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The Stability Analysis of Magnetohydrodynamic Equilibria: Comparing the Thermodynamic Approach to Bernstein's Energy Principle

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This paper contributes to stability analysis of current-carrying plasmas, i.e. plasma systems which are forced by external mechanisms to carry a nonrelaxable current. Under restriction on translational invariant configurations, the thermodynamic stability criterion for a multicomponent plasma [1,2] is rederived within the framework of nonideal magnetohydrodynamics. The considered dynamics neglects scalar resistivity, but allows for other types of dissipational effects both in Ohm's law and the equation of motion. In the second section of the paper the thermodynamic stability criterion is compared to the ideal MHD based energy principle of Bernstein et al.[3] By the help of Schwarz's inequality it is shown that the former criterion is always more "pessimistic" than the latter, i.e. that thermodynamic stability implies stability according to Bernsteins principle, but not vice versa. This result confirms the physically plausible idea that dissipational effects tend to weaken the stability properties of current-carrying plasma equilibria by breaking the constraints of ideal MHD and allowing for possibly destabilizing effects like magnetic field line reconfiguration.

I. Introduction

Not only in the laboratory, but also in space science and in astrophysics, one deals with plasma systems which are forced by external mechanisms to carry an electrical current and which are therefore prevented from relaxing into total thermodynamic equilibrium. A characteristic property of such systems is their tendency toward activity, i.e., to show an abrupt transition from a quiet state to dynamical behavior. The stability analysis of such systems is therefore particularly interesting; it is considerably facilitated if a stability criterion can be formulated in terms of an appropriate variational principle.

The most familiar example of such a criterion is probably the energy principle given by Bernstein et al.[3], which has found a wide variety of applications to plasma systems both in the laboratory and in space science.[4,5,6] However, as the underlying ideal magnetohydrodynamic theory (ideal MHD) does not include nonideal effects like particle collisions, charge carrier inertia or stochastic field fluctuations, Bernstein's principle is subject to the "frozen-in field line" constraint and therefore insensitive to possibly instable modes involving magnetic field line reconnection.

For this reason, many attempts have been made to formulate alternative approaches to the stability problem. Although so far no ansatz is known which takes into account all of the quoted nonideal phenomena at the same time, considerable progress has been made by considering the influence of the different effects individually. For instance, the role of charge carrier inertia has been studied within the framework of the collisionless Vlasov theory.[7,8] To treat the effects of stochastic field fluctuations (with a spectrum predominantly perpendicular to the current), Kiessling et al.[1] proposed an approach utilizing the principles of statistical mechanics. Recently, the same situation has also been studied within the framework of nonlinear thermodynamics.[2]

Having different approaches to the stability problem available, it is clearly necessary to gain a deeper understanding of their mutual connection. Of particular interest are relations that prove an ordering among the criteria, i.e. show that stability according to one criterion also implies stability to (one or more) other criteria. Such a relation has already been established between the statistical mechanics approach and the Vlasov criterion;[1] it turned out that stability with respect to the former also implies stability according to the latter (but not vice versa).

The subject of this paper is to discuss the relations between the statistical mechanics (resp. thermodynamics) approach and the energy principle by Bernstein et al. However, this comparison is not directly straight forward as Bernstein's principle is formulated within the theory of magnetohydrodynamics, i.e. in a one-fluid approximation involving the assumption of quasineutrality whereas the statistical mechanics criterion was already obtained for a multi-species plasma description using the exact form of Poisson's law.

For this reason, the following analysis is divided into two main sections. In the first one (section II), we present an alternative derivation of the thermodynamic stability criterion within the framework of (nonideal) magnetohydrodynamics, incorporating the above-mentioned assumption on the spectrum of the field fluctuations (being perpendicular to the current) by neglecting the resistivity in Ohm's law. Under the restriction of translationally invariant (i.e. z-independent) motion, we discuss the accessible equilibria of this model and formulate the linear form of the corresponding stability criterion in terms of a variational principle for the z-component of the vector potential perturbation $\delta \underline{A}$. Then we turn to compare this criterion with Bernstein's energy principle (section III). By specializing the latter likewise on translationally invariant configurations we can express it in a form comparable to the thermodynamic criterion, i.e. also as a variational principle in δA_z. Subsequently, we can apply Schwarz's inequality to achieve the desired definite relation between both criteria. In section IV we summarize our results and discuss their physical significance.

