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ZERO MODES AND ANOMALIES IN SUPERCONDUCTING STRINGS

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Abstract

Superconductivity in cosmic strings occurs when electrically charged fermions are trapped as massless particles (Jackiw-Rossi zero modes) in the core of a string. Currents are generated when an electric field is applied along the string, or more realistically, when the string moves through a cosmic magnetic field. In realistic models [e.g., those inspired by grand unified theories], the fermion-vortex systems that arise can be quite complicated and the question of whether or not superconductivity occurs is very model dependent. For example, in certain models, mixing between right and left moving zero modes gives rise to an effective mass for the fermions on the string. The currents in this case, at least for reasonable values for cosmic magnetic fields, are uninterestingly small. In this paper, we present a simple method for determining the number of true zero modes in a special class of fermion-vortex systems. These results are then applied to a particular particle physics model based on the gauge group E_6. We also consider the possibility that $\sum_{left\ movers} q^2 \neq \sum_{right\ movers} q^2$ where q is the electromagnetic charge of a zero mode. In this situation, which occurs in 'frustrated' as well as global strings, there is a gauge anomaly (and therefore charge non-conservation) in the effective $(1+1)$-dimensional theory for the fermion-string system. In the presence of an electric field, the string acquires both charge and current. Charge non-conservation on the string is accounted for by an inflow of charge from the world outside the string. However, both charge and current can be screened, either by polarization of the vacuum or be the surrounding plasma.

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Introduction

Superconducting cosmic strings [1], if they exist, may have important consequences for the Universe. For example, they may be responsible for the formation of large scale structure [2] or the production of ultra-high energy cosmic rays [3]. The electric current in a superconducting cosmic string can be carried by either fermions (Jackiw-Rossi zero modes [4] trapped in the string) or by bosonic zero modes associated with a charged condensate localized in the core of the string [1]. In either case, the question of whether or not the string is superconducting is very model dependent. In this paper, we address this question for the fermionic superconducting cosmic string by studying a variety of fermion-vortex systems. Some of these results are applied to the cosmic strings possible in a specific grand unified theory (GUT) based on the gauge group E_6. Superconductivity with bosonic charge carriers has been studied in detail elsewhere [5] and will not be discussed in this work.

The simplest model of a fermionic superconducting cosmic string and the one proposed by Witten [1] is based on the gauge group $U(1)' \times U(1)$ where $U(1)$ is the unbroken gauge symmetry of electromagnetism and $U(1)'$ is the spontaneously broken gauge symmetry that gives rise to the string. If ϕ is the scalar field responsible for the spontaneous symmetry breaking of $U(1)'$ then a cosmic string is a non-trivial topological configuration of ϕ and of the $U(1)'$ gauge field A'_μ.

If fermions acquire a mass through Yukawa interactions with ϕ, then in the presence of a string, there are zero-energy solutions to the Dirac equation. These zero modes are localized in the core of the string and can be described by an effective Lagrangian for massless fermions in 1+1 dimensions. Fermions that also carry ordinary electric charge couple to an external electromagnetic field through the axial vector anomaly. In particular, the divergence of the $(1+1)$-dimensional current for a given species trapped on the string, $j^i \equiv (r,\ j)$ is given by [6]

$$\partial^i j_i = \frac{\lambda |n| q^2 E}{2\pi} . \tag{1.1}$$

[Here and throughout, roman letter indices will refer to the (t, z) $(1 + 1)$-dimensional coordinate system where the z-axis is along the direction of the string. Greek letter indices will refer to the full $(3 + 1)$-dimensional coordinate system. Our metric conventions are $(- +)$ for $1 + 1$ dimensions and $(- \ + \ + +)$ for $3 + 1$ dimensions.] In Eqn(1.1), q is the electric charge of the trapped fermions, E is the electric field along the direction of the string, and $\lambda = +1$ (-1) for left (right) movers. [We refer to zero modes that travel in the $-z$ $(+z)$ direction as left (right) movers.] $|n|$ is the number of zero mode solutions for the particular species considered and in the simplest models, n is the winding number of ϕ. We use high energy physics units where $\hbar = c = k_B = 1$. For the superconducting cosmic string, one requires that there be both right and left moving zero modes with

$$\sum_{right\ movers} q^2 = \sum_{left\ movers} q^2. \tag{1.2}$$

Eqn(1.2) guarantees that the total contribution to the anomaly, Eqn(1.1), vanishes so that charge on the string is conserved:

$$\partial^i J_i = 0 \tag{1.3}$$

where $J^i \equiv (\rho,\ J)$ is the total (1+1)-dimensional electromagnetic current on the string found by summing over all zero modes present on the string. However, a current can be generated on the string. For example, in a constant electric field,

$$\frac{dJ}{dt} = -\left(\sum_{right\ movers} |n|q^2 + \sum_{left\ movers} |n|q^2\right)\frac{E}{2\pi}\,. \tag{1.4}$$

Eqn(1.4) indicates that currents persist in the absence an electric fields, i.e., the string is superconducting.

The discussion above can be recast in terms of currents and anomalies in the parent (3+1)-dimensional theory. In all of the models that we will consider, the gauged currents in the (3+1)-dimensional theory are assumed to be anomaly free. This implies that $\partial^\mu J_\mu = 0$ where J_μ is the four dimensional electromagnetic current. This equation holds regardless of whether or not there is a string present. [As discussed for example, by Witten [1], the requirement that the (3+1)-dimensional theory be free of anomalies places constraints on the particle content of the theory.] Eqn(1.3) states that for a Nielsen-Olesen string, the (1+1)-dimensional electromagnetic current is also conserved. This, together with the fact that $\partial^\mu J_\mu = 0$ implies that there are no currents flowing on to the string.

Eqn(1.4) expresses the fact that there is an anomaly in an ungauged current:

$$\partial^i \tilde{J}_i = \left(\sum_{right\ movers} |n|q^2 + \sum_{left\ movers} |n|q^2\right)\frac{E}{2\pi} \tag{1.5}$$

where $\tilde{J}^i \equiv \epsilon^{ij} J_j$ is the dual to the electromagnetic current and $\epsilon^{12} = -\epsilon^{21} = 1$ and $\epsilon^{11} = \epsilon^{22} = 0$. $\tilde{J}_i$ is classically conserved but quantum loop correction give rise to a non-zero divergence. Eqn(1.4) is just the special case of Eqn(1.5) where E is uniform but time-dependent.

$\tilde{J}^i$ can be derived from an ungauged (3+1)-dimensional current $\tilde{J}^\mu$ that also has an anomaly by integrating $\tilde{J}^\mu$ over the directions perpendicular to the string. By studying the anomaly equation for $\tilde{J}^\mu$ and comparing the result with Eqn(1.5) one can in fact determine the number of zero modes. This point has been discussed in detail by Hill and Lee [7].

In Witten's original model, left movers couple to ϕ while right movers couple to ϕ^*. No other Yukawa-type interactions are included. However, in 'realistic' models [e.g., those inspired by GUTs or superstrings] there can be many $U(1)'s$ that are broken and many scalar fields that acquire VEVs. This considerably complicates the situation and there are a number of issues that must be addressed if one is to determine whether superconductivity occurs in a given model. For example, there may be interactions between right and left movers due to the presence of a scalar field that is non-zero in the core of the string [i.e., a scalar field that does not wind where there is a string]. Only a detailed study of the Dirac equations for the fermions in the background field of the string can determine whether or not any zero modes exist. In the absence of zero modes, it is easy to show that in most astrophysically realistic situations, the currents are uninterestingly small [8,9].

A second possibility considered is that a cosmic string traps only right or only left movers [or more generally, that Eqn(1.2) is not satisfied]. This leads to an uncancelled anomaly in the $(1+1)$-dimensional theory for the string even though the full $(3+1)$-dimensional theory is anomaly free. This situation arises in global or axion strings [10] as well as 'frustrated' strings [11] but not for Nielsen-Olesen type strings. [As will be discussed below, frustrated strings arise when there are more than one $U(1)'$-charged scalar fields that acquire non-zero VEVs. In order that the string be a Nielsen-Olesen vortex or local cosmic string (i.e., that the energy of the string be localized in the core of the string) one requires special relations among the winding numbers of the fields. These conditions may not be satisfied for strings formed during a phase transition in the early Universe and strings can therefore have long-range contributions to their mass as in the case of the global string.] The breakdown of charge conservation on the string is accounted for by an inflow of charge from the $(3+1)$-dimensional world just as in the axion string studied by Callen and Harvey [10]. [We emphasize again that the four dimensional electromagnetic current *must* be conserved if the $(3+1)$-dimensional theory is free of gauge anomalies.] In the presence of an external electric field, the string acquires charge and current. However, both charge and current are screened by the surrounding medium. The screening is due to either polarization of the vacuum [12] or plasma effects.

In Section II we discuss the existence of zero modes first for Witten's model, and then for a more general model that allows for interactions between right and left movers. In Section III we discuss the effective low energy $(1+1)$-dimensional theory for the models considered in Section II. We find that superconductivity only occurs when there are zero modes present. In Section IV we apply these results to a specific model based on the gauge group E_6. In Section V we discuss anomalous superconductivity in frustrated strings as well as some electromagnetic properties of anomalous strings in astrophysical settings.

Finally, in Section VI we give a summary and some conclusions.

II Zero Modes in the Fermion Vortex System

We begin this section by discussing the existence of zero modes in the simple model for a superconducting cosmic string first proposed by Witten. In addition to the fields ϕ and A'_μ that make up the string, the model has 4 left-handed (2-component Weyl) spinors, Ψ, Γ, Λ, and Δ. The spinors are written in the Van der Waerden notation (see, for example, ref. [13]). The charge assignments for the model, in units of the elementary charge e, are given in Table I. As discussed by Witten [1], this collection of spinors represents the minimal set of spinors that is free of gauge anomalies in a $U(1) \times U(1)'$ gauge theory. For simplicity, we assume that the string is infinite, static, and straight and choose the z-axis of a cylindrical (r, θ, z) coordinate system to lie along the string. ϕ and A'_μ have the form [14]:

$$\phi = f(r)e^{in\theta} \qquad A'_x = \sin\theta \, \frac{a(r)}{er} \qquad A'_y = -\cos\theta \, \frac{a(r)}{er} \tag{2.1}$$

The asymptotic behavior of the fields is given by:

$$f(r) \to \eta \;\; for \;\; r \to \infty \qquad f(r) \to 0 \;\; for \;\; r \to 0 \tag{2.2a}$$

$$a(r) \to -\frac{n}{2} \;\; for \;\; r \to \infty \qquad a(r) \to 0 \;\; for \;\; r \to 0 \tag{2.2b}$$

where η is a constant determined by the scalar potential. We see that for $r \to \infty$, $D_\theta \phi \equiv (i\partial_\theta - 2a(r))\,\phi/r \to 0$ so that the mass per unit length μ $(= O(\eta^2))$ is finite and is localized within a core whose radius is $\propto \eta^{-1}$. This is characteristic of a Nielsen-Olesen vortex or local (gauge) string in contrast to either the global (axion) string or the frustrated string. As will be discussed below, for both global and frustrated strings, $D_\theta\phi \to O(\eta/r)$ for $r \to \infty$ so that there are long range contributions to the mass per unit length of these strings. For a Nielsen-Olesen string,

$$\begin{aligned} n &= -\frac{i}{2\pi}\int d\ln\phi \\ &= \frac{e}{\pi}\int dl_i \cdot A'_i \\ &= \frac{e}{\pi}\int d^2x F'_{12} \end{aligned} \tag{2.3}$$

where $F'_{12} = \partial_x A'_y - \partial_y A'_x$ is the z-component of the $U(1)'$ magnetic field. Eqn(2.3) holds *only* for Nielsen-Olesen strings.

The Lagrangian for the fermions in the model is

$$
\begin{aligned}
\mathcal{L} \;=\; & -\, i\bar{\Psi}_{\dot\alpha}\bar{\sigma}^{\mu\dot\alpha\alpha}D_\mu\Psi_\alpha \;-\; i\bar{\Gamma}_{\dot\alpha}\bar{\sigma}^{\mu\dot\alpha\alpha}D_\mu\Gamma_\alpha \;-i\;\bar{\Lambda}_{\dot\alpha}\bar{\sigma}^{\mu\dot\alpha\alpha}D_\mu\Lambda_\alpha \;-\; i\bar{\Delta}_{\dot\alpha}\bar{\sigma}^{\mu\dot\alpha\alpha}D_\mu\Delta_\alpha \\
& -\; ig_1\phi\Psi^\alpha\Gamma_\alpha \;-\; ig_2\phi^*\Lambda^\alpha\Delta_\alpha \;+\; h.c.
\end{aligned}
\tag{2.4}
$$

In Eqn(2.4), $\Gamma^\alpha = \epsilon^{\alpha\beta}\Gamma_\beta$, $\epsilon^{11} = \epsilon^{22} = 0$, and $\epsilon^{12} = -\epsilon^{21} = 1$. $\sigma^\mu = (I, \vec{\sigma})$ and $\bar{\sigma}^\mu = (I, -\vec{\sigma})$ where I is the 2×2 unit matrix and $\vec{\sigma}$ are the usual Pauli matrices. $D_\mu\psi = (\partial_\mu + iqA_\mu + iq'A'_\mu)\psi$ etc. where q and q' are the $U(1)$ and $U(1)'$ charges of ψ respectively. The conventions are those of, for example, Wess and Bagger [13]. Hereafter, we will suppress spinor indices.

