2} \; . \tag{2.19}$$

Its limiting forms are

$$\rho_t^{res}(\omega_t, p) \underset{p\to 0}{\sim} \frac{1}{2m_g} - \frac{2}{5}\frac{p^2}{m_g^3} + \frac{19}{35}\frac{p^4}{m_g^5} + \dots , \tag{2.20}$$

$$\rho_t^{res}(\omega_t, p) \underset{p\to\infty}{\sim} \frac{1}{2p} - \frac{3}{8}\frac{m_g^2}{p^3}\left(log\left(\frac{8}{3}\frac{p^2}{m_g^2}\right) - 2\right) + \dots . \tag{2.21}$$

Eqs. (2.17) - (2.21) show why I refer to m_g as a "mass", in quotes: both the mass shell, $\omega_t(p)$, and the residue, ρ_t, are more complicated than that for a relativistically invariant excitation.

The spectral discontinuity ρ_t^{disc} is found by using

$$L(\omega + i0^+, p) = i\pi + L(p, \omega) . \tag{2.22}$$

The result is

$$\rho_t^{disc}(\omega, p) = \frac{1}{\pi} Im\, \Delta_t$$

$$= \frac{3}{4}m_g^2 \frac{\omega(p^2-\omega^2)}{p^3} \Big/ \left\{ \left(p^2 - \omega^2 + \frac{3}{2}m_g^2\frac{\omega^2}{p^2}\left(1 + \frac{(p^2-\omega^2)}{2\omega p}L(p,\omega)\right)\right)^2 + \left(\frac{3\pi m_g^2\omega(p^2-\omega^2)}{4p^3}\right)^2 \right\} . \tag{2.23}$$

As $\omega \to p^-$,

$$\rho_t^{disc}(\omega, p) \underset{\omega\to p^-}{\sim} \frac{p^2-\omega^2}{3m_g^2p^2} , \tag{2.24}$$

while as $\omega \to 0$,

$$\rho_t^{disc}(\omega, p) \underset{\omega\to 0}{\sim} \frac{3}{4}\, m_g^2\, \omega p \Big/ \left(p^6 + \left(\frac{3\pi}{4}\, m_g^2 \omega\right)^2\right) . \tag{2.25}$$

For the longitudinal mode, the mass shell is the solution of

$$L(\omega_\ell, p) = \frac{2p}{3m_g^2\omega_\ell}\left(p^2 + 3m_g^2\right) ; \tag{2.26}$$

its limiting forms are

$$\omega_\ell(p) \underset{p\to 0}{\sim} m_g + \frac{3}{10}\frac{p^2}{m_g} - \frac{3}{280}\frac{p^4}{m_g^3} + \dots , \tag{2.27}$$

$$\omega_\ell(p) \underset{p\to\infty}{\sim} p\left(1 + 2x_\ell + \left(\frac{8}{3}\frac{p^2}{m_g^2} + 10\right)x_\ell^2 + \dots\right) , \tag{2.28}$$

where

$$x_\ell \equiv exp\left(-\frac{2}{3}\frac{p^2}{m_g^2}-2\right) . \qquad (2.29)$$

The residue is

$$\rho_\ell^{res}(\omega,p)=\left(-\frac{1}{p^2}\right)\frac{\omega(\omega^2-p^2)}{3m_g^2-\omega^2+p^2} , \qquad (2.30)$$

with limiting forms

$$\rho_\ell^{res}(\omega_\ell,p) \underset{p\to 0}{\sim} \left(-\frac{1}{p^2}\right)\left(\frac{m_g}{2}-\frac{3}{20}\frac{p^2}{m_g}+\frac{9}{560}\frac{p^4}{m_g^3}+\ldots\right) , \qquad (2.31)$$

$$\rho_\ell^{res}(\omega_\ell,p) \underset{p\to\infty}{\sim} \left(-\frac{1}{p^2}\right)\frac{4p}{3}\,x_\ell\left(1+\frac{8}{3}\left(\frac{p^2}{m_g^2}+3\right)x_\ell+\ldots\right) . \qquad (2.32)$$

Unlike the transverse field, the mass shell for the longitudinal mode is only similar to that for a field of mass $\sim m_g$ at small momentum, $p \sim m_g$, eq. (2.27). At large momentum, the mass shell of the longitudinal mode is exponentially close to the light cone, eq. (2.28). Even more striking is the behavior of the residue: ρ_t is of order one over all momenta, eqs. (2.20) and (2.21), up to kinematic factors. In contrast, ρ_ℓ is exponentially small at large momenta, eq. (2.32). Remember that large momenta means large compared with $\sim gT$: even for $p \sim T$, the residue for the longitudinal mode is $\sim exp(-\#/g^2)$! This is the signal for the longitudinal mode as a collective excitation.

As noted before, the longitudinal density is never positive; indeed, as seen from eqs. (2.30) – (2.35), there is always an overall factor of $-1/p^2$ in ρ_ℓ: about zero momentum, eq. (2.31), ρ_ℓ is only finite once this factor is extracted.