II. The MHD-Version of the Thermodynamic Stability Criterion

Magnetohydrodynamic theory can be viewed at as the effective description of a highly ionized plasma in the limit of low frequency ω and small wave numbers $\underline{k}$, obtained by integrating out the phenomena on smaller scales. These microphysical processes, however, give rise to the so called non-ideal effects in the dynamics, i.e. to phenomena like resistivity, viscosity or diffusion. As we have mentioned in the introduction, there are several such processes which, in a first approximation, may be considered independently. In this work we will discuss the influence of anomalous dissipation, i.e. the effects of a stochastic small scale fluctuation field which is usually described in terms of a density-density correlation spectrum $S_{\alpha,\alpha'}$. For simplicity we treat only the case of electrons and simply charged ions, so that $\alpha, \alpha' \in \{e, i\}$:

$$\langle n^{\alpha}(\underline{r},t)\, n^{\alpha'}(\underline{r}',t')\rangle = \frac{1}{8\pi^3}\int\!\!\int_{\omega\,\underline{k}} S^{\alpha\alpha'}(\underline{k},\omega,\frac{\underline{r}+\underline{r}'}{2},\frac{t+t'}{2})\exp(i\underline{k}\cdot(\underline{r}-\underline{r}')-i\omega(t-t'))\,d^3k\,d\omega. \quad (1)$$

We are especially interested in the case where the fluctuations are predominantly perpendicular to the direction of the current density $\underline{j}$. This situation can arise when the underlying microphysical processes are highly anisotropic – like for instance in the earth's magnetotail, where strong field aligned plasma beams are assumed to give rise to microinstabilities with wave vectors predominantly perpendicular to the drift velocity. (See reference 9, 10, 11 for a detailed discussion of this configuration). The anisotropy of the fluctuation field has the important consequence that the momentum exchange $\dot{p}^{(e,i)}$ between the electrons and the ions along the current direction vanishes. This conclusion can easily be drawn from an expression given by Tange and Ichimaru:[12]

$$\dot{\underline{p}}^{\alpha,\alpha'} = \frac{q^{\alpha}q^{\alpha'}}{4\pi\epsilon_0}\frac{1}{8\pi^3}\int\!\!\int_{\omega\,\underline{k}}\frac{4\pi\underline{k}}{k^2}\Im S^{\alpha,\alpha'}(\underline{k},\omega)\,d^3k\,d\omega. \quad (2)$$

In magnetohydrodynamics, we can represent this effect by neglecting the resistivity η in Ohm's law. Other dissipational effects, however, do not vanish, and therefore our considerations lead to the following version of nonideal magnetohydrodynamics:[13]

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\underline{v} = 0 \quad (3)$$

$$\rho\frac{d\underline{v}}{dt} = -\nabla p + \nabla\cdot(\underline{\underline{\pi}}_e + \underline{\underline{\pi}}_i) + \underline{j}\times\underline{B} \quad (4)$$

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho\underline{v}^2 + \frac{3}{2}p) + \nabla\cdot(\frac{1}{2}\rho\underline{v}^2 + \frac{5}{2}p)\underline{v} + \underline{\underline{\pi}}_e\cdot\underline{v}_e + \underline{\underline{\pi}}_i\cdot\underline{v}_i + \underline{q}_e + \underline{q}_i = \underline{j}\cdot\underline{E} \quad (5)$$