The existence of fermion zero modes in the background of the string may be demonstrated by directly solving the equations of motion for the fermions [4] and by calculating the index of the relevant Dirac operator [15]. We review these two techniques in some detail as both methods will be used to study more complicated systems. The equations for (Ψ, Γ) and (Λ, Δ) decouple. For Ψ and Γ we have:

$$\bar{\sigma}^\mu D_\mu\Psi \;-\; g_1\phi^*\bar{\Gamma} \;=\; 0 \tag{2.5a}$$

$$\sigma^\mu D_\mu\bar{\Gamma} \;+\; g_1\phi\Psi \;=\; 0 \tag{2.5b}$$

Zero modes are solutions to Eqns(2.5) of the form

$$\Psi_\alpha \;=\; \begin{pmatrix} \alpha_1(z-t)\psi_1(r,\;\theta) \\ \alpha_2(z+t)\psi_2(r,\;\theta) \end{pmatrix} \tag{2.6a}$$

$$\bar{\Gamma}_{\dot\alpha} \;=\; \begin{pmatrix} \alpha_1(z-t)\gamma_1^*(r,\;\theta) \\ \alpha_2(z+t)\gamma_2^*(r,\;\theta) \end{pmatrix} \tag{2.6b}$$

with $(\psi_1,\ \gamma_1)$ representing right movers and $(\psi_2,\ \gamma_2)$ representing left movers. With the zero mode ansatz Eqns(2.6), the equations for $(\psi_1,\ \gamma_1)$ and $(\psi_2,\ \gamma_2)$ decouple. For ψ_2 and γ_2 we have:

$$(\partial_- + ieA_-)\psi_2 \;+\; g_1\phi^*\gamma_2^* \;=\; 0 \tag{2.7a}$$

$$(\partial_+ - ieA_+)\gamma_2^* \;+\; g_1\phi\psi_2 \;=\; 0 \tag{2.7b}$$

where $\partial_\pm = \partial_x \pm i\partial_y$ and $A_\pm = A'_x \pm iA'_y$. We note, for future reference, that $\partial_\pm = e^{\pm i\theta}(\partial_r \pm \frac{i}{r}\partial_\theta)$ and $ieA_\pm = \pm e^{\pm i\theta}\ a(r)/r$. Similar equations can of course be written for the pairs of components $(\psi_1,\ \gamma_1)$, $(\lambda_1,\ \delta_1)$ and $(\lambda_2,\ \delta_2)$ where λ_α and δ_α are the (r, θ) dependent parts for the components of Λ and Δ as in Eqns(2.6). Eqns(2.7) can be written in the form $\mathcal{D}V_2 = 0$ where $V_2 = \begin{pmatrix} \psi_2 \\ \gamma_2^* \end{pmatrix}$ and $\mathcal{D}$ is a Dirac operator similar to the one studied by Weinberg. Explicitly, $\mathcal{D} = P_i\partial_i + Q$ where $Q = \begin{pmatrix} ieA_- & g_1\phi^* \\ g_1\phi & -ieA_+ \end{pmatrix}$. $P_1 = I$ and $P_2 = -i\sigma_3$ so that $P_iP_j^\dagger + P_j^\dagger P_i = 2\delta_{ij}I$ and $P_i^\dagger P_j + P_jP_i^\dagger = 2\delta_{ij}I$. The Dirac equation of $V_1 = \begin{pmatrix} \psi_2 \\ \gamma_2^* \end{pmatrix}$ is is $\mathcal{D}^\dagger V_1 = 0$ where $\mathcal{D}^\dagger$ is the Hermitian conjugate of $\mathcal{D}$.

The index $\mathcal{I}$ of $\mathcal{D}$ gives the number of normalizable solutions to the equation $\mathcal{D}V = 0$ minus the number of solutions to $\mathcal{D}^\dagger V = 0$. It follows that $\mathcal{I}$ gives the number of left moving zero modes (lower components) minus the number of right moving zero modes (upper components). The index is given by [15]:

$$\mathcal{I}(\mathcal{D}) \;=\; Tr\left(\frac{M^2}{\mathcal{D}\mathcal{D}^\dagger \;+\; M^2} \;-\; \frac{M^2}{\mathcal{D}^\dagger\mathcal{D} \;+\; M^2}\right) \tag{2.8}$$

where M^2 is an arbitrary constant. We will find it most convenient to calculate the index with $M^2 \to \infty$. A straightforward calculation shows that

$$\mathcal{D}\mathcal{D}^\dagger \;=\; (-\nabla^2 + |g_1\phi|^2 + e^2A'^2)I \;-\; 2ie\sigma^3(A'_x\partial_x + A'_y\partial_y) + eF_{12} + C \tag{2.9a}$$

$$\mathcal{D}^\dagger\mathcal{D} \;=\; (-\nabla^2 + |g_1\phi|^2 + e^2A'^2)I \;-\; 2ie\sigma^3(A'_x\partial_x + A'_y\partial_y) - eF_{12} + \tilde{C} \tag{2.9b}$$

where

$$C = g_1\begin{pmatrix} 0 & D_-\phi^* \\ D_+\phi & 0 \end{pmatrix}, \qquad \tilde{C} = -g_1\begin{pmatrix} 0 & D_+\phi^* \\ D_-\phi & 0 \end{pmatrix}, \tag{2.10}$$

$D_\pm\phi = (\partial_\pm - 2ieA_\pm)\phi$, and $D_\pm\phi^* = (\partial_\pm + 2ieA_\pm)\phi^*$. Following Weinberg we expand the terms in the trace:

$$\begin{aligned}\lim_{M^2\to\infty} \frac{M^2}{\mathcal{D}\mathcal{D}^\dagger + M^2} &= \frac{M^2}{(-\nabla^2 + |g_1\phi|^2 + M^2)^2} \\ &\quad + \frac{M^2}{(-\nabla^2 + |g_1\phi|^2 + M^2)^2}\, L\, \frac{M^2}{(-\nabla^2 + |g_1\phi|^2 + M^2)^2} + \ldots\end{aligned} \tag{2.11}$$

where $L = -2ie\sigma^3(A'_x\partial_x + A'_y\partial_y) + e^2A^2 + eF'_{12} + C$. For $M^2/(\mathcal{D}^\dagger\mathcal{D} + M^2)$, one replaces in Eqn(2.11) L with $\tilde{L}$ where $\tilde{L} = -2ie\sigma^3(A'_x\partial_x + A'_y\partial_y) + e^2A^2I - eF'_{12} + \tilde{C}$. We find, using Eqn(2.3) that

$$\begin{aligned}\mathcal{I} \;&=\; \lim_{M^2\to\infty}\int d^2x\; 4eF_{12}(x)\; M^2\; \langle x|\, \frac{1}{(-\nabla^2 + M^2)^2}\, |x\rangle \\ &=\; n\end{aligned} \tag{2.12}$$

Similar calculations show that the indices for (ψ_1, γ_1), (λ_1, δ_1), and (λ_2, δ_2) are $-n$, n, and $-n$ respectively.

The derivation outlined above is valid only for Nielsen-Olesen strings where $D_\pm\phi$ is exponentially damped away from the string. The analysis for a more general case where one requires that $D_\theta \phi$ fall off only as fast as $1/r$ for $r \to \infty$ is more difficult, but leads to the result that $\mathcal{I} = -\frac{i}{2\pi}\int d\ln\phi = n$ [15]. The results from the method outlined above agrees with this one for the Nielsen-Olesen string where Eqn(2.3) is valid.

What have we learned from these calculations? For $n > 0$, we know that the number of $(\Psi,\ \Gamma)$ left moving zero modes minus the number of $(\Psi,\ \Gamma)$ right moving zero modes is n. Similarly, the number of $(\Lambda,\ \Delta)$ right moving zero modes minus the number of $(\Lambda,\ \Delta)$ left moving zero modes is n. However, to determine the true number of zero modes in either case we must study the actual equations of motion.

Consider again, the equations of motion for ψ_2 and γ_2. With the ansatz $\psi_2(r,\theta) = e^{im\theta}\psi(r)$ and $\gamma_2^*(r,\theta) = e^{i(m+n-1)\theta}\gamma(r)$ we have

$$\left(\partial_r + \frac{m - a}{r}\right)\psi + g_1 f\gamma = 0 \tag{2.13a}$$

$$\left(\partial_r - \frac{m + n - 1 + a}{r}\right)\gamma + g_1 f\psi = 0 \ . \tag{2.13b}$$

Of the two solutions at infinity ($\psi, \gamma \propto e^{\pm g_1 \eta r}$) only the solution $\propto e^{-g_1\eta r}$ is acceptable. There are two solutions at the origin and some linear combination of these solutions will match the one acceptable solution at $r \to \infty$. Therefore, we require that both solutions at the origin be regular. To be more explicit, we have, for $r \to 0$,

$$\psi = a_1 r^{-m} + a_2 r^{m+n+p} \tag{2.14a}$$

$$\gamma = a_3 r^{m+n-1} + a_4 r^{-m+p+1} \tag{2.14b}$$

where $a_1 \ldots a_4$ are constant coefficients and $f(r) \to r^p$ for $r \to 0$. $p\ (>0)$ depends on the details of the scalar potential and for the original Nielsen-Olesen vortex [14], $p = |n|$. We note that $a(r) \to 0$ for $r \to 0$ so that a in Eqn(2.13) can be neglected for the purpose of determining the number of acceptable solutions to the Dirac equations. For both Eqn(2.14a) and Eqn(2.14b) to be acceptable solutions at the origin, we require that $0 \geq m \geq 1-n$. Therefore, for $n > 0$, there are n acceptable choices of m. Eqns(2.14) give two relations between the coefficients $a_1 \ldots a_4$ and the requirement of matching the solution at the origin to the solution at infinity places an additional constraint on the coefficients. The final parameter is fixed by normalization. It follows that there are n normalizable zero modes. These zero modes are effectively $(1+1)$-dimensional massless fermions constrained to move along the string in the $-z$ direction (left-movers). A similar analysis for (ψ_1, γ_1), (λ_1, δ_1), and (λ_2, δ_2) follows in exactly the same manner and one finds that there are n solutions (right-moving zero modes) for (λ_1, δ_1) and *no* normalizable solutions for either (ψ_1, γ_1) or (λ_2, δ_2). The result of the index theorem together with the fact that there are no $(\psi_1,\ \gamma_1)$ zero modes tells us that the n zero modes found above are *all* of the zero modes. Similarly, there are exactly n $(\lambda_1,\ \delta_1)$ zero modes (right movers). To summarize, we have learned that there are exactly n zero modes of the form:

$$\Psi = \begin{pmatrix} 0 \\ \alpha(z+t)\psi(r,\ \theta) \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 \\ \alpha(z+t)\gamma(r,\ \theta) \end{pmatrix} \tag{2.15a}$$

$$\Lambda = \begin{pmatrix} \beta(z-t)\lambda(r,\ \theta) \\ 0 \end{pmatrix} \qquad \Delta = \begin{pmatrix} \beta(z-t)\delta(r,\ \theta) \\ 0 \end{pmatrix} \tag{2.15b}$$

The effective $(1+1)$-dimensional physics of the system described above will be discussed in a later section. We now discuss the existence of zero modes for a more general Lagrangian, and one that can be readily adapted to realistic models (see also ref. 16). As before, we consider a model with 4 left-handed spinors Ψ, Γ, Λ, and Δ though now the fermions can each carry charge under more than one $U(1)'$ symmetry. The Yukawa terms in the model are assumed to be of the form

$$\mathcal{L}_{Yukawa} \;=\; -i\left[\phi_1\Psi\Gamma \;+\; \phi_2\Lambda\Delta \;+\; \phi_3\Psi\Delta \;+\; e^{i\epsilon}\phi_4\Gamma\Lambda\right] \tag{2.16}$$

where $\phi_1,\ \phi_2,\ \phi_3$ and ϕ_4 are complex scalar fields that can acquire VEVs and ϵ is a CP violating phase. Coupling constants have been absorbed into the definition of the scalar fields. A cosmic string exists when at least one of the fields acquires a VEV that has a winding number. In general, one has $\phi_i = f_i(r)e^{in_i\theta}$ where the n_i are integers and at least one of the n_i is non-zero. The case discussed above corresponds to $\langle\phi_1\rangle = \langle\phi_2^*\rangle \neq 0$, $\langle\phi_3\rangle = \langle\phi_4\rangle = 0$, and $n_1 = -n_2 = n$. Eqn(2.16) is of course invariant under the gauge group for the model. In what follows, g will refer to the $U(1)'$ factors that are broken. $q(g, X)$ will denote $g'th$ charge of the field X. It follows that, for example:

$$q(g,\ \phi_1) \;+\; q(g,\ \Psi) \;+\; q(g,\ \Gamma) \;=\; 0. \tag{2.17}$$

The analysis of this model closely follows our previous discussion of Witten's model though here, the Dirac operator is a 4×4 matrix. We begin by assuming a zero mode ansatz for the fermions:

$$\Lambda \;=\; \begin{pmatrix} \beta_1(z-t)\lambda_1(r,\ \theta) \\ \beta_2(z+t)\lambda_2(r,\ \theta) \end{pmatrix} \tag{2.18a}$$

$$\bar{\Delta} \;=\; \begin{pmatrix} \beta_1(z-t)\delta_1^*(r,\ \theta) \\ \beta_2(z+t)\delta_2^*(r,\ \theta) \end{pmatrix} \tag{2.18.b}$$

and Eqns(2.6) for Ψ and Γ. With this ansatz, the equations for the lower components separate from the equations for the upper components. For the lower components, we have that $\mathcal{D}V_2 = 0$ where

$$\mathcal{D} = \begin{pmatrix} D_- & \phi_1^* & 0 & \phi_3^* \\ \phi_1 & D_+ & e^{i\eta}\phi_4 & 0 \\ 0 & e^{-i\eta}\phi_4^* & D_- & \phi_2^* \\ \phi_3 & 0 & \phi_2 & D_+ \end{pmatrix} \tag{2.19}$$

and

$$V_2 = \begin{pmatrix} \psi_2 \\ \gamma_2^* \\ \lambda_2 \\ \delta_2^* \end{pmatrix} \tag{2.20}$$

In these equations, $D_\pm \psi_2 = \Big(\partial_\pm + \sum_g iq(g,\ \Psi)A_\pm(g)\Big)\psi_2$ etc. where $A_\mu(g)$ is the gauge field for the $g'th\ U(1)'$ factor and $A_\pm(g) = A_x(g) \pm iA_y(g)$. As before, the Dirac equation for the upper components is $\mathcal{D}^\dagger V_1 = 0$ where $V_1 = (\psi_1,\ \gamma_1^*,\ \lambda_1,\ \delta_1^*)$.