The longitudinal discontinuity is given by

$$\rho_\ell^{disc}(\omega,p)=\left(-\frac{1}{p^2}\right)\frac{3m_g^2\omega p}{2}\Bigg/\left\{\left(p^2+3m_g^2-\frac{3}{2}\,m_g^2\,\frac{\omega}{p}\,L(p,\omega)\right)^2+\left(\frac{3\pi m_g^2}{2}\,\frac{\omega}{p}\right)^2\right\} , \qquad (2.33)$$

where

$$\rho_\ell^{disc}(\omega,p) \underset{\omega\to p^-}{\sim} \left(-\frac{1}{p^2}\right)\frac{2}{3}\,\frac{p^2}{m_g^2\,log^2(2p/(p-\omega))} , \qquad (2.34)$$

$$\rho_\ell^{disc}(\omega,p) \underset{\omega\to 0}{\sim} \left(-\frac{1}{p^2}\right)\frac{3}{2}\,\frac{m_g^2\omega p}{(p^2+3m_g^2)^2} . \qquad (2.35)$$

While the self-energies to $\sim gT$ are independent of the choice of gauge, the propagator itself does change. In Coulomb gauges with gauge-fixing parameter ξ_c, the renormalized propagator is:

$$\Delta_{00} = \Delta_\ell + \xi_c \frac{p_0^2}{p^4}\ , \ \Delta_{0i} = \xi_c\ \frac{p^0 p^i}{p^4}\ , \ \Delta_{ij} = (\delta_{ij} - \hat{p}_i \hat{p}_j)\, \Delta_t + \xi_c\ \frac{\hat{p}^i \hat{p}^j}{p^2}\ . \tag{2.36}$$

In covariant gauges with gauge-fixing parameter ξ,

$$\Delta_{00} = \frac{p^4}{(P^2)^2}\ \Delta_\ell + \xi\ \frac{p_0^2}{(P^2)^2}\ , \ \Delta_{0i} = -\ \frac{p_0 p^i p^2}{(P^2)^2}\ \Delta_\ell + \xi\ \frac{p_0 p^i}{(P^2)^2}\ ,$$

$$\Delta_{ij} = \left(\delta^{ij} - \hat{p}^i \hat{p}^j\right) \Delta_t + \frac{p^i p^j p_0^2}{(P^2)^2}\ \Delta_\ell + \xi\ \frac{p^i p^j}{(P^2)^2}\ , \tag{2.37}$$

with Δ_t and Δ_ℓ as in eq. (2.2). The transverse propagator Δ_t is always the same, only appearing through the term $\sim \delta^{ij} - \hat{p}^i \hat{p}^j$. Also, terms involving the gauge fixing parameters, ξ or ξ_c, are not renormalized, in that they do not involve either Δ_t or Δ_ℓ.

What is surprising is how the physical longitudinal mode, Δ_ℓ, appears. At zero temperature, the propagator represents two physical, transverse modes, and two unphysical, longitudinal modes. At non-zero temperature, there are three physical modes: the two transverse modes, and one physical longitudinal mode, with the remaining longitudinal mode unphysical. But as eqs. (2.36) and (2.37) show, the way in which the physical longitudinal mode shows up is in covariant gauges is very different from Coulomb gauges. It is not apparent how Δ_ℓ contributes to physical quantities in the same fashion.

To understand this, consider the interaction with a conserved, external current J^μ, $P_\mu J^\mu = 0$, in the abelian theory. The J dependent part of the effective action is

$$S_{eff} = +\frac{1}{2}\, tr_{(+)}\ J^\mu(P) \Delta_{\mu\nu}(P) J^\nu(P)\ . \tag{2.38}$$

Using current conservation, this reduces to

$$S_{eff} = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \int_0^\infty d\omega\ \left(1 + 2n\,(\omega)\right) \left(\rho_t\,(\omega,p)\, \vec{J}_{tr}^{\,2} + \rho_\ell\,(\omega,p)\, J_0^2\right)\ ; \tag{2.39}$$

the transverse current satisfies $\vec{p} \cdot \vec{J}_{tr} = 0$.

This result is independent of gauge: both ρ_t and ρ_ℓ appear in the same manner. For this to hold, all that matters is that J^μ is a conserved current. This is why T-matrix elements, such as self-energies on mass shell, are gauge invariant.

To compute S_{eff} explicitly, I assume that about zero momentum,

$$\bar{J}_{tr}^2(\omega,p) \underset{p\to 0}{\sim} +\frac{\mathcal{J}_{tr}^2}{p^2}\ ,\ J_0^2(\omega,p) \underset{p\to 0}{\sim} -\mathcal{J}_0^2\ , \tag{2.40}$$

with $\mathcal{J}_{tr}^2$ and $\mathcal{J}_0^2$ positive constants. The behavior of $\bar{J}_{tr}$ is standard for an accelerated charge. As the charge density, J_0 behaves smoothly about zero momentum. With Euclidean conventions, J_0 is imaginary: by current conservation, $p_0 J^0 = -\bar{p}\cdot\bar{J}$, so J_0 is imaginary when p_0 is.

Then

$$S_{eff} = +\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\left(+\frac{1}{p^2}\right)\int_0^\infty d\omega\ \left(1+2n\left(\omega\right)\right)\left(\rho_t\left(\omega,p\right)\mathcal{J}_{tr}^2 + \left(-p^2\rho_\ell\left(\omega,p\right)\right)\mathcal{J}_0^2\right)\ . \tag{2.41}$$

The $-p^2$ above cancels the $-1/p^2$ in ρ_ℓ, so that in all the longitudinal density contributes with a positive weight to S_{eff}, in a manner no more infrared divergent than that for the transverse density. Thus ρ_t is positive, and ρ_ℓ negative, because ρ_t couples to the space-like part of the current, while ρ_ℓ couples to the time-like part.