$$\underline{E} + \underline{v}\times\underline{B} = \frac{m_e m_i}{\rho e}(\frac{1}{m_e}(\nabla p_e + \nabla\cdot\underline{\underline{\pi}}_e) - \frac{1}{m_i}(\nabla p_i + \nabla\cdot\underline{\underline{\pi}}_i)) + \frac{m_i - m_e}{\rho e}\underline{j}\times\underline{B} \quad (6)$$

$$\frac{\partial\underline{B}}{\partial t} = -\nabla\times\underline{E} \quad (7)$$

$$\frac{1}{\mu_0}\nabla\times\underline{B} = \underline{j} \tag{8}$$

$$\nabla\cdot\underline{B} = 0. \tag{9}$$

This set of equations can be viewed at as a closed system for the dynamical variables ρ, $\underline{v}$, p and $\underline{B}$ (denoting mass density, bulk velocity, scalar pressure and magnetic field, respectively). The current density $\underline{j}$ and the electric field $\underline{E}$ can be eliminated directly with the help of Ohm's and Ampere's laws (eq. (6) and (8)); the dissipative quantities $\underline{q}$ and $\underline{\underline{\pi}}$, i.e. the heat flux and the nondiagonal part of the pressure tensor, can – at least in principle – be obtained as functions of the dynamical variables. (Explicit expressions, of course, would require the determination of the fluctuation spectrum S.)

For the rest of this paper we restrict ourselves on translationally invariant systems, i.e. on configurations which are independent of the z-axis of a suitable chosen coordinate system. This assumption, although it does not impose the most general symmetry we could consider, simplifies the following calculations considerably. The treatment of more general configurations is essentially analogous; for an extensive discussion of this point see reference 2.

In order to derive definite statements on the stability properties of the considered plasma dynamics, we have to fix appropriate boundary conditions (any equilibrium can be destroyed by uncontrolled external influences). For simplicity, we assume the plasma to be confined in a finite cylinder volume $V = F \times [0, L]$ oriented parallel to the invariant direction, i.e. the z-axis of the coordinate system chosen above and demand

i) Vanishing bulk velocity at the cylinder mantle M:

$$\underline{v}\,|_M = \frac{m_e\underline{v}_e + m_i\underline{v}_i}{m_e + m_i}\,|_M = 0 \tag{10}$$

ii) Fixed the drift velocity of the particles at the mantle:

$$\underline{v}_i - \underline{v}_e\,|_M = \frac{\underline{j}}{\frac{1}{2}ne}\,|_M = w\underline{e}_z \tag{11}$$

iii) Fixed (electron and ion) temperature $\bar{T}$ at the mantle ($T \equiv p/nk$):

$$T_i\,|_M = T_e\,|_M = \bar{T} \tag{12),}$$

iv) "Ideal-mirror" boundary conditions for the electromagnetic quantities (which can be achieved by assuming the cylinder mantle to have infinite electrical conductivity):

$$\underline{E}\times \underline{d\Omega}\,\Big|_M = 0 \tag{13}$$

$$\frac{\partial \underline{B}}{\partial t}\cdot \underline{d\Omega}\,\Big|_M = 0 \tag{14}$$

Although the postulated boundary conditions clearly correspond to a highly idealized situation, they seem to be reasonable in important physical situations. For instance, in the case of the earth's magnetotail the mantle M may be represented by the magnetopause.[9]

It turns out to be appropriate for our purpose to express the magnetic field $\underline{B}$ by the vector potential $\underline{A}$ via $\underline{B} = \nabla \times \underline{A}$ and $\nabla \cdot \underline{A} = 0$ (Coulomb gauge). This substitution replaces the induction equation (7) by

$$\frac{\partial \underline{A}}{\partial t} = -\underline{E} + \nabla \int_F G(\underline{r}, \underline{r}') \nabla \cdot \underline{E}(\underline{r}') d^3 r'. \tag{15}$$

Here, $G(\underline{r}, \underline{r}')$ denotes the Green's function for the two-dimensional Laplace operator on the cross-section F, i.e. the solution to the problem

$$\nabla^2 G(\underline{r}, \underline{r}') = \delta(\underline{r} - \underline{r}'), \quad G(\underline{r}, \underline{r}') = 0 \ \ for \ \ \underline{r}' \in \partial F. \tag{16}$$