We now derive the index of the Dirac operator Eqn(2.19) for the simple case where the string is a Nielsen-Olesen type string. For a Nielsen-Olesen-type string, if ϕ_i acquires a non-zero VEV, then $D_\theta \phi_i \to 0$ for $r \to \infty$. This implies that

$$n_i - \sum_g q(g, \phi_i)\alpha(g) = 0 \tag{2.21}$$

where $iA_\pm(g) \to \pm e^{\pm i\theta}\alpha(g)/r$ for $r \to \infty$ and $\alpha(g)$ is a constant. [This equation is of course meaningless if $\langle \phi_i \rangle = 0$.] Eqns(2.21) can actually place a constraint on the allowed winding numbers for the scalar fields. For example, if there is only one $U(1)'$ factor that is broken, but four scalars that acquire non-zero VEVs, then

$$\frac{n_1}{q(g,\ \phi_1)} = \frac{n_2}{q(g,\ \phi_2)} = \frac{n_3}{q(g,\ \phi_3)} = \frac{n_4}{q(g,\ \phi_4)} \tag{2.22}$$

in order for the string to be a Nielsen-Olesen string. For a Nielsen-Olesen string, we see that:

$$\begin{aligned} \sum_g q(g, \phi_i) \int F_{12}(g) d^2x &= \sum_g q(g, \phi_i) \int \vec{A}(g) \cdot d\vec{l} \\ &= -2\pi n_i \end{aligned} \tag{2.23}$$

where $F_{12}(g) = \partial_x A_y(g) - \partial_y A_x(g)$.

The calculation of the index proceeds as before. Again, we make use of the fact that we are dealing with a Nielsen-Olesen string where the covariant derivatives of the scalars that enter into the expressions for $\mathcal{D}\mathcal{D}^\dagger$ and $\mathcal{D}^\dagger\mathcal{D}$ are exponentially damped outside the string. It is straightforward to show that

$$\mathcal{I} \;=\; \frac{1}{4\pi}\int d^2x \left(2 \sum_{X=\Psi,\Gamma,\Lambda,\Delta} \sum_g q(g,\, X)\; F_{12}(g) \right). \tag{2.24}$$

From Eqn(2.17) [and three similar equations] it follows that:

$$-2 \sum_{X=\Psi,\Gamma,\Lambda,\Delta} q(g,\, X) \;=\; q(g,\, \phi_1) \;+\; q(g,\, \phi_2) \;+\; q(g,\, \phi_3) \;+\; q(g,\, \phi_4). \tag{2.25}$$

Using this and Eqn(2.21) we see that

$$\mathcal{I} \;=\; \frac{1}{2}(n_1 \;+\; n_2 \;+\; n_3 \;+\; n_4). \tag{2.26}$$

This is the index for the class of models described by the Lagrangian Eqn(2.16) given that one has a Nielsen-Olesen string. Though this appears to be a fairly restrictive class of models, we shall see in the next section, that our results are directly applicable to some 'realistic' examples. Furthermore, it should be easy to generalize the methods outlined here to a broader class of models.

As before, we note that an index calculation such as Eqn(2.26) contains only partial information concerning the number of zero modes for a Dirac operator. We now attempt to explicitly construct solutions to the Dirac equations for the model. As before, we investigate the possible solutions at the origin and at infinity. The Dirac equations for the model are:

$$\begin{pmatrix} e^{-i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta\right) & f_1 e^{-in_1\theta} & 0 & f_3 e^{-in_3\theta} \\ f_1 e^{in_1\theta} & e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta\right) & e^{i\epsilon} f_4 e^{in_4\theta} & 0 \\ 0 & e^{-i\epsilon} f_4 e^{-in_4\theta} & e^{-i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta\right) & f_2 e^{-in_2\theta} \\ f_3 e^{in_3\theta} & 0 & f_2 e^{in_2\theta} & e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta\right) \end{pmatrix} \begin{pmatrix} \psi_2 \\ \gamma_2^* \\ \lambda_2 \\ \delta_2^* \end{pmatrix} \;=\; 0. \tag{2.27}$$

[Again, we note that the gauge fields are irrelevant for counting solutions at either $r \to 0$ or $r \to \infty$ and so we have omitted them from the above equation.] A separation of variables is possible only when

$$n_1 + n_2 - n_3 - n_4 = 0 \tag{2.28}$$

and we now consider this case. It is interesting and important to note that all Nielsen-Olesen strings satisfy Eqn(2.28); i.e., using Eqn(2.17) plus three similar equations and

Eqn(2.21) we can show that Eqn(2.28) is satisfied. Therefore, all results derived below should agree with the index calculation above though the results below are in principle applicable to a more general class of cosmic strings.

With the ansatz:

$$\psi_2(r,\ \theta) \ = \ e^{im\theta}\psi(r) \qquad \gamma_2^*(r,\ \theta) \ = \ e^{i(m+n_1-1)\theta}\gamma(r)$$
$$\lambda_2(r,\ \theta) \ = \ e^{i(m+n_1-n_4)\theta}\lambda(r) \qquad \delta_2^*(r,\ \theta) \ = \ e^{i(m+n_3-1)\theta}\delta(r) \tag{2.29}$$

we have, for Eqns(2.27)

$$\begin{pmatrix} (\partial_r + \frac{m}{r}) & f_1 & 0 & f_3 \\ f_1 & (\partial_r - \frac{m+n_1-1}{r}) & e^{i\epsilon}f_4 & 0 \\ 0 & e^{-i\epsilon}f_4 & (\partial_r + \frac{m+n_1-n_4}{r}) & f_2 \\ f_3 & 0 & f_2 & (\partial_r - \frac{m+n_3-1}{r}) \end{pmatrix} \begin{pmatrix} \psi \\ \gamma \\ \lambda \\ \delta \end{pmatrix} = 0. \tag{2.30}$$

For $r \to \infty$ there are four solutions:

$$\psi,\ \gamma,\ \lambda,\ \delta \ \propto \ e^{\kappa_+ r},\ e^{\kappa_- r},\ e^{-\kappa_+ r},\ e^{-\kappa_- r} \tag{2.31}$$

Here,

$$\kappa_\pm \ = \ \frac{1}{2}\left(F \ \pm \ \left[F^2 \ - \ 4\Omega\right]^{1/2}\right) \tag{2.32}$$

where $F = \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2$ and $\Omega = \left(\eta_1^2\eta_2^2 - 2\eta_1\eta_2\eta_3\eta_4\cos\epsilon + \eta_3^2\eta_4^2\right)$. In general, two and only two of these solutions will be acceptable. In order to have a single normalizable solution one requires *three* acceptable solutions at the origin. The most general solution at the origin would then have three arbitrary constants. Matching this solution to an acceptable solution at $r \to \infty$ places two constraints on these constants. Normalization then completely fixes the solution. [Four acceptable solutions at the origin would lead to a one-parameter family of normalizable solutions, (i.e., an infinite number of solutions) while only two acceptable solutions at the origin will not in general be adequate for matching.] The requirement that three solutions at the origin be regular implies that of the four conditions:

$$m \ \leq \ 0 \qquad m \ + \ n_1 \ - \ 1 \ \geq \ 0$$
$$m \ + \ n_1 \ - \ n_4 \ \leq \ 0 \qquad m \ + \ n_3 \ - \ 1 \ \geq \ 0 \tag{2.33}$$

three must be satisfied.

Let us now consider some specific examples.

Case 1: $n_1 \ = \ -n_2 \ = \ n, \ \ n_3 \ = \ n_4 \ = \ 0$

In this case, the index is zero. Furthermore, we see that only two of the four conditions in Eqns(2.33) can be satisfied with a single choice of m and we conclude that there are no normalizable zero modes. However, this conclusion is based on the assumption that all of the VEVs of the scalars be non zero. If, for example, $\langle\phi_3\rangle = \langle\phi_4\rangle = 0$ then one is back to the Witten string studied in the beginning of the section and in this case, there are n normalizable solutions with $\gamma_2 = \delta_2 = 0$. Here, the two regular solutions at the origin have matched on to one of the acceptable solutions at infinity. [The index is still zero owing to the fact that there are now n normalizable solutions to $\mathcal{D}^\dagger V = 0$]. However, when $\langle\phi_3\rangle \neq 0$ and/or $\langle\phi_4\rangle \neq 0$, this type of matching does not work, and we conclude that there are no zero modes.

Case 2: $n_1 = n_4 = n, \quad n_2 = n_3 = 0$

Here, the index is n and for $n > 0$ this tells us that there are at least n zero modes. With $1 - n \leq m \leq 0$ we see that three of the conditions in Eqns(2.33) are satisfied and we conclude that for $n > 0$, there are exactly n zero modes. Furthermore, an analysis of the equations of motion reveals that there are no normalizable solutions to $\mathcal{D}^\dagger V = 0$. This, together with the index theorem tells us that there are exactly n left moving zero modes of the form

$$\Psi = \begin{pmatrix} 0 \\ \alpha(z+t)\psi(r,\ \theta) \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 \\ \alpha(z+t)\gamma(r,\ \theta) \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 0 \\ \alpha(z+t)\lambda(r,\ \theta) \end{pmatrix} \qquad \Delta = \begin{pmatrix} 0 \\ \alpha(z+t)\delta(r,\ \theta) \end{pmatrix} \tag{2.34}$$

and that these are the only zero modes.

III Superconductivity and Massive Bound States

In Section II we studied the existence of zero modes for a class of models involving four Weyl spinors and four complex scalar fields. Given a certain relation among the winding numbers of the scalars, Eqn(2.28), we found that the number of zero modes could be best determined by directly studying the equations of motion for the fermions. Consistency and completeness are checked by calculating the index of the appropriate Dirac operator. In this section, we discuss the effective $(1+1)$-dimensional theory for the various cases considered in Section II.

An effective Lagrangian for the superconducting string discussed at the beginning of Section II can be written and has been studied by a number of authors. Superconductivity can be understood (1) from the bosonized equations of motion for the electric current [1]; (2) from the anomaly equations for the $(1+1)$-dimensional fermions [17]; and (3) from the presence of a mixed anomaly in the full $(3+1)$-dimensional theory [7]. [By mixed anomaly, we mean that $\partial^\mu j_\mu \sim \epsilon_{\mu\nu\lambda\kappa} F_{\mu\nu} F'_{\lambda\kappa}$ where $F_{\mu\nu}$ is the field strength tensor for ordinary electromagnetism and $F'_{\mu\nu}$ is the field strength tensor for the $U(1)'$ gauge group that is broken. In the present discussion, we will follow the second of these approaches.

We begin by reviewing the derivation of superconductivity for Witten's original model. Recall that for $n > 0$ we have n right moving and n left moving zero modes of the form given in Eqn(2.17). [The generalization to the case where there are more than one species of right or left movers is straightforward. In the equations that follow, one would simply sum over the different species that are trapped as zero modes on the string. To keep the notation simple, we ignore this complication in the present discussion.]

We assume that only the zero modes contribute to the low energy $(1+1)$-dimensional effective Lagrangian, $\mathcal{L}_{eff}$. $\mathcal{L}_{eff}$ is found by integrating the full $(3+1)$-dimensional Lagrangian, Eqn(2.4), over r and θ:

$$\begin{aligned} \mathcal{L}_{eff} \;=\; \sum_{m=1}^{n} \Big[& i\alpha^*_m(z,\ t)\Big[\partial_0 - \partial_3 + ie(A_0 - A_3)\Big]\alpha_m(z,\ t) \\ & + \; i\beta^*_m(z,\ t)\Big[\partial_0 + \partial_3 + ie(A_0 + A_3)\Big]\beta_m(z,\ t)\Big] \;. \end{aligned} \tag{3.1}$$

Here, m labels the different Jackiw-Rossi zero modes by, say their angular momentum [though this m is not precisely the same m as in Eqn(2.13)]. In deriving Eqn(3.1) we have assumed proper normalization for the zero modes.

Consider the total electric current for the fermions, $J^i \equiv \delta\mathcal{L}_{eff}/\delta A_i \equiv (\rho,\ J)$ where

$$\rho \;=\; -\sum_{m=1}^{n}\Big[\alpha^*_m(z,\ t)\alpha_m(z,\ t) \;+\; \beta^*_m(z,\ t)\beta_m(z,\ t)\Big] \tag{3.2a}$$

$$J \;=\; \sum_{m=1}^{n}\Big[\alpha^*_m(z,\ t)\alpha_m(z,\ t) \;-\; \beta^*_m(z,\ t)\beta_m(z,\ t)\Big] \tag{3.2b}$$

The divergence of this current is zero because the contributions from $(\Psi,\ \Gamma)$ and $(\Lambda,\ \Delta)$ exactly cancel. From Eqn(1.1) we see that

$$(\partial_0 \;-\; \partial_3)\alpha^*_m(z,\ t)\alpha_m(z,\ t) \;=\; -\frac{e^2}{2\pi}E \tag{3.3a}$$

$$(\partial_0 \;+\; \partial_3)\beta^*_m(z,\ t)\beta_m(z,\ t) \;=\; \frac{e^2}{2\pi}E \tag{3.3b}$$

(no sum on m) so that

$$\partial_0\rho \;+\; \partial_3 J \;=\; 0 \tag{3.4}$$

However, it also follows that

$$\partial_0 J \;+\; \partial_3\rho \;=\; -\frac{ne^2}{\pi}E \tag{3.5}$$

In a constant electric field, dJ/dt is just proportional to the electric field and furthermore, when the field is turned off the current persists. These results imply that the string is superconducting.