I first evaluate S_{eff} using the tree level propagators; then I can ignore ρ_ℓ, and take

$$\rho_t(\omega,p) = \frac{1}{2p}\,\delta(\omega-p)\ . \tag{2.42}$$

With an infrared cutoff μ_{cutoff}, at zero temperature there is the usual infrared logarithm:

$$S_{eff} \sim +\frac{\mathcal{J}_{tr}^2}{8\pi^2}\,log\left(\frac{1}{\mu_{cutoff}}\right) + \ldots,\ T=0. \tag{2.43}$$

At $T\neq 0$, the Bose–Einstein distribution function enters, $1+2n(\omega)\approx 2T/\omega + 0(\omega)$, and the logarithm turns into a power:

$$S_{eff} \sim +\frac{\mathcal{J}_{tr}^2}{4\pi^2}\,\frac{T}{\mu_{cutoff}} + \ldots,\ T\neq 0. \tag{2.44}$$

By construction, $exp(-S_{eff})$ is the probability to emit zero photons in the presence of the source J^μ; the divergences in S_{eff} are typical of those found in scattering processes. Eq. (2.44) applies to cold QED, when the temperature is much less than the mass of any matter field.

In hot QED, or QCD, most contributions to S_{eff} are infrared finite. For instance, over momenta $p \sim m_g$, the tranverse pole contributes

$$\sim +\mathcal{J}_{tr}^2 \int d^3p \, \frac{1}{p^2} \, n\left(\omega_{tr}(p)\right) \rho_t^{res}\left(\omega_{tr}(p)\right)_{\widetilde{p \sim gT}} + \frac{1}{g} \, \mathcal{J}_{tr}^2 \, . \tag{2.45}$$

Similarly, the longitudinal pole contributes $\sim +gT^2\mathcal{J}_0^2$. The longitudinal discontinuity also produces a finite result, as ρ_ℓ^{disc} is finite about zero momentum, eq. (2.35).

The only infrared divergence arises from the transverse discontinuity. As $\omega \to 0, n(\omega) \approx T/\omega$, and $\int d\omega(\rho_t/\omega)$ enters in eq. (2.41). But as $p \to 0$,

$$\int_0^\infty \frac{d\omega}{\omega} \, \rho_t^{disc}(\omega, p) \, \widetilde{_{p \to 0}} \, \frac{1}{2p^2} \, . \tag{2.46}$$

To show this, only the form of ρ_t^{disc} as $\omega \to 0$ is needed, eq. (2.25). Because ρ_t^{disc} is so singular for $\omega \ll p$, the integral in eq. (2.46) is dominated by very small values of $\omega, \sim p^3/m_g^2$, which produces the $1/p^2$ on the right hand side.

Using eq. (2.46),

$$S_{eff} \sim +\frac{\mathcal{J}_{tr}^2}{4\pi^2} \, \frac{T}{\mu_{cutoff}} + \ldots \, , T \neq 0. \tag{2.47}$$

This applies to hot QED, when T is much larger than any mass. Why the result should be the same as for cold QED, eq. (2.44), escapes me.

This example exemplifies what happens in the calculation of damping rates [1]. The largest contribution is from the effects of the discontinuity in the transverse distribution, concentrated in the region about zero frequency, $\omega \sim 0$. The identity of eq. (2.46) is useful is picking out the dominant contribution to damping rates at non-zero momentum.

It is not difficult to understand why the transverse distribution is singular in the limit of small ω, and the longitudinal distribution not. Consider first the imaginary part of Π, for either Π_ℓ or Π_t. At $\omega = 0, Im\,\Pi$ must vanish, so $Im\,\Pi \sim \omega$. Since we are keeping terms $\sim (gT)^2$ in Π, and as it has dimensions of $(\text{mass})^2$, when $\omega \ll p$, $\Pi \sim m_g^2\omega/p$. Thus the difference between ρ_ℓ^{disc} and ρ_t^{disc} arises from the behavior of the real part of Π in the limit of $\omega \to 0$.

At $\omega = 0$, though, I can calculate $Re\,\Pi$ equally well in Euclidean space-time, without bothering with analytic continuation. In this case, the behavior of $Re\,\Pi$ in the static limit is familiar; for terms $\sim (gT)^2$, $Re\,\Pi_\ell \sim m_g^2$, while $Re\,\Pi_t$ vanishes. This difference arises because to $\sim gT$, static electric fields are screened, while static magnetic fields are not.

For $\omega \ll p$, then,

$$\rho_{\ell,t}^{disc} \sim Im\frac{1}{p^2 - \Pi_{\ell,t}} \sim \frac{Im\,\Pi_{\ell,t}}{(p^2 - Re\,\Pi_{\ell,t})^2 + (Im\,\Pi_{\ell,t})^2}\,. \tag{2.48}$$

Thus about zero frequency, $\omega \ll p$, ρ_t^{disc}, eq. (2.25), is so much more singular than ρ_ℓ^{disc}, eq. (2.35), because static magnetic fields are not screened: $Re\,\Pi_\ell \neq 0$, $Re\,\Pi_t \approx 0$.

It is worth considering corrections to these results. Although to $\sim gT$ $Re\,\Pi_t$ vanishes, there are terms in $Re\,\Pi_t$ at $\sim g^2T$: $Re\,\Pi_t \sim g^2Tp$ [6]. Because these terms vanish at zero momentum, static magnetic fields remain unscreened, but as $Re\,\Pi_t$ vanishes less slowly than p^2, these terms act to soften the behavior of ρ_t for $p \sim g^2T$.

This is crucial to understanding damping rates at non-zero momentum [1]. Computing to $\sim gT$, the effects of ρ_t for $\omega \ll p \ll m_g$ turn out to produce a logarithmic divergence. The cutoff for these terms only arises when terms $\sim g^2T$ in $Re\,\Pi_t$ are considered. Given the mild logarithmic divergence, any change in the behavior of $Re\,\Pi_t$, and so ρ_t, is sufficient to provide convergence at a scale $\sim g^2T$.