An appropriate boundary condition for the vector potential $\underline{A}$ compatible with equation (15) can be easily derived from (13) and (14) as

$$\frac{\partial \underline{A}}{\partial t} \times \underline{d\Omega} \Big|_M = 0. \tag{17}$$

It is convenient for our following considerations to introduce a state space Z as the set of all configurations accessible to the considered plasma under the restrictions of the imposed symmetry and boundary conditions. Taking into account that the total particle number N in the volume is conserved (which follows from (3) and (10)), we can define Z as

$$Z = \left\{ (n, \underline{v}, p, \underline{A}) \Big| \ \ \tfrac{\partial}{\partial z} = 0 \wedge \textstyle\int_V n d^3 r = N \wedge \text{boundary conditions (10 - 44) and (17)} \right\}. \tag{18}$$

Note that Z indeed is a closed space in the sense, that if the plasma system belongs to it at a time t_0, it remains therein for all $t > t_0$.

After formulating the necessary prerequisites, we can now proceed to derive the thermodynamic stability criterion. The basic idea is to utilize the second law of thermodynamics, according to which the net production rate of the entropy, i.e. the sum of the temporal change and the flux through the boundary, is a non-negative quantity. With respect to the volume V, the second law reads (k being Boltzmann's constant and C an arbitrary normalization):[14]

$$\frac{d}{dt} \int_V kn \ln C p^{\frac{3}{2}} n^{-\frac{5}{2}} d^3 r + \frac{1}{T} \int_M (\underline{q}_e + \underline{q}_i) \cdot \underline{d\Omega} \ \geq 0 \, . \tag{19}$$

Our ansatz is to use the inequality of the second law (19) to construct a candidate for a Lyapunov functional on the state space Z, i.e. a functional $I\!F\{z\}$ with $\frac{d}{dt} I\!F \leq 0$ for all $z \in Z$, by transforming the remaining surface integral into a total time derivative as well. This can be done by using the balance equations of momentum, canonical momentum and energy, which we obtain by integrating the equation of motion (4), the induction law (15) combined with Ohm's law (6), and the energy equation (5) over the total volume V. Note that we have used for (21) and (22) the translational invariance of the configuration, and for (21) also the assumption of vanishing resistivity:

$$\frac{d}{dt}\int_V \frac{1}{2}(m_e+m_i)nv_z d^3r + \int_{\partial V} \underline{e}_z \cdot (\underline{\underline{\pi}}_i + \underline{\underline{\pi}}_e)\cdot \underline{d\Omega} = 0 \tag{20}$$

$$\frac{d}{dt}\int_V \frac{e}{m_e m_i}\rho A_z d^3r + \int_M \underline{e}_z \cdot (\frac{1}{m_i}\underline{\underline{\pi}}_i - \frac{1}{m_e}\underline{\underline{\pi}}_e)\cdot \underline{d\Omega} = 0 \tag{21}$$

$$\frac{d}{dt}\int_V \frac{3}{2}p + \frac{1}{2}\rho \underline{v}^2 + \frac{1}{2\mu_0}\underline{B}^2 d^3r + \int_{\partial V} (\underline{\underline{\pi}}_e \cdot \underline{v}_e + \underline{\underline{\pi}}_i \cdot \underline{v}_i + \underline{q}_e + \underline{q}_i)\cdot \underline{d\Omega} = 0. \tag{22}$$