Superconductivity has a simple physical interpretation in terms of particles and holes in the Dirac sea [1,17]. In the absence of an electric field, the negative energy states of right and left movers are filled while positive energy states are empty (Fig. 1). In an applied electric field, the fermi level of the left movers is shifted upward while the fermi level of the right movers is shifted downward. The total numbers of right and left movers are separately conserved as is required by the axial symmetry of the model. However, at the top of the Dirac sea, where one actually measures the current, the shift in fermi levels amounts to (has the appearance of) negative energy particles (in this case, right movers) being excited into positive energy states with opposite chirality (here, left moving states). This is schematically shown in Fig. 2. The result is a net electric current and this current can relax only when an electric field is applied in the opposite direction.

As discussed by Witten, we note that the string cannot build up charge indefinitely. Eventually, the zero modes gain enough energy to move off the string as massive fermions. This transition is shown in Fig. 3. The energy at which the transition can first occur is given by the mass of the fermions off the string and is $g\eta$ for the superconducting string of Eqn(2.4). In the four fermion models, this energy is given by $min(|\kappa_+|,\ |\kappa_-|) = |\kappa_-|$ (cf. Eqns(2.31-2)). Once the energy of the zero modes becomes greater than $|\kappa_-|$, bound state solutions to the Dirac equations no longer exist. If M denotes the energy at which the fermions can first move off the string, then the maximum or critical current is given by [1]

$$J_{max} \;=\; \frac{q}{e}\,\frac{M}{1\ GeV}\;4\times 10^{4}\ Amps. \tag{3.6}$$

With $10^{16}\ GeV$ fermions (GUT scale) for example, the currents can be as large as $4\times 10^{20}\ Amps.$

In Case 2 of Section II we found that there were n left moving zero modes of the form given in Eqn(2.34). These zero modes alone would lead to an uncanceled anomaly on the string. Anomalous superconductivity will be discussed in Section V. Here, we assume that there are also right moving modes such that Eqn(1.2) is satisfied. [As we shall see in the next section, this naturally occurs in some GUT based models.] The analysis now proceeds in exactly the same manner as in Witten's model and we conclude that the string is superconducting.

We now discuss the behavior of the string in the absence of zero modes. Qualitatively, this case is easy to understand. Suppose we introduce into Witten's model, small couplings between right and left movers. This is Case 1 of Section II with $(\langle\phi_3\rangle,\ \langle\phi_4\rangle) \ll (\langle\phi_1\rangle,\ \langle\phi_2\rangle)$.

We have seen that there are no zero modes in the presence of these couplings. However, we expect bound state solutions with a non-zero groundstate energy (or effective mass) that is determined by the interaction terms in the Lagrangian [i.e., by $|\langle\phi_3\rangle|$ and $|\langle\phi_4\rangle|$]. This mass induces a gap between positive and negative energy levels as in Fig. 4. In order for a current to be excited, the energy due to the applied electric field must be greater than this mass scale. Let m denote this mass, B the field strength of some cosmic magnetic field, and v the velocity of the string relative to the B-field. Astrophysically interesting currents can be generating only if [8]

$$v\left(\frac{B}{1\ Gauss}\right)\left(\frac{1\ eV}{m}\right)^2 \geq 200. \tag{3.7}$$

Since galactic magnetic fields are typically $O(10^{-6})$ $Gauss$ and intergalactic fields are at least a few orders of magnitude smaller than this, we see that the inequality Eqn(3.7) will not be satisfied for most reasonable choices of m.

A detailed and quantitative analysis of Case 1 in its most general form is very difficult. However, we can gain some insight into how one can approach this problem by studying the very simple case where $\phi_1 = \phi_2^*$ and $\phi_3 = \phi_4 = im$. This model was analyzed in ref.[8] using the bosonized equations of motion. Here, we directly study the effective $(1+1)$-dimensional theory for the fermions without resorting to bosonization.

For the case at hand, there are indeed bound state solutions [8]. These solutions have exactly the same form as in Eqn(2.15) but now

$$[\partial_0^2 \ - \ \partial_3^2 \ + \ m^2)]\alpha(z,\ t) \ = \ 0 \tag{3.8a}$$

$$[\partial_0^2 \ - \ \partial_3^2 \ + \ m^2)]\beta(z,\ t) \ = \ 0 \tag{3.8b}$$

The effective Lagrangian has, in addition to the kinetic energy terms (Eqn(3.1)), the mass terms:

$$\Delta\mathcal{L}_{eff} \ = \ m\,[\alpha(z,\ t)\beta(z,\ t) \ + \ h.c.] \tag{3.9}$$

This is precisely the Lagrangian studied in a rather different context in ref. [18]. These authors calculate the creation of particle-antiparticle pairs (or equivalently, the production of electric current) in an external electric field using the Bogoliubov method. Here, we review their calculation and discuss their results as they apply to superconducting strings. Consider the case where an electric field is applied for a finite period of time, i.e., the electromagnetic potential $A_3 = A(t)$ is given by

$$A(t) \ = \ \begin{cases} 0 & t < 0 \\ A(t) & 0 < t < \tau \\ A_o & t < \tau \end{cases} \tag{3.10}$$

The two component, (1+1)-dimensional spinor $\psi = \begin{pmatrix} \alpha(z,\ t) \\ \beta(z,\ t) \end{pmatrix}$ is expanded in the following way:

$$\psi \;=\; \int \frac{dk}{2\pi}\, e^{ik\cdot x}\left[u_k(t)b_k \;+\; v_{-k}(t)d_k\right] \tag{3.11}$$

where u and v are spinors that satisfy the following Dirac equations:

$$i\partial_t u_k \;=\; \left[\sigma^3(k - A(t)) \;-\; \sigma^1 m\right] u_k \tag{3.12a}$$

$$i\partial_t v_{-k} \;=\; \left[\sigma^3(k - A(t)) \;-\; \sigma^1 m\right] v_{-k}. \tag{3.12b}$$

For $t < 0$, u and v correspond to positive and negative energy plane-wave solutions which we can write as $u_k^o(t) = e^{-i\omega_k t}u_k^o$ and $v_{-k}^o(t) = e^{i\omega_k t}v_{-k}^o$ where $\omega_k = \sqrt{k^2+m^2}$. The true vacuum for $t < 0$ is the state in which all of the negative energy states are filled. For $t > \tau$, the solutions have the form:

$$u_k(t) \;=\; A_k u^{(0)}_{k-eA_o}(t) \;+\; B_k v_{-(k-eA_o)}(t) \tag{3.13a}$$

$$v_k(t) \;=\; C_k u^{(0)}_{k-eA_o}(t) \;+\; D_k v_{-(k-eA_o)}(t) \tag{3.13b}$$

where $u^o_{k-eA_o}(t)$ and $v^o_{-(k-eA_o)}(t)$ are positive and negative energy plane-wave solutions for the true vacuum with $t > \tau$. $A_k \ldots D_k$ are known as Bogoliubov coefficients [see, for example, ref. [19]]. Non-zero B_k and D_k imply that negative energy particles have been excited into positive energy states. Specifically, $|B_k|^2$ is the probability that a particle with momentum k has been shifted up to a positive energy level. The total electric current generated can be expressed in terms of the Bogoliubov coefficients and we refer the reader to ref. [18] for the details of this calculation.

The results from this method agree with the analysis found previously using the bosonized equations of motion [8] as well as the qualitative discussion presented above. There is a current generated if and only if the energy in the electric field is greater than the mass m (sudden approximation in ref. [18]) whereas there is no current generated when the electric field is much smaller than this mass (adiabatic approximation of ref. [18]).

The Bogoliubov method can in principle be applied to the case where the VEVs of the scalars are arbitrary. However, this case is a computational nightmare. One must first study the Dirac equations in order to determine the groundstate energy of the bound state solutions. [This amounts to finding the eigenvalues of an 8×8 matrix equation!] Concentrating on the lowest energy solution, one would then derive an effective $(1+1)$-dimensional Lagrangian. As before, the functions are then expanded in terms of Bogoliubov coefficients and the coefficients are computed in a specified electric field. Though the exact value of the effective mass for the lowest energy boundstates can be determined only by solving the Dirac equations for the model, we expect that the groundstate energy of the lowest bound state will be at least $O[min(\langle\phi_3\rangle,\ \langle\phi_4\rangle) \times\ coupling\ constant]$. This energy is the mass that enters into a formula similar to Eqn(3.7). The inequality Eqn.(3.7) will ***not*** be satisfied (and in fact misses by many orders of magnitude) for most reasonable choices of m and B [e.g., $m > 1\ eV$ and $B \leq 10^{-6}\ Gauss$] and therefore in most cases, it is possible to determine that in the absence of zero modes on the string, superconducting currents will not be generated.

IV An E_6 Example

In Section II, we described a method for determining the number of zero modes in a certain class of fermion-vortex systems. We then discussed the effective $(1+1)$-dimensional theory for systems with zero modes and then for ones with massive bound states but no zero modes and found that astrophysically interesting currents were generated only when zero modes were present. In this section we apply these results to a specific particle physics model based on the gauge group E_6. [E_6 is introduced to provide a set of scalars and fermions within the framework of a viable GUTs theory that is interesting in terms of the possible fermion-vortex systems. The results apply equally well to $SU(5) \times U(1)'' \times U(1)'$ and $SO(10) \times U(1)'$.]

Superconducting strings in an E_6 model were first considered by Witten [1]. His analysis was a proof of existence one in which he showed that given cosmic strings formed from certain E_6 scalars and certain E_6 fermions, superconducting strings can arise. Here we consider a more complete set of fields found in an E_6 based GUT. As we shall see, both superconducting and nonsuperconducting strings (Case 1 and Case 2 of Section II) arise.

E_6 is broken to $O(10) \times U(1)'$ by a Higgs field in the adjoint representation of E_6 (i.e., the ϕ_{78}) and to $SU(5) \times U(1)' \times U(1)''$ by a Higgs field in the adjoint of $O(10)$ (i.e., the ϕ_{45}). The remaining Higgs fields and all of the fermions that will be consider are contained in the **27** of E_6. We first consider the electrically neutral scalars in the model, $\phi_i,\ i = 1 \div 5$. The charge assignments for the scalars are given in Table II. [To make contact with notation found elsewhere, we note that the scalars can also be labeled by the letter used for the fermion field that has the same quantum numbers, i.e., $\phi_1 \equiv n,\ \phi_2 \equiv N,\ \phi_3 \equiv \nu_E,\ \phi_4 \equiv N_E,\ \phi_5 \equiv \nu$.] We assume that ϕ_1 and ϕ_2 obtain large ($\geq 300GeV$) VEVs and break $U(1)'$ and $U(1)''$. The symmetry breaking gives rise to almost stable strings. The strings can actually decay through the formation of monopole-antimonopole pairs. However, if the scale at which the $U(1)'$ and $U(1)''$ are broken is much less (by a factor of 10 or so) than the scale at which E_6 is broken to $SU(5) \times U(1)' \times U(1)''$ then the decay rate is so slow that the strings are effectively stable [1]. [In any case, our purpose here is not to construct a complete and detailed GUT based cosmology, but to study superconducting strings in an interesting gauge theory.] The VEVs of ϕ_3, ϕ_4, and ϕ_5 are $\leq O(M_W)$. Actually, $\langle\phi_5\rangle$ may be zero but for now, let us assume that $\langle\phi_5\rangle = O(M_W)$. There are then five neutral scalars that acquire VEVs and effectively three magnetic $U(1)$ fluxes; $U(1)'$, $U(1)''$, and a flux T of electroweak fields that can be present in the core of any given string. [As discussed in [1] the core radii of the different fluxes are not equal but we can ignore this fact in the present discussion.] Put another way, there are three gauge fields that enter into the covariant derivatives for the scalars when we construct vortex solutions. For example,

$$iD_\mu\phi_3 \;=\; i\partial_\mu \;+\; 2A'_\mu \;+\; 2A''_\mu \;-\; T_\mu. \tag{4.1}$$

As usual, $\phi_i \to \eta_i e^{in_i\theta}, A'_\theta \to -\alpha'/r,\ A''_\theta \to -\alpha''/r$, and $T_\theta \to -t/r$ for $r \to \infty$. In the present discussion, we will assume that we have Nielsen-Olesen strings. For Nielsen-Olesen strings, $D_\theta\phi_i \to 0$ and, together with Eqn(4.1), this implies that

$$n_3 \;+\; 2\alpha' \;+\; 2\alpha'' \;-\; t \;=\; 0 \tag{4.2}$$

With n_i fixed but arbitrary, Eqn(4.2) and the corresponding equations for the other scalars are simultaneous equations for α', α'', t. Clearly, by solving these equations we minimize the energy of the string as this eliminates long range (log-infinite) contributions to the mass per unit length. However, there are five equations similar to Eqn(4.2) (one for each scalar) for the three unknowns $(\alpha'\ ,\alpha''\ ,t)$ and in general one cannot find a consistent solution.