I emphasize that this cutoff comes naturally from considering the corrections to the leading $\sim gT$ approximation. It is *not* related to the appearance of "magnetic mass" in QCD. For instance, for QED with scalar matter fields, the analogous scale is $\sim e^2T$ — yet here a "magnetic mass" probably does not arise.

Indeed, in an unusual circumstance, I do not understand what cutoffs the infrared logarithm for QED with only fermion matter fields. In this case, $Re\,\Pi_t \sim p^2$ order by order in perturbation theory, so it must be higher order corrections to $Im\,\Pi_t$ that provide the cutoff. This scale is probably smaller than e^2T.

III. Fermions

In this section I compute the the renormalized propagator for both massless [2] and light, massive fermions.

Consider massless, isodoublet fermions coupled to an $SU(N)$ gauge field. In Feynman gauge, the self-energy of the fermion is

$$\Sigma_f = 2ig^2C_f\, tr_+\frac{\not K}{K^2(P-K)^2} = 2ig^2C_f\int\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k2E_{p-k}}$$

$$\Big\{(1-\bar n(E_k)+n(E_{p-k}))\left(\frac{-\not K}{ip_0-E_k-E_{p-k}}+\frac{\not K^*}{ip_0+E_k+E_{p-k}}\right)$$

$$(\bar n(E_k)+n(E_{p-k}))\left(\frac{-\not K}{ip_0-E_k+E_{p-k}}+\frac{\not K^*}{ip_0+E_k-E_{p-k}}\right)\Big\}\ ; \qquad (3.1)$$

$C_f=(N^2-1)/(2N), \not K = -iE_k\gamma_0+\bar k\cdot\bar\gamma$, $\not K^*=+iE_k\gamma_0+\bar k\cdot\bar\gamma$.

Unlike $\Pi_{\mu\nu}$, Σ has dimensions of mass, so it is not apparent how terms $\sim (gT)^2$ can appear. Again, though, let $k\sim T$ in eq. (3.1): the first two terms $\sim g^2\int d^3k/k^2\sim g(gT)$, and so can be ignored. Terms $\sim (gT)^2$ only arise from the second two terms, where the energy denominators $ip_0\pm(E_k-E_{p-k})\sim ip_0\pm p\cos\theta$, and the k integral is $\sim g^2\int d^3k/k\sim(gT)^2$. The renormalized propagator,

$$\Delta_f^{-1}(p_0,\bar p) = -i\not p - \Sigma_f = -(\gamma^0D_0+i\,\not\bar pD_s)\ . \qquad (3.2)$$

is

$$D_0=\omega-\frac{m_f^2}{2p}L(ip_0,p)\ ,\ D_s=p+\frac{m_f^2}{p}\left(1-\frac{ip_0}{2p}L(ip_0,p)\right)\ , \qquad (3.3)$$

with the $L(\omega,p)$ of eq. (2.8). m_f is the fermion "mass" induced by temperature:

$$m_f^2=\frac{e^2}{8}T^2\ (QED)\ ,\ m_f^2=\frac{g^2C_f}{8}T^2\ (SU(N))\ . \qquad (3.4)$$

Like the gauge field propagator, once the sum over virtual k_0 is performed, it is easy to pick out the terms that contribute $\sim gT$: they only arise from virtual momenta that are hard, $k\sim T$. By similar means, it can be shown that the the terms $\sim gT$ are independent of gauge.

I start by discussing the quasi-particle excitations of massless fermions. On mass shell, spinors satisfy

$$\left(\gamma^0 D_0 + i \not{\hat{p}} D_s\right) \psi = 0 \; . \tag{3.5}$$

Multiply this by $\gamma^5\gamma^0$, and define $\bar{\sigma} = -i\gamma^5\gamma^0\bar{\gamma}$. Then ψ obeys

$$\gamma^5 \psi = \left(\frac{D_s}{D_0}\right) \bar{\sigma} \cdot \hat{p} \, \psi \; . \tag{3.6}$$

From this relation, it is apparent that

$$\chi \equiv \frac{chirality(\psi)}{helicity(\psi)} = sign\left(\frac{D_s}{D_0}\right) \; . \tag{3.7}$$

At zero temperature, sign(D_s/D_0) =sign(ω): positive energy states have $\chi = +$, while negative energy states, $\chi = -$.

At non-zero temperature, there is not one but *two* poles in Δ_f, for a given $p \neq 0$ and positive ω. There is the usual pole,

$$D_0 = +D_s, \tag{3.8}$$

along which $\chi = +$, as at zero temperature. The second solution is given by

$$D_0 = -D_s, \tag{3.9}$$

with $\chi = -$. This second solution represents a collective excitation, where the usual relation between chirality and helicity is *flipped*.

This collective excitation with flipped helicity gives rise to some interesting phenomena. For example, the ratio of the decay rate for a charged pion into $e\nu$ versus $\mu\nu$ is $\sim m_e^2/m_\mu^2$ at zero temperature. Since the decay into electrons is suppressed by the standard relation between chirality and helicity, at sufficiently high temperatures decay into electrons can proceed via the χ_- mode, with the ratio of decay rates $\sim (eT)^2/m_\mu^2$.