We can add these three equations to the second law such that all surface integrals cancel. (Actually, the balance of momentum (20) is not needed due to the boundary condition (10); however, it would be necessary in order to treat the slightly more general case $\underline{v}|_M = \bar{v}\underline{e}_z \neq 0$.) The resulting equation defines a functional $I\!F$ of the state space Z, called the (generalized) free energy

$$I\!F\{n,\underline{v},p,\underline{A}\} = \int \frac{1}{2\mu_0}(\nabla\times\underline{A})^2 + \frac{1}{2}\frac{m_e+m_i}{2}n\underline{v}^2 + \frac{3}{2}p - \bar{T}k\ln(Cp^{\frac{3}{2}}n^{-\frac{5}{2}})n - \frac{1}{2}wenA_z d^3r \tag{23}$$

which has the desired property, i.e. a definite sign of the total time derivative:

$$\frac{d}{dt}I\!F\{n,\underline{v},p,\underline{A}\} \leq 0. \tag{24}$$

Because of its monotonical decrease in time, the generalized free energy $I\!F$ already has one of the properties of a Lyapunov functional. The remaining property of positive definiteness therefore provides us with a sufficient stability criterion: A stationary configuration z_0 of the discussed plasma system is locally stable against translational invariant perturbations if the functional $I\!F$ has a local minimum at this point. (Of course, the set of suitable test functions is given by the state space Z as defined in (18); therefore, the particle number fixation has to be taken into account as a constraint.)

As we have the fully non-linear form of the functional $I\!F$ available, it may serve as a starting point for a non-linear stability analysis of the system, e.g., by the method of bifurcation analysis. First steps in that direction have already been taken; [15,16] however, for the rest of this paper we restrict ourselves to the problem of local stability, expanding the free energy in a series according to

$$I\!F\{z_o+\delta z\} = I\!F + \delta I\!F + \delta^{(2)} I\!F + \cdots . \tag{25}$$

As usual, the condition of the first variation to vanish determines the structure of the equilibria. Taking particle conservation into account with the help of a Lagrangian multiplier Λ, the Euler Lagrange equations for $I\!F$ read

$$-\frac{1}{\mu_0}\nabla\times\nabla\times\underline{A}-\frac{1}{2}wen\underline{e}_z=0 \tag{26}$$

$$\frac{1}{2}\frac{m_e+m_i}{2}\underline{v}^2-\bar{T}k\ln(Cp^{\frac{3}{2}})n^{-\frac{5}{2}}+\frac{5}{2}k\bar{T}-\frac{1}{2}weA_z+\Lambda=0 \tag{27}$$

$$\frac{1}{2}(m_e+m_i)\underline{v}=0 \tag{28}$$

$$\frac{3}{2}-\frac{3}{2}\frac{nk\bar{T}}{p}=0. \tag{29}$$

From these equation follows $\underline{v}=0$, $p=nk\bar{T}$ and $B_z\equiv\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}=const.(:=0$ for simplicity); the rest of the information can be easily reduced to one remaining equation of Grad-Shafranov type for the z-component of the vector potential $\underline{A}\equiv A\underline{e}_z$

$$-\nabla^2A(x,y)=\frac{\partial p}{\partial A}, \tag{30}$$

where the scalar pressure p is given in terms of A via the following expression:

$$p(A)=NkT\frac{\exp(\frac{\frac{1}{2}weA}{kT})}{\int_V\exp(\frac{\frac{1}{2}weA}{kT})d^3r}. \tag{31}$$

Note that p does not only depend on the local value of A but also on an integral over the whole volume (insuring invariance against the remaining gauge transformation $A\rightarrow A+const.$). However, the partial derivative in (30) is meant to affect only the local dependence, i.e.:

$$\frac{\partial p}{\partial A}=\frac{1}{2}weN\frac{\exp(\frac{\frac{1}{2}weA}{kT})}{\int_V\exp(\frac{\frac{1}{2}weA}{kT})d^3r}. \tag{32}$$

As is well-known, the equilibria given by (30) and (31) are by no means trivial. In particular, they cover a wide class of weakly two-dimensional solutions describing stretched configurations such as the earth's magnetotail.[17,18,19]