If we impose the conditions

$$n_1 \;+\; n_3 \;+\; n_4 \;=\; 0 \tag{4.3a}$$

$$n_2 \;+\; n_4 \;+\; n_5 \;=\; 0 \tag{4.3b}$$

then there will be a unique choice for $(\alpha',\ \alpha'',\ t)$ such that Eqn.(4.2) and the corresponding equations for the other scalars are satisfied. In particular, one has

$$\alpha' \;=\; \frac{n_1}{4} \qquad \alpha'' \;=\; \frac{n_1 \;-\; 4n_2}{20} \qquad t \;=\; \frac{3n_1 \;-\; 2n_2 \;+\; 5n_3}{5}. \tag{4.4}$$

Strings whose winding numbers satisfy Eqns(4.3) have magnetic fluxes given by Eqn(4.4) and are Nielsen-Olesen vortices.

There is good theoretical motivation for Eqns(4.3). In general, one expects the terms $\phi_1\phi_3\phi_4$ and $\phi_2\phi_4\phi_5$ to be present in the scalar potential, either as terms in the original Lagrangian, or as induced terms in some effective Lagrangian. Minimizing these terms leads to the conditions Eqns(4.3). We note that these are the only trilinear couplings allowed given the charge assignments in Table II. As for higher order terms, only terms that *cannot* be written as an absolute square of scalar fields can affect the phases and hence winding numbers of the fields [e.g., $|\phi_1\phi_2|^2$ cannot affect the winding numbers of either ϕ_1 or ϕ_2.]. Moreover, minimizing the other higher order terms that might be present such as $\phi_1\phi_2^*\phi_3\phi_5^*$ is already achieved by minimizing the two trilinears. Indeed, the counting of solutions makes sense! There are five equations (Eqn(4.2) etc.), three unknowns, and two constraints. The questions of whether or not the trilinear terms are present (and important) in the potential and whether or not Eqns(4.3) are satisfied are interesting in their own right and have been addressed in a previous paper [9]. For the present discussion we will make the assumption that these conditions are in fact satisfied so that we have Nielsen-Olesen strings.

The charged fermions we consider are in the **27** of E_6 and their charges are given in Table II. It is easy to verify that the set of fermions is free of triangle anomalies such as QQA', QQA'', $QA'A''$, *etc.* . The Yukawa potential consists of the terms

$$\begin{aligned} \mathcal{L} \;=\; & \phi_1 EE^c \;+\; \phi_2 eE^c \;+\; \phi_3 ee^c \;+\; \phi_5 e^c E \\ & +\; \phi_1 hh^c \;+\; \phi_2 hd^c \;+\; \phi_3 dd^c \;+\; \phi_5 h^c d \;+\; \phi_4 uu^c \end{aligned} \tag{4.5}$$

where the coupling constants are implicit. The Dirac equations for the fermions separate into three groups of coupled equations; one for the leptons, one for the d and h quarks, and one for the u quarks. The Dirac equations for the u quarks are essentially the same as those considered by Jackiw and Rossi and by Weinberg. If ϕ_4 has winding number n_4 then there are $|n_4|$ u-quark zero modes. The Dirac equations for the leptons are the same (save only the values of the coupling constants) as those and for the d and h quarks and are precisely of the form studied in Section II (cf. Eqn(2.16)) so that the application of our previous results is immediate.

Presently, we consider what we shall refer to as minimal strings. These are Nielsen-Olesen strings [i.e., strings whose winding numbers satisfy Eqns(4.3)] that have no more than three fields that wind. We view these as the most likely strings to form as they involve the simplest Higgs configurations consistent with the requirement that they be Nielsen-Olesen vortices. The extension of this analysis to other Nielsen-Olesen strings is straightforward. The winding numbers for the minimal strings are listed in Table III. The zero modes in each of these cases are easily found using the results in Section II. For example, for strings A and D (case 1 of section II) there are no zero modes and hence no superconductivity. For strings B, C, and E, there are left moving zero modes composed of $(e,\ e^c, E,\ E^c)$ and $(d,\ d^c,\ h,\ h^c)$ [cf. Eqn(2.34)] and a right moving zero mode for $(u,\ u^c)$ (Fig. 5). It is easy to check that the total axial vector anomaly for these zero modes is zero (i.e., Eqn(1.1) is satisfied:

$$\sum_{left\ movers} = (-1)^2 + 3\left(-\frac{1}{3}\right)^2 = \sum_{right\ movers} = 3\left(\frac{2}{3}\right)^2 \tag{4.6}$$

where the factor of 3 for the quarks comes from summing over color indices. This is to be expected of the $(1+1)$-dimensional theory that is the effective theory for a Nielsen-Olesen string when the parent theory is anomaly free.

As mentioned above, $\langle\phi_5\rangle$ may be zero. In this case, there are four neutral scalars that acquire VEVs and therefore four equations of the form Eqn(4.2). Now, only Eqn(4.3a) is necessary to insure a Nielsen-Olesen vortex [Eqn(4.3b) no longer makes sense!] Again, when we consider the minimal strings, we see that A and D have no zero modes while B, C, and E still have zero modes. However, the left movers in string C and E are now pure E and h particles rather than admixtures of $(E,\ e)$ and $(d,\ h)$.

Let us compare our results with those found by Witten [1]. Witten assumed that only ϕ_1, ϕ_3, and ϕ_4 acquired VEVs [i.e., he did not consider the breaking of $U(1)''$.] He finds two superconducting strings (cf. his Fig. 11), one with E and h particles as left movers and u quarks as right movers and the other with E and h as left movers and d and e as right movers. The first case is akin to our strings B and C. Though we have also found that the strings are superconducting, we see that the left movers may now involve ordinary particles [either string B or string C with $\langle\phi_5\rangle \neq 0$]. Witten's second case is akin to our string A and here we see that the strings do *not* superconduct due to the presence of additional neutral scalars that due not wind in the presence of the vortex.

V. Anomalous Superconductivity

In this section we discuss anomalous superconductivity in frustrated cosmic strings. For frustrated strings there can be an uncanceled contribution to the axial vector anomaly in the effective $(1+1)$-dimensional theory (i.e., Eqn(1.2) does not hold) even though the underlying $(3+1)$-dimensional theory is anomaly free. This is in contrast to the Nielsen-Olesen vortex where the effective theory on the string is always anomaly free if the parent theory is anomaly free. An uncanceled anomaly indicates that the electric charge on the string is not conserved. The charge that appears on the string can be accounted for by an inflow of charge (a radially directed current) from the world outside the string. This current is due to the interaction of the fermions with the neutral scalars in the model and is very similar to the effect studied by Callen and Harvey [10] for the axion string.

Consider again Witten's $U(1)\times U(1)'$ model (Eqn(2.4)) but now with $(\Psi,\ \Gamma)$ coupling to ϕ_1 and $(\Lambda,\ \Delta)$ coupling to ϕ_2 where ϕ_1 and ϕ_2 are independent complex scalar fields. The charge assignments for the model are given in Table IV.

We first discuss the various string configurations possible. We assume that after symmetry breaking, ϕ_1 and ϕ_2 acquire non-zero VEVs. The phases of ϕ_1 and ϕ_2 at the time of symmetry breaking are correlated only within distances less than the correlation length for the phase transition. As the fields evolve, much of the initial variations in the fields die out. However, vortices remain. For a given vortex, $\phi_1 = f_1(r)e^{in_1\theta}$ and $\phi_2 = f_2(r)e^{in_2\theta}$. As usual, for $r \to \infty$, $f_1(r) \to \eta_1$, $f_2 \to \eta_2$, and $A'_\theta \to -\alpha/r$ where α is a constant. The mass per unit length μ of the string is given by

$$\mu = \int d^2r \left(|D_\mu\phi_1|^2 + |D_\mu\phi_2|^2 + V(\phi_1, \phi_2)\right) \tag{5.1}$$

where $V(\phi_1, \phi_2)$ is the ($U(1)'$-invariant) scalar potential, $D_\mu\phi_1 = (\partial_\mu + isA'_\mu)\phi_1$, and $D_\mu\phi_2 = (\partial_\mu - isA'_\mu)\phi_2$. For $r \to \infty$, there is a contribution to μ due to the kinetic energy terms that approaches $\left[(n_1 - s\alpha)^2\eta_1^2 + (n_2 + s\alpha)^2\eta_2^2\right] \int dr/r$. When $n_1 = -n_2 = s\alpha$ this long-range contribution to the mass vanishes so that the mass is localized in the core of the string and in fact $\mu \sim (\eta_1^2 + \eta_2^2)$. However, for $n_1 \neq -n_2$, there is a contribution to μ of the form $\left[(n_1 - s\alpha)^2\eta_1^2 + (n_2 + s\alpha)^2\eta_2^2\right] \ln(R/r_o)$ where R is an astrophysical scale (either the typical size of a loop or the mean separation between strings) and r_o is the core radius of the string. Energetically, the Nielsen-Olesen strings (those with $n_1 = -n_2$) are clearly favored. However, strings form during a time when the phases of the different fields are not completely correlated and the Nielsen-Olesen configuration may not, at least initially, be realized. It seems that the most likely situation, at least for the initial field configuration, is one in which $n_1 = 1$ and $n_2 = 0$ as this is the simplest configuration possible; i.e., involves the least number of fields winding about a given point. [Numerical simulations support this claim though work is still in progress [20].] The possibility that strings in a local $U(1)'$ theory might not be Nielsen-Olesen strings is referred to as frustration [11,21].

We now consider the fermions in the model. For $n_1 > 0$ ($n_1 < 0$) there are $|n_1|$ left-moving (right-moving) (Ψ, Γ)$-$zero modes. Similarly, for $n_2 > 0$ ($n_2 < 0$) there are $|n_2|$ left-moving (right-moving) (Λ, Δ)-zero modes. The electric current $J^i \equiv (\rho, J)$ is therefore

$$\rho = -\sum_{J-R\ zero\ modes} \left[\alpha^*(z, t)\alpha(z, t) + \beta^*(z, t)\beta(z, t)\right] \tag{5.2a}$$

$$J = \sum_{J-R\ zero\ modes} \left[sgn(n_1)\alpha^*(z, t)\alpha(z, t) + sgn(n_2)\beta^*(z, t)\beta(z, t)\right. \tag{5.2b}$$

where $sgn(x) = x/|x|$. Here, $\alpha(z, t)$ and $\beta(z, t)$ are the (z, t) parts of the zero mode solutions for (Ψ, Γ) and (Λ, Δ) respectively. Suppose that there is a constant electric field applied along the string so that J_i depends only on t. From Eqn(1.1) we see that

$$\frac{d}{dt}\,\alpha^*\alpha \;=\; -sgn(n_1)\frac{e^2E}{2\pi} \tag{5.3a}$$

$$\frac{d}{dt}\,\beta^*\beta \;=\; -sgn(n_2)\frac{e^2E}{2\pi}\;. \tag{5.3b}$$

It follows that

$$\frac{d\rho}{dt} \;=\; \Big(n_1 \;+\; n_2\Big)\frac{e^2E}{2\pi} \tag{5.4a}$$

$$\frac{dJ}{dt} \;=\; -\Big(|n_1| \;+\; |n_2|\Big)\frac{e^2E}{2\pi} \tag{5.4b}$$

For the Nielsen-Olesen vortex ($n_1 = -n_2$), we recover our previous results (Eqns(3.4-5)). Eqn(5.4a) indicates that in the presence of an applied electric field, charge appears on the string at a rate of $(n_1+n_2)e^2E/2\pi$. As we now demonstrate, the appearance of this charge can be accounted for by an inflow of charge from the world outside the string.