Unfortunately, none of decays that open up due to the χ_- mode appear to be of dramatic significance for the quark-gluon plasma. Like the above example, chirality suppressed modes are typically of importance only for weak decays; for the strong interactions, such effects are rather more indirect. This does suggest, however, that the effects of the χ_- mode might be of importance for the early universe.

The fourier transform of the fermion propagator is found from

$$\Delta_f(\tau,\bar{p}) = \sum_{\substack{n=-\infty \\ p_0=(2n+1)\pi T}}^{+\infty} e^{-ip_0\tau}\Delta_f(p_0,\bar{p}) = \frac{1}{2\pi}\oint_C dz \, \frac{e^{-iz\tau}}{e^{-iz/T}+1}\Delta_f(z,\bar{p}) \,, \tag{3.10}$$

which is convergent at infinity in the complex z plane for $0 \le \tau \le 1/T$. Deforming the contour C as before, to wrap around infinity and the imaginary z axis, generates the spectral representation of Δ_f:

$$\Delta_f(\tau,\bar{p}) = \int_0^\infty d\omega \, \Big[\rho_0(\omega,p)\gamma^0\left((1-\tilde{n}(\omega))e^{-\omega\tau} + \tilde{n}(\omega)e^{\omega\tau}\right)$$

$$+\rho_s(\omega,p) i \, \hat{\not{p}}\left((1-\tilde{n}(\omega))e^{-\omega\tau} - \tilde{n}(\omega)e^{\omega\tau}\right)\Big] \,. \tag{3.11}$$

The difference in signs for the terms $\sim exp(\omega\tau)$ is because the term $\sim \gamma^0$ involves $\partial/\partial\tau \; exp(\mp\omega\tau) \sim \mp\omega \; exp(\mp\omega\tau)$.

As for the gauge field, since the renormalized fermion propagator to $\sim gT$ is independent of gauge, it exhibits some general properties. The density $\sim \gamma^0$ is never negative: $\rho_0(\omega,p) \ge 0$ for all $\omega \ge 0$ and p. Secondly, the equal time anti-commutation relations determine the anti-commutator between ψ and $\psi^\dagger$; thus ρ_0 satisfies a sum rule,

$$\int_0^\infty d\omega \; 2\rho_0(\omega,p) = 1 \,. \tag{3.12}$$

Using the propagator of eqs. (3.2) and (3.3), I checked numerically that this sum rule was satisfied. The equal time anti-commutation relations do not determine the anti-commutator between ψ and $\psi^\dagger\bar{\gamma}$; numerically, I did not find any sum rule, like that of eq. (3.12), which was satisfied by ρ_s.

There are three contributions to the spectral densities:

$$\rho_{0,s}(\omega,p) = \rho_{0,s}^{res}(\omega,p)\left(\delta(\omega-\omega_+(p)) + \delta(\omega-\omega_-(p))\right) + \theta(p-\omega)\rho_{0,s}^{disc}(\omega,p) \,. \tag{3.13}$$

$\omega_\pm(p)$ are the mass shells for the $\chi_\pm$ modes. $\rho^{res}(\omega_\pm,p)$ are the residues of Δ_f at these poles, which has the same functional form for each, eq. (3.18). As these modes are eigenstates of χ,

$$\rho_s^{res}(\omega_\pm,p) = \pm\rho_0^{res}(\omega_\pm,p) \,. \tag{3.14}$$

This discontinuity due to Landau damping does not contribute with a fixed Dirac structure: generally $\rho_0^{disc} \neq \pm\rho_s^{disc}$. This is because the discontinuity represents the effect of a pair of particles, a virtual fermion plus gauge field, and this pair need not be in an eigenstate of χ.

I begin with the poles of Δ_f. The mass shell for the χ_+ mode, $D_0 = +D_s$, is the solution of

$$L(\omega_+, p) = \frac{2p^2}{m_f^2(p-\omega_+)}\left(\omega_+ - p - \frac{m_f^2}{p}\right) ; \tag{3.15}$$

with limiting forms

$$\omega_+(p) \underset{p\to 0}{\simeq} m_f + \frac{p}{3} + \frac{1}{3}\frac{p^2}{m_f} + \dots , \tag{3.16}$$

$$\omega_+(p) \underset{p\to\infty}{\simeq} p + \frac{m_f^2}{p} - \frac{1}{2}\frac{m_f^4}{p^3}\, log\left(2\frac{p^2}{m_f^2}\right) + \dots . \tag{3.17}$$

The residue is:

$$\rho_0^{res}(\omega_\pm, p) = \frac{1}{4}\frac{\omega_\pm^2 - p^2}{m_f^2} . \tag{3.18}$$

For the χ_+ mode,

$$\rho_0^{res}(\omega_+, p) \underset{p\to 0}{\simeq} \frac{1}{4} + \frac{1}{6}\frac{p}{m_f} - \frac{1}{18}\frac{p^2}{m_f^2} + \dots , \tag{3.19}$$

$$\rho_0^{res}(\omega_+, p) \underset{p\to\infty}{\simeq} \frac{1}{2} - \frac{1}{4}\frac{m_f^2}{p^2}\left(log\left(\frac{2p^2}{m_f^2}\right) - 1\right) + \dots . \tag{3.20}$$