To give a stability criterion for the equilibria defined by (30) and (31), we proceed considering the next order of the free energy, i.e. the second variation $\delta^2 I\!F$. It has the form

$$\delta^{(2)}I\!F\{\delta\underline{A},\delta n,\delta\underline{v},\delta p\}=\int_V\frac{1}{2\mu_0}(\nabla\times\delta\underline{A})^2d^3r+\frac{1}{2}(m_e+m_i)\frac{n}{2}(\delta\underline{v})^2$$
$$+\frac{3}{4}p(\frac{\delta p}{p})^2-\frac{3}{2}p\frac{\delta n}{n}\frac{\delta p}{p}+\frac{5}{4}p(\frac{\delta n}{n})^2-\frac{1}{2}we\delta n\delta A_zd^3r. \tag{33}$$

The equilibria discussed above are linear stable if this functional is positive definite under the constraint of constant particle number, i.e. under

$$\int_V \delta n \, d^3r = 0. \tag{34}$$

It is obvious that minimizing perturbations have $\delta \underline{v} = 0$, $\delta p = k\bar{T}\delta n$ and $\delta B_z = \nabla \times \delta \underline{A} \cdot \underline{e}_z = 0$. Minimizing for δn under the constraint (34) gives:

$$\delta n = \frac{1}{2}\frac{we}{k\bar{T}} n \, (\delta A_z - \frac{1}{N}\int_V n \delta A_z d^3r). \tag{35}$$

Inserting these results into expression (33) and using the form of pressure function (31), we can finally formulate the thermodynamic stability criterion in terms of a variational principle for the perturbation of the vector potential δA_z:

An equilibrium given by (30) and (31) is stable against sufficiently small translationally invariant perturbations in the frame of nonresistive MHD, if the following functional is positive for all nonvanishing test functions $\delta A_z(x, y)$ with $\delta A_z(\underline{r}) = 0$ for $\underline{r} \in \partial V$:

$$\delta I\!F\{\delta A_z\} = \int_V \frac{1}{2\mu_0}(\nabla \delta A_z)^2 - \frac{1}{2}\frac{\partial^2 p}{\partial A^2}(\delta A_z)^2 d^3r + \frac{1}{2}\frac{(\int \frac{\partial p}{\partial A}\delta A_z d^3r)^2}{\int_V p d^3r}. \tag{36}$$

This is exactly the stability criterion which has already been formulated by ref. 1 and 2 for multi-fluid case (specialized on the case of two particle species and assumed quasi-neutrality).

III. Comparison to Bernsteins Energy Principle

We now proceed with the second part of our program, the comparison of the thermodynamic stability criterion to Bernstein's energy principle for ideal magnetohydrodynamics. For that purpose, we first sketch the derivation of Bernstein's principle and bring it into a form which is appropriate for our argumentation; i.e., we formulate it like the thermodynamic criterion in terms of a variational principle for the z-component of the vector potential.

The starting point for the derivation of Bernstein's energy principle is given by the equations of ideal MHD, which can be obtained from the nonideal description (eq. 3 - 9) by neglecting all dissipative effects in the equation of motion, the energy balance and in Ohm's law[15]:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{v} = 0 \tag{37}$$

$$\rho \frac{d\underline{v}}{dt} = -\nabla p + \frac{1}{\mu_0}(\nabla \times \underline{B}) \times \underline{B} \tag{38}$$

$$\frac{d}{dt} p \rho^{-\gamma} = 0 \tag{39}$$

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B}). \tag{40}$$

The adiabatic index γ in equation can be adjusted to cover two different assumptions on the heat flux $\underline{q}$ resp. the temperature gradient ∇T: the choice $\gamma = 5/3$ corresponds to the adiabatic case $\underline{q} = 0$, $\gamma = 1$ represents the isothermal case $\nabla T = 0$. Formally, "polytropic" behavior (values of γ between 1 and 5/3) is also allowed.