It is convenient to rewrite the Lagrangian for the model in terms of the four-component Dirac spinors $\mathcal{X}$ and $\mathcal{Y}$:

$$\begin{aligned}\mathcal{L} \;=\; & i\bar{\mathcal{X}}\gamma^\mu\left[\partial_\mu \;+\; ieA_\mu \;+\; \left(s+\frac{r}{2}-\frac{r}{2}\gamma_5\right)A'_\mu\right]\mathcal{X} \\ & +\; i\bar{\mathcal{Y}}\gamma^\mu\left[\partial_\mu \;+\; ieA_\mu \;+\; \left(s+\frac{r}{2}+\frac{r}{2}\gamma_5\right)A'_\mu\right]\mathcal{Y} \\ & +\; g_1f_1(r)e^{i\gamma_5 n_1\theta}\bar{\mathcal{X}}\mathcal{X} \;+\; g_2f_2(r)e^{i\gamma_5 n_2\theta}\bar{\mathcal{Y}}\mathcal{Y}\end{aligned} \tag{5.5}$$

where

$$\mathcal{X} \;=\; \begin{pmatrix}\Psi_\alpha \\ \bar{\Gamma}^{\dot\alpha}\end{pmatrix} \qquad \mathcal{Y} \;=\; \begin{pmatrix}\Lambda_\alpha \\ \bar{\Delta}^{\dot\alpha}\end{pmatrix}. \tag{5.6}$$

Here we are using the chiral representation for the γ matrices. Namely

$$\gamma^\mu \;=\; \begin{pmatrix}0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix} \qquad \gamma_5 \;=\; \begin{pmatrix}I & 0 \\ 0 & -I\end{pmatrix}. \tag{5.7}$$

The electric current is

$$J^\mu \;=\; e\left[\bar{\mathcal{X}}\gamma^\mu\mathcal{X} \;+\; \bar{\mathcal{Y}}\gamma^\mu\mathcal{Y}\right] \tag{5.8}$$

and the expectation value of this current far from the core of the string is found to be (see for e.g. ref. 10):

$$\langle J_\mu\rangle \;=\; \frac{(n_1+n_2)e^2}{8\pi^2}\epsilon_{\mu\nu\lambda\kappa}\partial^\nu\theta F^{\lambda\kappa}. \tag{5.9}$$

The calculation of Eqn(5.9) deserves some explanation. We have calculated $\langle J_\mu\rangle$ using a point split procedure. To see how this works, consider a point split 'gauge invariant' definition of the quantity $K^\mu = \bar{\mathcal{X}}\gamma^\mu\mathcal{X}$:

$$\begin{aligned} K^\mu(x+\epsilon/2, x-\epsilon/2) &= \bar{\mathcal{X}}(x+\epsilon/2)\gamma^\mu exp\left\{ie\int_{x-\epsilon/2}^{x+\epsilon/2} A_\nu(y)dy^\nu\right\} \\ &\times exp\left\{i(r+s/2-\gamma_5 s/2)\int_{x-\epsilon/2}^{x+\epsilon/2} A'_\nu(y)dy^\nu\right\}\mathcal{X}(x-\epsilon/2) \end{aligned} \tag{5.10}$$

Here, ϵ is small and we will let $\epsilon^2 \to 0$ (averaging over all directions of ϵ) at the end of the calculation. Naively, K^μ is separately invariant under the two transformations:

$$\mathcal{X} \to \tilde{\mathcal{X}} = e^{ie\lambda}\mathcal{X} \qquad A_\nu \to \tilde{A}_\nu = A_\nu + \partial_\nu\lambda \tag{5.11a}$$

$$\mathcal{X} \to \tilde{\mathcal{X}} = e^{i(r+s/2-\gamma_5 s/2)\lambda}\mathcal{X} \qquad A'_\nu \to \tilde{A}'_\nu = A'_\nu + \partial_\nu\lambda \tag{5.11b}$$

However, K^μ itself cannot be associated with a proper electromagnetic current as $\partial_\mu K^\mu \neq 0$. Contributions to the divergence of K^μ arise through the triangle anomalies that mix $U(1)$ and $U(1)'$ vertices. These anomalies are exactly canceled by the corresponding anomalies for the current associated with $\mathcal{Y}$ so that the full electric current, Eqn(5.8), is conserved. [Our choice of charges in Table IV was chosen so that this would be true!]

To see this in more detail, and to see how one derives Eqn(5.9) we expand K^μ to first order in ϵ and take its vacuum expectation value:

$$\begin{aligned} \langle K^\mu\rangle &= Tr\Big[(1+ie\epsilon^\alpha A_\alpha(x)+i(r+s/2)\epsilon^\alpha A'_\alpha(x))\gamma^\mu G(x-\epsilon/2,\ x+\epsilon/2)\Big] \\ &+ \frac{s}{2}Tr\Big[\epsilon^\alpha A'_\alpha(x)\gamma_5\gamma^\mu G(x-\epsilon/2,\ x+\epsilon/2)\Big] \end{aligned} \tag{5.12}$$

where

$$G(x-\epsilon/2,\ x+\epsilon/2) = \langle 0|T\psi(x-\epsilon/2,\ x+\epsilon/2)|0\rangle \tag{5.13}$$

is the two point function and T denotes time ordering. $G(x,\ y)$ can be written as an expansion in the interaction terms (see Fig. 6). For the $U(1)$ and $U(1)'$ gauge fields, these are $e\gamma^\mu A_\mu$ and $(r+s/2-\gamma_5 s/2)\gamma^\mu A'_\mu$. For the interaction of $\mathcal{X}$ with ϕ_1 we use the adiabatic approximation [22] valid far from the string. We expand $\phi_1(y)$ about the point x

$$\begin{aligned} \phi_1(y) &= \left(f_1(r)e^{i\gamma_5 n_1\theta} - \eta_1\right) + \eta_1 \\ &= \eta_1 + i\eta_1\gamma_5(y-x)^\lambda\partial_\lambda\theta(x) \end{aligned} \tag{5.14}$$

where $\theta(x) = 0$. The interaction of $\mathcal{X}$ with ϕ_1 therefore splits into a mass term and an interaction term.

There are many non-zero contributions to $\langle K^\mu\rangle$. However, most of these are canceled by the corresponding diagrams for the current associated with $\mathcal{Y}$. For example, there are two contributions to $\langle K^\mu \rangle$ that lead to terms proportional to $\epsilon^{\mu\nu\lambda\kappa} A_\nu(x)\partial_\lambda A'_\kappa(x)$, one in the term $Tr[\gamma^\mu G(x-\epsilon/2,\ x+\epsilon/2)]$ coming from diagram e in Fig. 6 and the other in the term $Tr[ie\epsilon^\alpha A_\alpha(x)\gamma^\mu G(x-\epsilon/2,\ x+\epsilon/2)$ coming from diagram c. The coefficient of this term is proportional to $es/2$ while the corresponding contribution from the $\mathcal{Y}$ current has a coefficient proportional to $-es/2$. The uncanceled contributions to $\langle J^\mu \rangle$ come from diagrams involving the scalar fields [e.g., diagram f of Fig. 6] and the calculation which follows closely calculations that appear elsewhere in the literature [10,23], leads directly to Eqn(5.9).

From Eqn(5.9) we see that when a constant electric field is applied in the direction of the string, there is an inward radial current $J_r = -(n_1+n_2)e^2E/4\pi^2 r$ that is precisely what is needed to account for the appearance of charge on the string in Eqn(5.4a).

One can in fact analytically solve Maxwell's equations with a source given by Eqn(5.9) for simple string and field configurations. This has been done for the axion string ($n_1 = 1,\ n_2 = 0$) in the case where there is a charge and current present in the string but no applied fields [10]. One finds that the charge and current on the string are screened by polarization of the vacuum outside the string. The screening is such that the charge (current) measured by an observer a distance R from an infinite straight string is $(R/r_o)^{-e^2/4\pi^2}$ of the total charge (current) on the string. In the above expression r_o is the core radius of the zero mode where $r_o \sim M$ and $M\ (=g\eta)$ is the mass of the fermion of the string. The screening is very weak as $e^2/4\pi^2 \simeq 0.002$. For example, with $M = 10^{16}\ GeV$ (GUT scale zero modes) and $R =$ one light year, only 20% of the charge is screened. To screen 90% of the charge, one would have to go a distance $R \simeq 10^{400}\ cm$ from the string! Presently, we solve Maxwells equations for arbitrary choices of n_1 and n_2. We also calculate (in a very rough way) the screening effects due to a conducting plasma for the case where $(n_1 = 1,\ n_2 = 0)$ and find that screening due to plasma effects is much more dramatic.

We begin by writing Maxwells equation with the source given by Eqn(5.9) in cylindrical coordinates and assuming that the fields depend only on r and t. The relevant equations are

$$\frac{\partial E_r}{\partial t} = \beta\,\frac{E_z}{r} \qquad\qquad \frac{1}{r}\frac{\partial}{\partial r} rE_r = \beta\,\frac{B_\theta}{r}$$
$$-\frac{\partial E_z}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r} rB_\theta = \beta\,\frac{E_z}{r} \tag{5.15}$$

where $\beta = (n_1+n_2)e^2/4\pi^2$. The static solutions to the above equations are [12]

$$E_r = \frac{C_+}{r}\left(\frac{r}{r_o}\right)^{\beta} + \frac{C_-}{r}\left(\frac{r}{r_o}\right)^{-\beta} \tag{5.16a}$$

$$B_\theta = \frac{C_+}{r}\left(\frac{r}{r_o}\right)^{\beta} - \frac{C_-}{r}\left(\frac{r}{r_o}\right)^{-\beta} \tag{5.16b}$$

As before, r_o is the radius of the charge and current on the string. $C_\pm$ are determined by matching these solutions to the solutions at r_o. If ρ_o and J_o are the (fixed) charge and current per unit length on the string, then it follows from Eqn(5.4) that $\rho_o/J_o = -(n_1+n_2)/(|n_1|+|n_2|)$. $E_r(r_o) = \rho_o/2\pi r_o$ and $B_\theta(r_o) = J_o/2\pi r_o$ and we find that

$$C_+ \; = \; -\frac{J_o}{4\pi}\left(\frac{n_1 \; + \; n_2 \; - \; |n_1| \; - \; |n_2|}{|n_1| \; + \; |n_2|}\right) \tag{5.17a}$$

$$C_- \; = \; -\frac{J_o}{4\pi}\left(\frac{n_1 \; + \; n_2 \; + \; |n_1| \; + \; |n_2|}{|n_1| \; + \; |n_2|}\right) \tag{5.17b}$$

For the axion string ($n_1 = 1,\; n_2 = 0$) we recover the result in ref. [10] that $C_+ = 0,\; C_= - J_o/2\pi$. For the superconducting cosmic string ($n_1 = -n_2 = n$), we find that $C_+ = -C_- = J_o/2\pi,\; E_r = 0$ and $B_\theta = J_o n/\pi$ as expected for a neutral current carrying wire.

For $n_1 > 0$ *and* $n_2 < 0$ (or $n_1 < 0$ *and* $n_2 > 0$) we see that both C_+ and C_- are non-zero. This is a rather curious result. From Gauss's law we find that the charge per unit length inside a cylinder of radius R increases as $R^{|\beta|}$ and is therefore infinite for $R \to \infty$. This seemingly unphysical result should not concern us too much. Infinite global (or frustrated) strings have a mass per unit length that diverges logarithmically. Furthermore, the energy per unit length in the electromagnetic field outside an infinite wire is also log-divergent. It is in fact the magnetic field B_θ that gives rise to the polarization charge (cf. Eqns(5.14)). The infinities associated with global strings are cut off by introducing some astrophysical scale R that is either the radius of a loop or half the distance to a neighboring string. We also note that the dependence of charge per unit length on R is exceedingly weak ($\beta = (n_1+n_2)/400$). Suppose that in the present Universe, we have one string per Hubble volume. With $r_o = (10^{16} GeV)^{-1}$ and $n_1 = 2,\; n_2 = -1$, the charge per unit length as measured by an observer 6000 Mpc from the string is actually about 20% of the charge as detected at $R = r_o$.

In the above discussion, we assumed that the strings were in vacuum. However, the environment of a cosmic string is typically some electrically conducting plasma such as the interstellar or intergalactic medium. One therefore expects a charged string to attract oppositely charged particles that in turn screen the electric field of the string. Here we present a simplified discussion of how this effect might be realized. A detailed study of a charged, current carrying string in a conducting plasma is very difficult and is beyond the scope of the present work. In particular, we will regard the string as static though typically the string moves at some fraction of order $O(0.5)$ of the speed of light relative to the plasma. The dynamical problem may actually be very different from the one considered here. [Electrically neutral superconducting strings in astrophysical plasmas were considered in ref. 2 and in ref. 24.]

As an interesting prelude to our discussion of plasma effects, we consider the motion of a test charge outside an infinite, straight, and static string. We consider the case where $n_1 = 1$ and $n_2 = 0$ so that there are only left moving zero modes and $\rho_o = -J_o$. Outside the string, the electromagnetic fields are

$$E_r = -B_\theta = \frac{\rho_o}{2\pi r}\left(\frac{r_o}{r}\right)^\beta \tag{5.18}$$

For definiteness, we take $\rho_o > 0$ so that the electric field points away from the string. The equations of motion for a particle of mass m and charge q in the field of the string are:

$$\begin{aligned} \frac{dp_r}{dt} &= \alpha(r)(1 + v_z) \qquad & \frac{dp_\theta}{dt} &= 0 \\ \frac{dp_z}{dt} &= -\alpha(r)v_r \qquad & \frac{dU}{dt} &= \alpha(r)v_r \end{aligned} \tag{5.19}$$

Here, $\vec{v}$ is the ordinary velocity of the test particle, $(U,\ \vec{p}) = m\gamma(1,\ \vec{v})$, $\gamma = \left(1 - |\vec{v}|^2\right)^{-1/2}$, and $\alpha(r) = qE_r/m$. Qualitatively, a particle's motion is easy to understand. Consider particles that start at rest some distance from the string. Positively charged particles (hereafter called ions) are accelerated away from the string while negatively charged particles (electrons) move towards the string. With a radial velocity, both the electrons and the ions are accelerated in the $-z$ direction (this is just the Hall effect). [Actually, from the last two of Eqns(5.18) we find that $1/\gamma = C(1+v_z)$ where C is an integration constant determined by the initial conditions. For the case where initially $v_r = v_z = v_\theta = 0$, we find that $v_z^2 - v_z = v_x^2/2$.] We have solved Eqns(5.18) numerically and find that the motion of an electron is periodic in the plane perpendicular to the string while its velocity in the z direction is always less than or equal to zero. The trajectory of an electron is shown in Fig. 7.

Recall that in vacuum, a positively charged string is surrounded by a negative vacuum polarization charge. Roughly speaking, we can say that with free charges present, the positive ions neutralize this vacuum charge while the electrons screen the string itself. Also, the motion of the electron along the string screens the current.