The mass shell for the χ_- mode, $D_0 = -D_s$, is the solution of

$$L(\omega_-, p) = \frac{2p^2}{m_f(\omega_- + p)}\left(\omega_- + p + \frac{m_f^2}{p}\right) , \tag{3.21}$$

where

$$\omega_-(p) \underset{p\to 0}{\simeq} m_f - \frac{p}{3} + \frac{1}{3}\frac{p^2}{m_f} + \dots , \tag{3.22}$$

$$\omega_-(p) \underset{p\to\infty}{\simeq} p\left(1 + 2x_- + 4x_-^2 + \dots\right) , \tag{3.23}$$

$$x_- = exp\left(-2\frac{p^2}{m_f^2} - 1\right) . \tag{3.24}$$

The residue for the χ_- mode has the limits,

$$\rho_0^{res}(\omega_-, p) \underset{p\to 0}{\simeq} \frac{1}{4} - \frac{1}{6}\frac{p}{m_f} - \frac{1}{18}\frac{p^2}{m_f^2} + \dots , , \tag{3.25}$$

$$\rho_0^{res}(\omega_-, p) \underset{p\to\infty}{\simeq} \frac{p^2}{m_f^2} x_- (1 + 3x_- + \ldots) . \tag{3.26}$$

The χ_+ mode is a renormalized version of the usual fermion. Finite temperature effects push the mass shell above the light cone, as if it had a mass $\sim gT$; its residue is of order one over all momenta, eqs. (3.19) and (3.20). In contrast, while the χ_- mode is a collective excitation, like the longitudinal mode of the gauge field: it acts like a massive field over $p \sim m_f$, but at large momenta, its residue is exponentially small, eq. (3.26).

One property of the χ_- mode is unique. $\omega_+(p)$ increases monotonically with p. $\omega_-(p)$ first decreases, has a minimum at

$$p \equiv p_{min} \approx .408 m_f , \tag{3.27}$$

and then increases. About p^{min},

$$\omega_-(p) \underset{p\sim p_{min}}{\simeq} .928 m_f + .966 \frac{(p - p^{min})^2}{m_f} + \ldots . \tag{3.28}$$

I do not attach much significance to this minimum in $\omega_-(p)$. Certainly it means that for a pair of fermions with zero net momentum, the smallest total energy is not $2m_f$, from fermions with $p = 0$, but $.928(2m_f)$, from fermions with $p = p^{min}$. Still, I do not believe the existence of p^{min} signals a transition to a new ground state, etc. [2]. After all, the minimum occurs at energies $\sim gT$, which is far below the temperature T. This is exactly opposite to what happens for rotons in liquid helium; there a minimum in the energy at $p \neq 0$ produces superfluidity, but only when the temperature is much *less* than the roton energy.

There is one limit in which the sum rule of eq. (3.12) is evident. At non-zero momentum, the sum rule receives contributions from the discontinuities due to Landau damping, but these vanish at $p = 0$. At zero momentum, from eqs. (3.19) and (3.25) each mode contributes 1/4 to ρ_0, to give 1/2 overall.

The discontinuities are given by

$$\rho_0^{disc}\gamma_0 + \rho_s^{disc} i\, \hat{\not p} = \frac{1}{\pi}\, Im \left\{ \frac{D_0\gamma^0 + D_s i\, \hat{\not p}}{-D_0^2 + D_s^2} \right\} , \tag{3.29}$$

where the imaginary part is evaluated for $\omega > 0$, using eq. (2.22). The general form of ρ_0^{disc} and ρ_s^{disc} are not very enlightening, so I give just their forms for $\omega \ll p \ll m_f$:

$$\rho_0^{disc} \underset{\omega \ll p \ll m_f}{\sim} \frac{2}{\pi^2+4}\,\frac{p}{m_f^2} + \dots\,, \tag{3.30}$$

$$\rho_s^{disc} \underset{\omega \ll p \ll m_f}{\sim} \frac{2(\pi^2-12)}{(\pi^2+4)^2}\,\frac{\omega}{m_f^2} + \dots\,, \tag{3.31}$$

while as $\omega \to p^- \ll m_f$,

$$\rho_0^{disc} \sim -\rho_s^{disc} \underset{\omega \to p^- \ll m_f}{\sim} \frac{1}{2m_f^2}\,\frac{p}{((L(p,\omega)-1)^2+\pi^2)}\,. \tag{3.32}$$

For $p \gg m_f$, ρ_0^{disc} and ρ_s^{disc} fall off as $\sim m_f^2/p^3$. For general ω, the discontinuity does not contribute with definite helicity: *e.g.*, ρ_s^{disc} vanishes as $\omega \to 0$, while ρ_0^{disc} does not. As $\omega \to p^-$, though, $\rho_0^{disc} \sim -\rho_s^{disc}$, so the cut contributes like an excitation with $\chi = -$.

The important property of ρ_0^{disc} and ρ_s^{disc} is that, like the longitudinal discontinuity of a gauge field, ρ_ℓ^{disc}, they are smoothly behaved about zero momentum, eqs. (3.30) and (3.31). This is the limit, $\omega \ll p \ll gT$, in which the transverse discontinuity of the gauge field is singular, eq. (3.22). That the fermion densities are smoothly behaved in this limit follows from

$$D_0 \underset{\omega \ll p \ll m_f}{\sim} \frac{i\pi m_f^2}{2p}\,,\; D_s \underset{\omega \ll p \ll m_f}{\sim} \frac{m_f^2}{p} + \frac{i\pi m_f^2\omega}{2p^2}\,, \tag{3.33}$$

and eq. (3.29).