We think of the ideal plasma to be confined in the same cylinder volume as the nonideal plasma we have treated in the previous chapter. This assumption corresponds to the following set of boundary conditions:

i) Vanishing bulk velocity at the cylinder mantle M:

$$\underline{v}\,|_M = 0. \tag{41}$$

ii) "Ideal-mirror" boundary conditions for the electromagnetic quantities:

$$\frac{\partial \underline{B}}{\partial t} \cdot \underline{d\Omega}\,\Big|_M = 0. \tag{42}$$

As it is well known (e.g. ref. 20), static ($\underline{v} = 0$) and translationally invariant ($\frac{\partial}{\partial z} = 0$) equilibria of the ideal MHD equations can be obtained by choosing an arbitrary pressure function $p = p(A)$, i.e. by specifying the pressure as a function of the z-component of the vector potential $\underline{A} = A\underline{e}_z$. This ansatz reduces the system (37 - 40) to a single partial differential equation for A, the Grad-Shafranov equation, the same which already has appeared in the previous section:

$$\frac{1}{\mu_0}\nabla^2 A = -\frac{\partial p}{\partial A}. \tag{43}$$

However, in contrast with the nonresistive case, the ideal MHD equations give no information on the form of the pressure function $p(A)$. In order to compare both theories, we will later specialize $p(A)$ to form (31) also in the nonideal case, but for the moment let us only assume that the pressure function is positive and invertible, i.e. that $p(A) \geq 0$ for all A and that from $p(A_1) = p(A_2)$ follows $A_1 = A_2$.

Linearizing the ideal MHD equations around an equilibrium given by (43) and introducing the Lagrangian coordinate $\xi = \int_0^t \underline{v}\, dt$ (the infinitesimal displacement of a fluid element), one obtains the equation of motion for an infinitesimal perturbation:

$$\rho\frac{\partial^2 \underline{\xi}}{\partial t^2} = \nabla(\gamma p \nabla\cdot\underline{\xi} + \underline{\xi}\cdot\nabla p) + \frac{1}{\mu_0}(\nabla\times\underline{B})\times(\nabla\times(\underline{\xi}\times\underline{B}) + \frac{1}{\mu_0}\nabla\times(\nabla\times(\underline{\xi}\times\underline{B}))\times\underline{B} \equiv F\underline{\xi}. \tag{44}$$

By multiplying by $\frac{\partial \xi}{\partial t}$, integrating by parts and using that F is Hermitian, one derives a conservation law for the energy of the perturbation in the form:

$$\frac{d}{dt}\int_V \frac{1}{2}\rho\dot{\underline{\xi}}^2 - \frac{1}{2}\underline{\xi}\cdot F\underline{\xi}\, d^3r \equiv \int_V \frac{1}{2}\rho\dot{\underline{\xi}}^2\, d^3r + \delta^2 W = 0. \tag{45}$$

This equation leads directly to Bernstein's energy principle:[3] a MHD-equilibrium is stable against small perturbations $\underline{\xi} = \underline{\xi}(x, y, z)$, when the following functional is positive definite:

$$\delta^2 W = -\frac{1}{2}\int_V \underline{\xi}\cdot(\nabla(\xi\cdot\nabla p + \gamma p\nabla\cdot\xi) + \frac{1}{\mu_0}(\nabla\times\underline{B})\times\nabla\times(\xi\times\underline{B}) + \frac{1}{\mu_0}\nabla\times(\nabla\times(\underline{\xi}\times\underline{B}))\times\underline{B})\, d^3r. \tag{46}$$

As in the previous chapter, we now concentrate on the case of translationally invariant perturbations ($\frac{\partial \xi}{\partial z} = 0$). This assumptions simplifies the expression for $\delta^2 W$ to:

$$\delta^2 W = \int_V \frac{1}{2\mu_0}(\nabla(\underline{\xi}\cdot\nabla A)) + \frac{1}{2\mu_0}(\nabla\times(\xi_z\nabla A))^2 + \frac{1}{2}\gamma(\nabla\cdot\underline{\xi})^2 - \frac{1}{2}\frac{\partial^2 p}{\partial A^2}(\underline{\xi}\cdot\nabla A)^2 d^3r. \tag{47}$$