We now attempt to determine the screening radius, i.e., the distance at which the electric field of the string is screened. We assume that one has a thermal distribution of electrons at some temperature T and further assume that the thermal energy of the electrons is greater than the electrostatic energy. Our derivation then closely follows the arguments of Debye and Hückel for electrostatic screening. The electrostatic potential Φ is determined by solving the Poisson equation for the string and electrons:

$$\nabla^2\Phi = -\rho_o\delta(\vec{r}) - en_e\left[e^{-e\Phi/T} - 1\right] \tag{5.20}$$

where n_e is the average electron number density. $en_e\left[e^{-e\Phi/T}-1\right]\simeq e^2 n_e\Phi/T$ is the excess charge density in a region where the electrostatic potential is Φ. Both the Laplacian and the delta function in the above equation are two-dimensional owing to the cylindrical symmetry in the problem. [As we shall see, the screening radius is quite small so that a curved string or loop can be considered straight for the present discussion.] The solution to the above equation is $\Phi=\rho_o K_o(\kappa_D r)$ where $\kappa_D=\left(e^2 n_e/T\right)^{1/2}$ and K_o is a modified Bessel function. In the interstellar medium, $n_e=10^{-2}\ cm^{-3}$ and $T=100\ \deg K$ so that $\kappa_D^{-1}=10^3\ cm$. In the intergalactic medium, $n_e=10^{-5}\ cm^{-3}$ and $T=10^5\ \deg K$ so that $\kappa_D^{-1}=10^6\ cm$. [This astrophysical data is found in ref. 25.] Close to the string, where $\kappa_D r\ll 1$, $\Phi\approx\rho_o ln(r)$ [as it should be]. Far from the string, $\Phi\approx\rho_o e^{-\kappa_D r}/(\kappa_D r)^{1/2}$. Therefore κ_D^{-1} sets the scale for the electrostatic screening and is in fact just the Debye screening length. For example, 90% of the charge is screened at a distance of $2.3\kappa_D^{-1}$. Clearly, plasma effects are much more efficient at screening the charge on an anomalous string than the vacuum polarization effects discussed at the beginning of this section.

VI Summary and Conclusions

In this paper, we have focused, for the most part, on two aspects of fermion-vortex systems that may be relevant in determining whether or not superconducting cosmic strings occur in realistic models. First, we have shown that the existence of zero modes is essential for superconductivity as (almost) any effective mass for the bound state fermions inhibits the generation of an electric current. The existence of zero modes can best be determined by directly studying the Dirac equations for the fermions in the field of a vortex though additional or complimentary information can be obtained by constructing an index theorem. However, neither method seems to be directly applicable to the most general fermion-vortex system. In particular, we have not been able to determine whether or not zero modes exist for frustrated strings in the four fermion systems considered in Section II (e.g., the model given by Eqn(2.16) with $n_1=1$ and $n_2=n_3=n_4=0$).

We have also studied some properties of frustrated or anomalous superconducting strings. These strings are superconducting in the sense that in the absence of an external electric field, currents persist. They are anomalous in that the charge per unit length on the string is not conserved; a string that carries current is also charged. In vacuum, the region around such a string has both charge and current due to polarization of the vacuum. In the simple case where there are only left (or only right) movers present, the polarization charge and current are such that an observer very far from the string sees a completely neutral string. However, the screening due to vacuum polarization is extremely weak. A much more dramatic screening takes place when there is a conducting plasma present as would be the case in the interstellar or intergalactic meduim.

To conclude, let us return to our original question: Do fermionic superconducting cosmic strings naturally arise in realistic models? The present analysis indicates that superconductivity can occur in certain realistic models but that the results are very model dependent. Clearly, the most sensible procedure is to begin with a compelling (or at least viable) particle physics theory and determine first whether or not cosmic strings occur and then whether or not they are superconducting. Here, we have studied two alternative scenarios to the original superconducting cosmic string that may occur in realistic models and that therefore should be kept in mind when carrying out the above program.

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16. While this work was in preparation, we learned of similar work concerning the number of zero modes in 'realistic' models by N. Ganoulis and G. Lazarides [University of Thessaloniki Preprint, UT-STPD-2/88]. They derive a general formula for determining the number of zero modes in more complicated models though it appears that their formulas are readily applicable only in special cases similar to the ones considered here.
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21. The field configurations for frustrated cosmic strings may be more complicated than the simple $n_1 = 1$, $n_2 = 0$ vortex described above. Terms proportional to $\phi_1\phi_2$ and

$\phi_1^2\phi_2^2$ are typically present in the scalar potential [either at tree level or as induced terms in some effective potential] and these terms correlate the phases of ϕ_1 and ϕ_2. One may be lead to a situation where domain walls become attached to the strings [though unstable, these walls can be long-lived]. The question of whether or not these structures from is very model dependent and has been discussed in detail in ref [11]. For the present discussion we will ignore these effects with the understanding that this may be an oversimplification. A study of superconductivity in these more complicated structures will be the subject of future investigations.

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Figure Captions

Figure 1: Occupation of energy levels for $t < 0$ (zero electric field). Negative energy levels are filled while positive energy levels are unoccupied.

Figure 2: Occupation of states for $t > \tau$. An electric field has been applied for $0 < t < \tau$. For $t > \tau$, $A(t) = A_o$. Positive energy left moving states are occupied; negative energy right moving states are empty and there is a net current in the $-z$ direction.

Figure 3: Process by which a particle moves off the string. A zero mode particle with enough energy can make a transition to become a massive particle moves off the string.

Figure 4: Energy levels in the case where there are no zero modes but only massive bound states. There is now a mass gap between positive and negative energy levels.

Figure 5: E_6 example of a superconducting cosmic string.

Figure 6: Diagramatic representation of the perturbation series for the two point function $G(x, y) = \langle 0|T\mathcal{X}(x)\bar{\mathcal{X}}(y)|0\rangle$.

Figure 7: Motion of an electron in the field of a charged current carrying (i.e., anomalous) superconducting cosmic string. The electron starts at rest outside the string. The units, which we leave as arbitrary, depend on the initial position of the particle as well as the charge on the string.

Tables

Table 1: Charge assignments for the superconducting string. Q refers to the electromagnetic $U(1)$ charge while Q' refers the $U(1)'$ charge.

Table I		
	Q	Q'
ϕ	0	-2
Ψ	1	1
Γ	-1	1
Λ	1	-1
Δ	-1	-1

Table II: Charge assignments for the E^6 superconducting strings

Table II			
scalars	$U(1)'' \times U(1)' \times T$	fermions	$U(1)'' \times U(1)' \times T$
ϕ_1	(4, 0, 0)	$E,\ E^c$	(-2, 2, 2), (-2, -2,-2)
ϕ_2	(1, -5, 0)	$d,\ d^c$	(1, -1, -4), (1, 3, 2)
ϕ_3	(-2, -2, 2)	$h,\ h^c$	(-2, 2, -2), (-2, -2, 2)
ϕ_4	(-2, 2, -2)	$u,\ u^c$	(1, -1, 1), (1, -1, 1)
ϕ_5	(1, 3, 2)	$e,\ e^c$	(1, 3, 2), (1, -1, -4)

Table III: Winding numbers for the 'minimal' cosmic strings for the E^6 model in Table II. Minimal strings are Nielsen-Olesen strings (i.e., satisfy Eqn()) but have at most three fields that wind.

Table III					
string	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
A	1	0	-1	0	0
B	1	1	0	-1	0
C	1	0	0	-1	1
D	0	1	0	0	-1
E	0	1	1	-1	0

Table IV: Charge assignments for the frustrated string.