The extension to light, massive fermions is immediate. Assume that the mass, as defined at zero temperture, is no greater than $\sim gT$. Then because terms in the self-energy only arise from virtual momenta that are hard, $k \sim T$, m can be ignored in the functions of eq. (3.3). At non-zero mass, there is also a term in the self-energy $\sim m$, but this is no larger than $\sim m\; g(gT)$, and so can be ignored. Thus in the renormalized propagator of eq. (3.2), m merely tags along:

$$\Delta_f^{-1} = -\left(\gamma^0 D_0 + i\,\not{p} D_s\right) + m\,. \tag{3.34}$$

As the mass m is increased to become $\sim T$, one enters a complicated crossover regime, with many terms of the same order. Of course the limit of a very heavy fermion, $m \gg T$, is simple, as then all self-energy corrections are $\sim g^2T/m$, and it suffices to use the bare propagator at leading order.

At non-zero mass, it remains true that that Δ_f exhibits two branches for $\omega > 0$. I refer to the one where $D_0 = +(D_s + m)$ as the χ_+^m mode, and $D_0 = -(D_s + m)$ as the χ_-^m mode. When $m \neq 0$, of course, states do not have definite chirality. Nevertheless, each $\chi_\pm^m$ made smoothly connects with $\chi_\pm$ as $m \to 0$, so the notation is convenient.

For simplicity, I concentrate on zero momentum. At $p = 0$,

$$\omega_\pm^m(0) = \frac{1}{2}\left(\sqrt{m^2 + 4m_f^2} \pm m\right) . \tag{3.35}$$

As m increases from $0 \to \infty$, $\omega_+^m(0)$ moves up, from m_f to m:

$$\omega_+^m(0) \underset{m \ll m_f}{\sim} m_f + \frac{m}{2} + \ldots \ , \ \omega_+^m(0) \underset{m \gg m_f}{\sim} m + \frac{m_f^2}{m} + \ldots \ , \tag{3.36}$$

while $\omega_-^m(0)$ moves down, from m_f to 0:

$$\omega_-^m(0) \underset{m \ll m_f}{\sim} m_f - \frac{m}{2} + \ldots \ , \ \omega_-^m(0) \underset{m \gg m_f}{\sim} \frac{m_f^2}{m} - \frac{m_f^4}{m^3} + \ldots . \tag{3.37}$$

The residues behave similarly: that for χ_+^m increases as m does:

$$\rho_0\left(\omega_+^m, 0\right) \underset{m \ll m_f}{\sim} \frac{1}{4} + \frac{1}{8}\frac{m}{m_f} + \ldots \ , \ \rho_0\left(\omega_+^m, 0\right) \underset{m \gg m_f}{\sim} \frac{1}{2} - \frac{1}{2}\frac{m_f^2}{m^2} + \ldots \ , \tag{3.38}$$

while that for χ_-^m decreases:

$$\rho_0\left(\omega_-^m, 0\right) \underset{m \ll m_f}{\sim} \frac{1}{4} - \frac{1}{8}\frac{m}{m_f} + \ldots \ , \ \rho_0\left(\omega_-^m, 0\right) \underset{m \gg m_f}{\sim} \frac{1}{2}\frac{m_f^2}{m^2} - \frac{3}{2}\frac{m_f^4}{m^4} + \ldots \tag{3.39}$$

As at zero mass, ρ_0 satisfies the sum rule of eq. (3.12).

These results can be used to determine when the collective mode, χ_-^m, is important. At first, one might hope for dramatic effects from χ_-^m at large m, since it becomes very light when m is heavy: $\omega_-^m(0) \sim m_f^2/m$ for $m \gg m_f$, eq. (3.37). But as the energy of χ_-^m becomes small, so does its residue; $\rho_0(\omega_-^m, 0)$ vanishes as $\sim m_f^2/m^2$ when $m \to \infty$, eq. (3.39). In particular, let $m \sim T$; the above doesn't get factors of one right, but should give the correct powers of g. With $m_f \sim gT$ and $m \sim T$, $\omega_-^m(0) \sim g^2T$ and $\rho_0(\omega_-^m, 0) \sim g^2$. Since the residue of χ_-^m is $\sim g^2$ by the time that $m \sim T$, its effects can be safely ignored.

This is of consequence for the quark–gluon plasma. The relevant regime of temperatures is $T \sim 100 - 200$ MeV, so at these temperatures, the up and down quarks are (essentially) massless $m_{u,d} \ll T$, while for the strange quark, $m_s \sim T$. Thus at small momentum, $p \sim gT$, the propagation of up and down quarks is strongly renormalized by the medium, and they develop a collective made with flipped chirality/helicity. For the strange quark, though, even at small momentum any renormalization effects are strictly $\sim g^2$, and its collective mode can be ignored. Of course I am assuming that I can use perturbation theory over such T, which is surely wrong; but perhaps these perturbative results are qualitatively correct.

Finally, I mention the results of a similar analysis for massless fermions at zero temperature, but non-zero chemical potential μ. Following the approach at $T \neq 0$, I compute only the leading terms in the high density limit, $\sim (g\mu)^2$, neglecting those $\sim g(g\mu)$, *etc.*. In the tree propagator, to go to $\mu \neq 0$, one merely replaces $\omega \rightarrow \omega + \mu$. It is not obvious, but in the high density limit this also holds for the renormalized propagator of eq. (1): everywhere ω becomes $\omega + \mu$. It is also necessary to redefine the fermion "mass"; for QCD, $m_f^2 = g^2 C_f \mu^2/(8\pi^2)$.

Thus one would naively think that the same phenomena present at $T \neq 0$, $\mu = 0$ would arise at $\mu \neq 0$, $T = 0$ (Klimov, ref. (2)). For instance, at zero momentum, there is not one pole at $\omega = -\mu$, but two, for $\omega = -\mu \pm m_f$; further, for $\omega > -\mu + p$, there are two branches, with one a collective mode like χ_-.

But at non-zero chemical potential, however, what is of interest is not the region about zero momentum, but that about the Fermi surface, when $p \sim \mu$. In this case, the collective mode is completely negligible, since its residue is $\sim exp(-\#/g^2)$; similarly, the renormalization of the χ_+ mode is uniformly small, $\sim g^2$. In particular, due to χ_+ states near the Fermi surface, there are still excitations with arbitrarily small energy — at least to leading order in $\sim g\mu$, there is no superconductivity.

Thus when computing at $\mu \neq 0$, $T = 0$, the bare fermion propagator can be used in computing scattering amplitudes to lowest order in g; likewise, the bare vertex with a gauge field is adequate. This is a simplification from $T \neq 0$ [1], where it is necessary to use renormalized quantities for the fermion propagator and its vertex. To leading order at $\mu \neq 0$, $T = 0$, the only effects $\sim g\mu$ reside exclusively in the gauge field: in its propagator, and for a non-abelian field, in its self-interactions.

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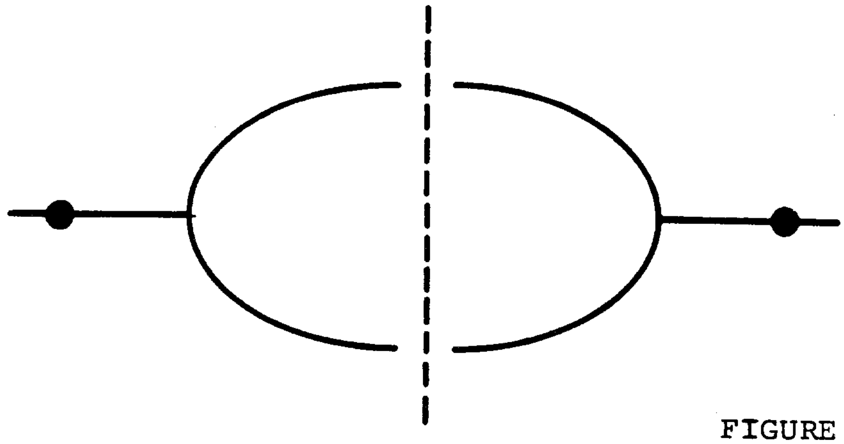

FIGURE 1

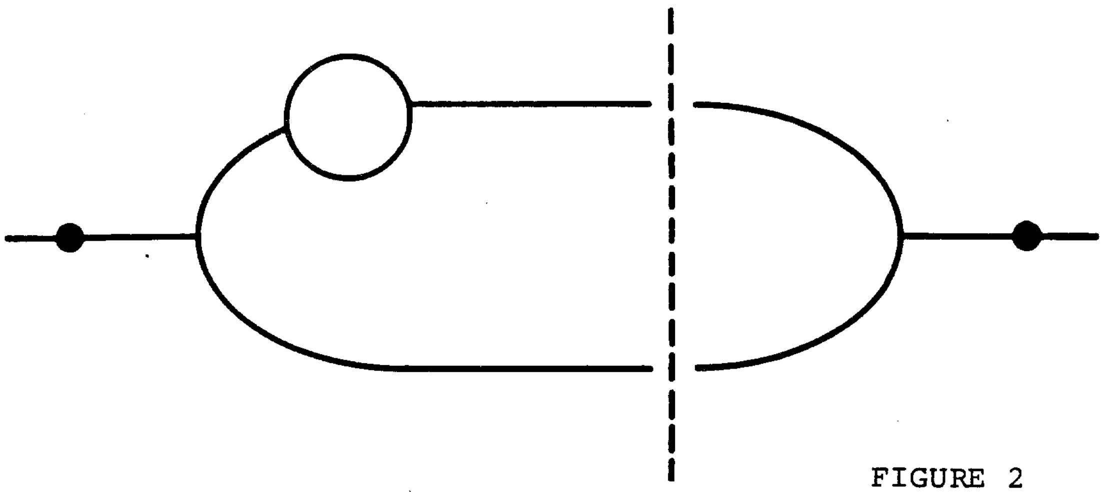

FIGURE 2

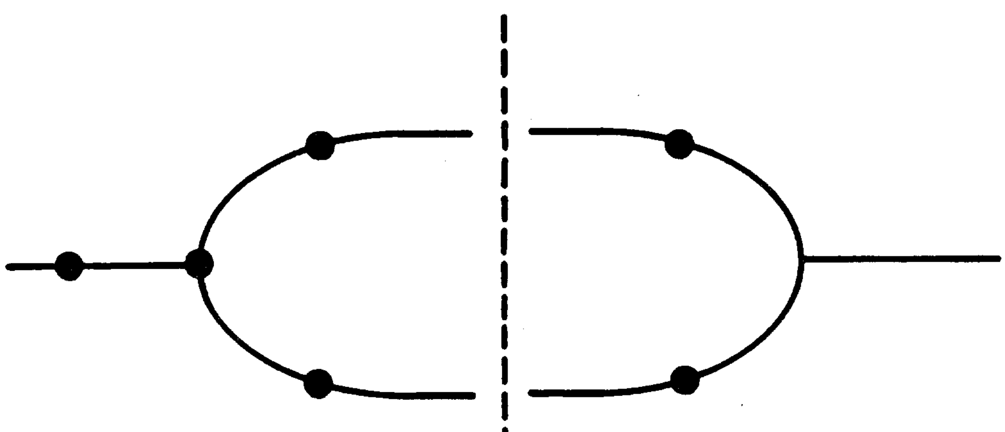

FIGURE 3