Table IV		
	Q	Q'
ϕ_1	0	s
ϕ_2	0	$-s$
Ψ	1	r
Γ	-1	$-r-s$
Λ	1	$r+s$
Δ	-1	$-r$

Fig 1

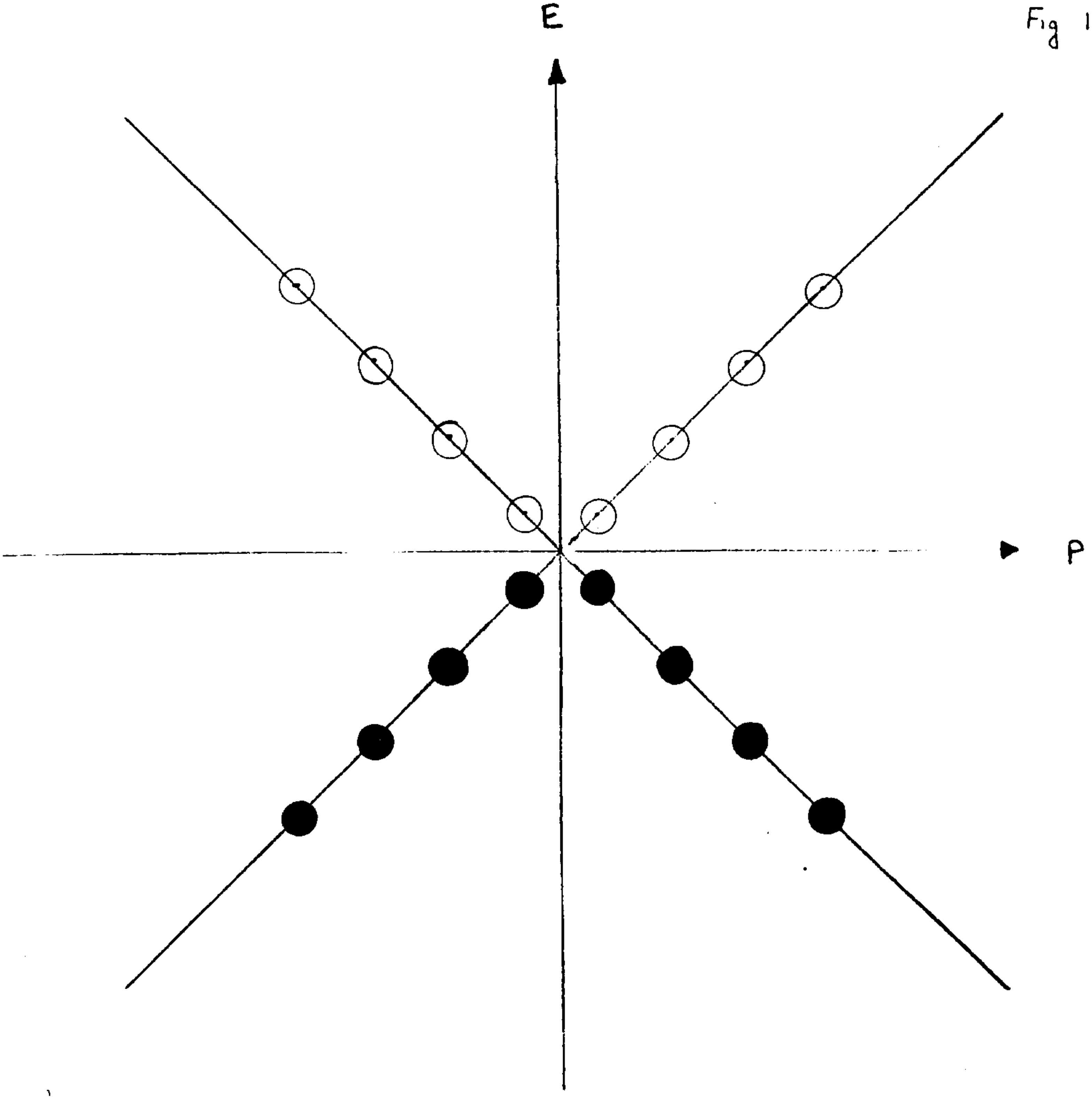


Fig 2

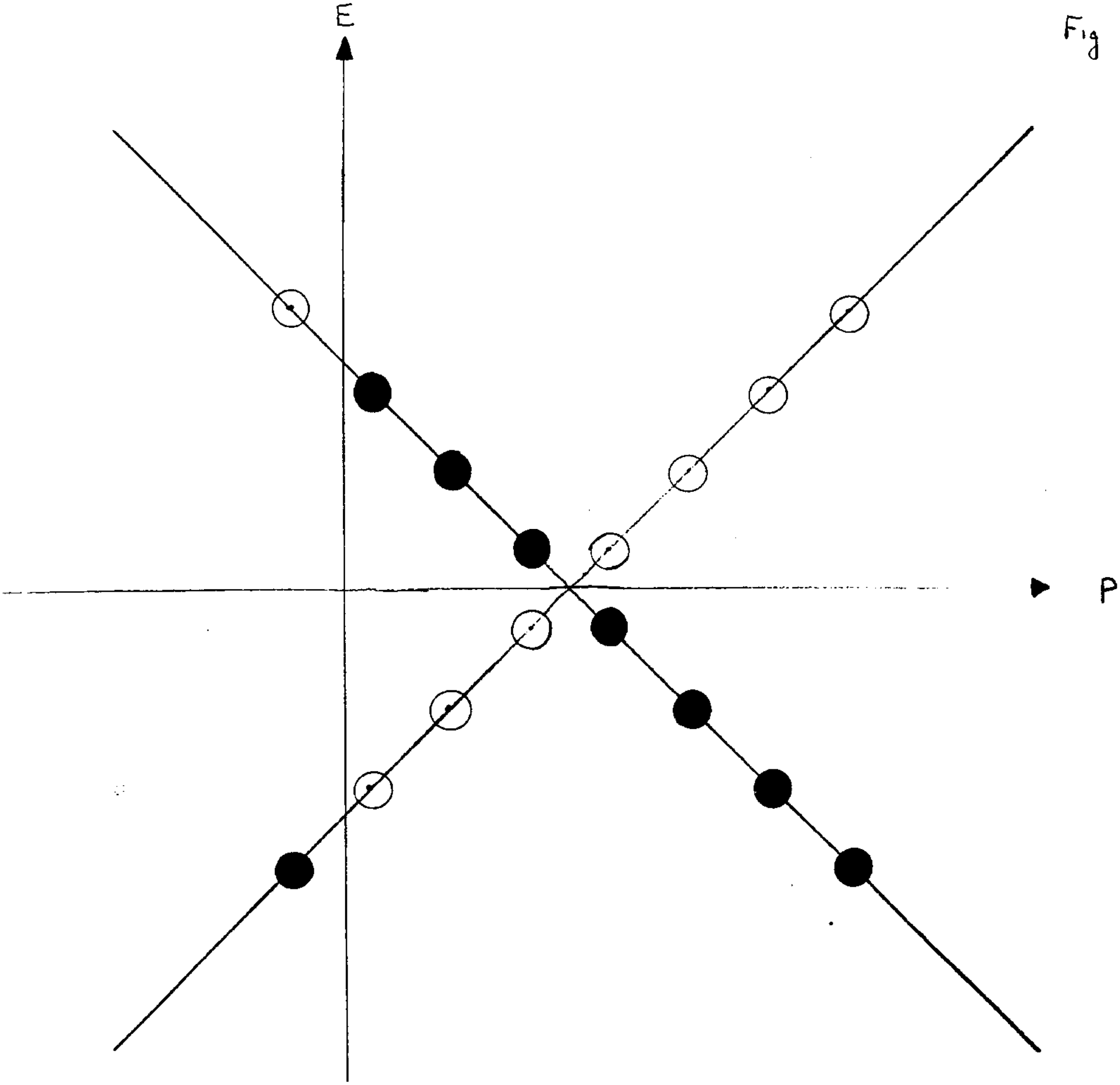
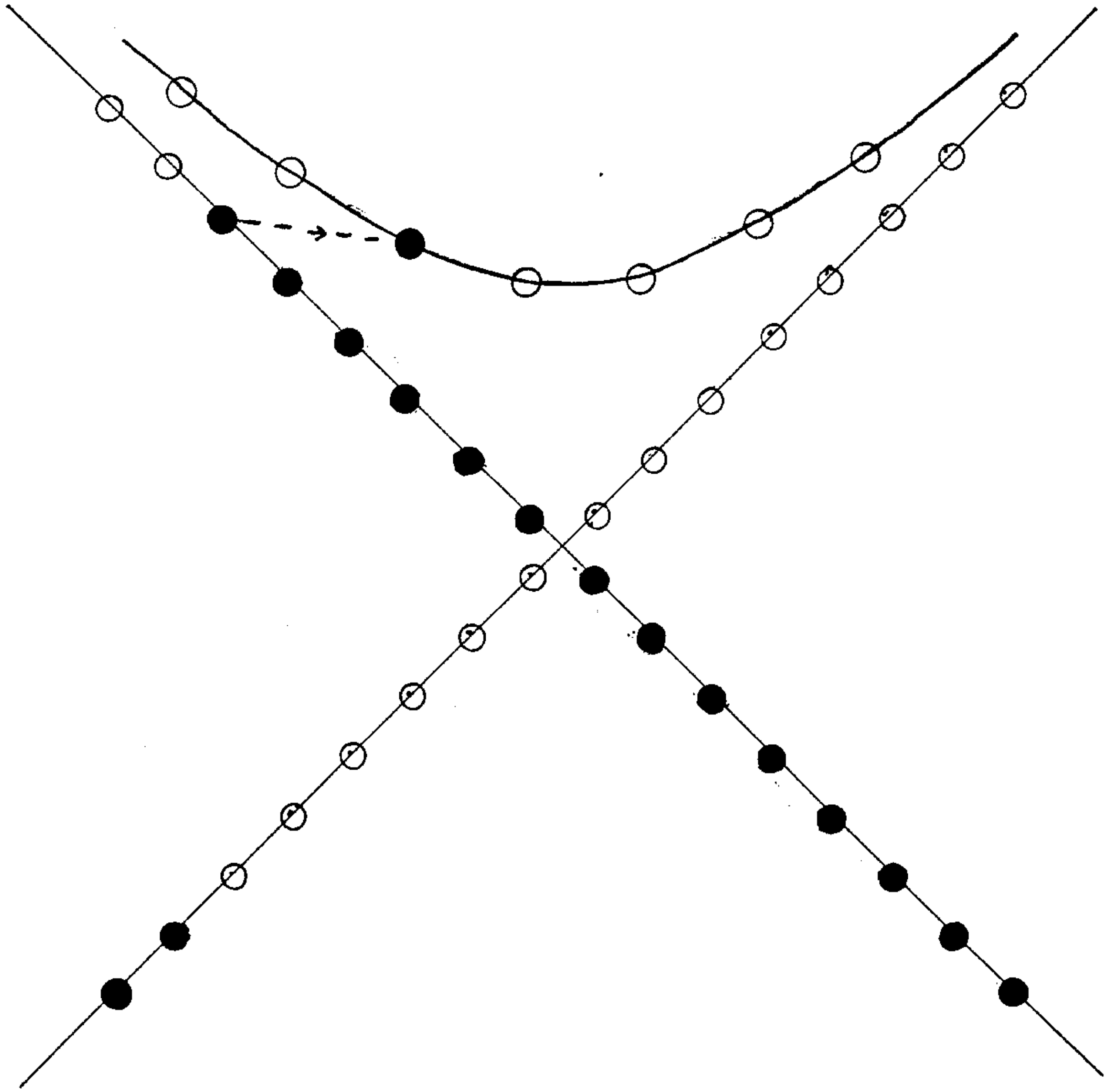


Fig 3



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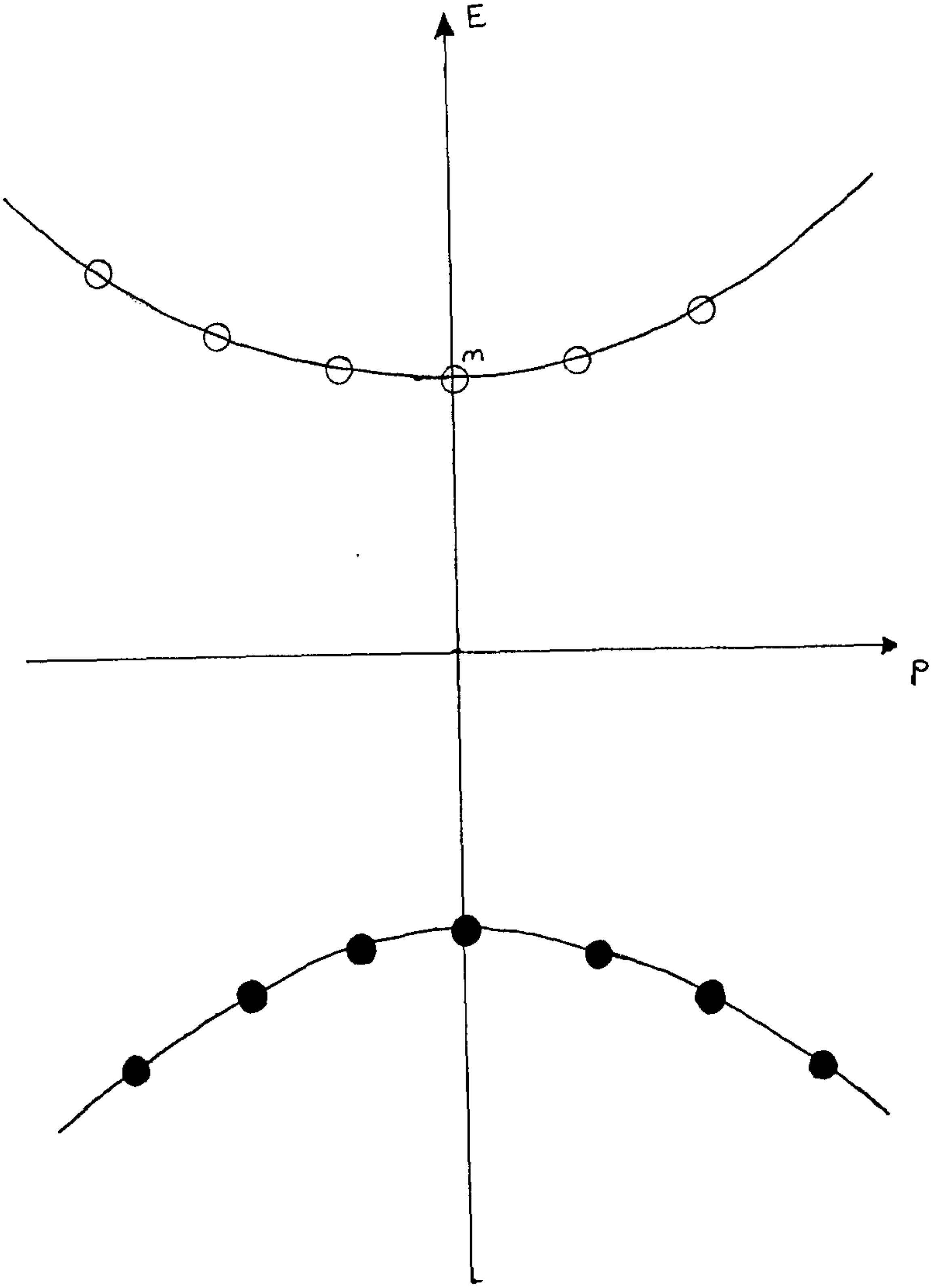

Fig 5

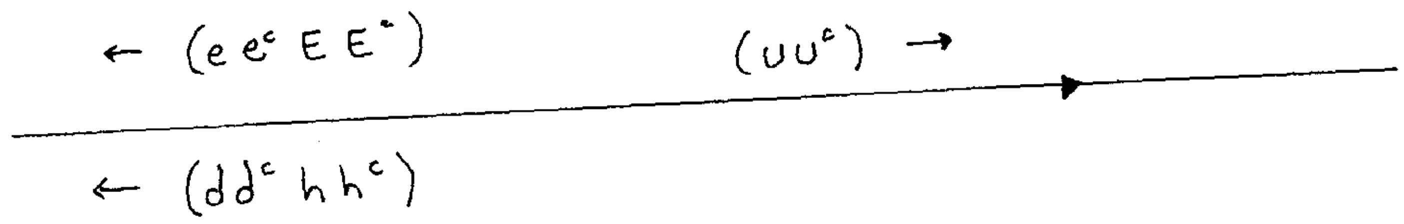

Fig 6

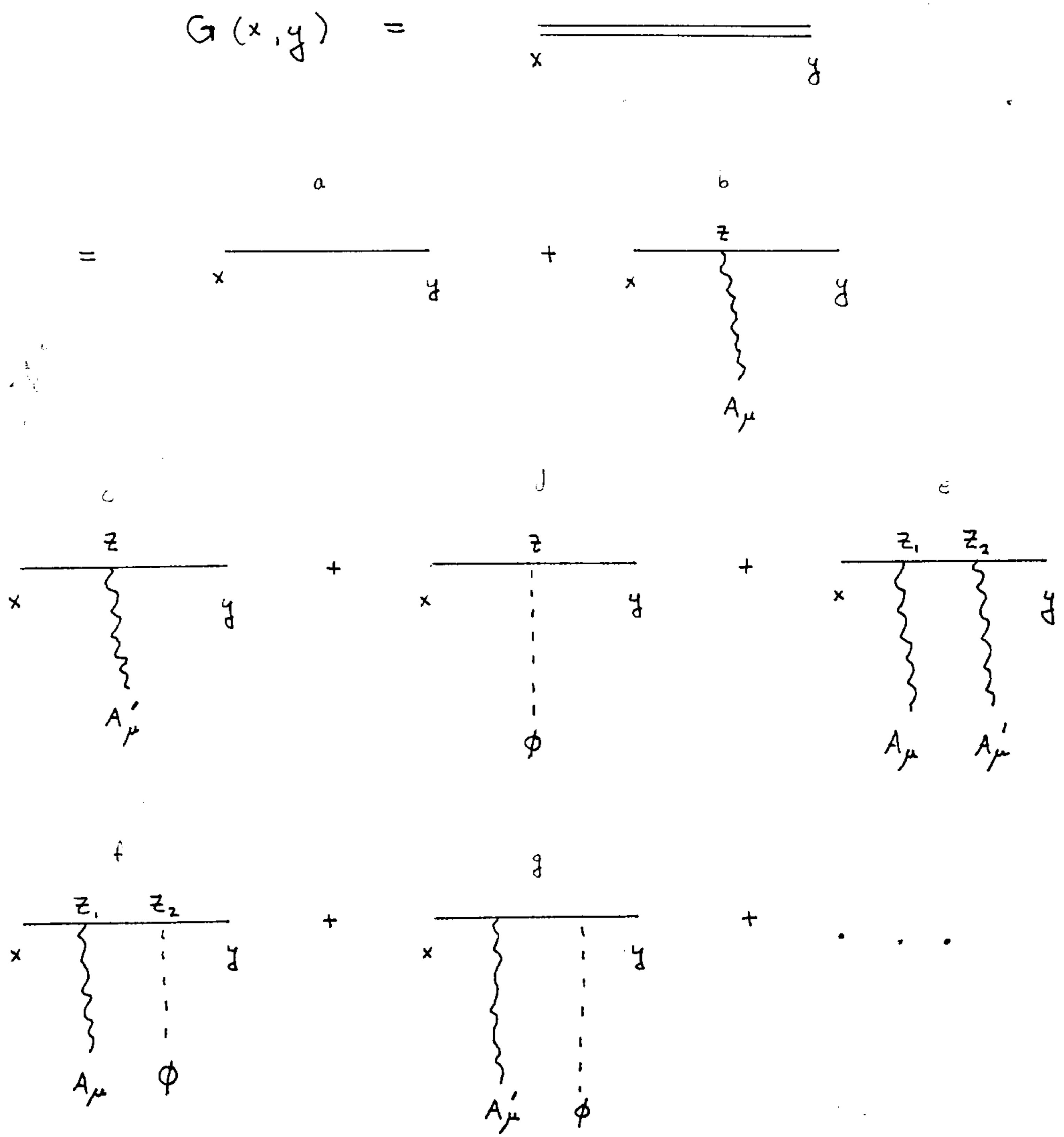

Fig 7