Evidently, minimizing perturbations have no ξ_z component, so that the variational principle can be evaluated with the following form of δW, subject to variations $\underline{\xi} = (\xi_x, \xi_y)$:

$$\delta^2 W = \int_V \frac{1}{2\mu_0}(\nabla(\underline{\xi}\cdot\nabla A))^2 + \frac{1}{2}\gamma(\nabla\cdot\underline{\xi})^2 - \frac{1}{2}\frac{\partial^2 p}{\partial A^2}(\underline{\xi}\cdot\nabla A)^2 d^3r. \tag{48}$$

To compare Bernsteins principle to the thermodynamic stability criterion, we have to reduce this functional further so that it can be formulated in terms of the z-component of the vector potential, δA_z: we observe that in all terms except the second one the displacement $\underline{\xi}$ only occurs in the form $\underline{\xi} \cdot \nabla A$. An inspection of the ideal MHD equations shows that this quantity can be identified with $-\delta A_z$:

$$\xi \cdot \nabla A = -\delta A_z. \tag{49}$$

Therefore, we try to minimize the functional (or equivalently the term $\gamma \int_V (\nabla \cdot \xi)^2 d^3r$) with respect to the component orthogonal to the gradient of A. We make the following ansatz, where ψ is an abbreviation for $-\delta A_z/(\nabla A)^2$:

$$\underline{\xi} = \psi \nabla A + \chi \nabla A \times \underline{e}_z. \tag{50}$$

It is easy to see that this variational problem has a solution: the Euler-Lagrange equations with respect to the orthogonal component χ are

$$\nabla(\gamma p \nabla \cdot (\psi \nabla A + \chi \nabla A \times \underline{e}_z)) \times \nabla A = 0. \tag{51}$$

As the equilibrium pressure is a function of A, i.e. $p = p(A)$, it follows that

$$\nabla \nabla \cdot (\psi \nabla A + \chi \nabla A \times \underline{e}_z)) \times \nabla A = 0, \tag{52}$$

which shows that the term subject to the gradient is a function $F(A)$ of the vector potential:

$$\nabla \cdot (\psi \nabla A + \chi \nabla A \times \underline{e}_z) = F(A). \tag{53}$$

Our next step is to determine the function F in terms of ψ (or, equivalently, in terms of δA_z). We can write the last equation in form of a partial differential equation for χ:

$$\nabla \cdot (\chi \nabla A \times \underline{e}_z) \equiv \frac{\partial \chi}{\partial x}\frac{\partial A}{\partial y} - \frac{\partial \chi}{\partial y}\frac{\partial A}{\partial x} = F(A) - \nabla \cdot (\psi \nabla A). \tag{54}$$

A necessary and sufficient condition (Fredholm condition[21]) for this linear equation to have a solution is that the right-hand-side has to be orthogonal to the kernel of the operator on the left hand side, which is given by the set of all functions g of the potential A:

$$\int_V g(A)\,(F(p) - \nabla \cdot (\psi \nabla A))d^3r = 0 \; for \; all \; g = g(A). \tag{55}$$

This condition determines $F(A)$ uniquely:

$$F(A) = \frac{\int_{V'} \delta(A - A')\nabla'(\psi(\underline{r}')\nabla' A')d^3r'}{\int_{V'} \delta(A - A')d^3r'}. \tag{56}$$

Using our assumption that the pressure p is given by an invertible function of the vector potential A, we can equivalently write:

$$F(p) = \frac{\int_{V'} \delta(p - p')\nabla'(\psi(\underline{r}')\nabla' A')d^3r'}{\int_{V'} \delta(p - p')d^3r'} \tag{57}$$

or, after partial integration and re-inserting of $\psi = -\frac{\delta A_z}{(\nabla A)^2}$:

$$F(p) = -\frac{\frac{d}{dp}\int_{V'}\delta(p-p')\delta A_z(\underline{r}')d^3r'}{\int_{V'}\delta(p-p')d^3r'}. \tag{58}$$

With this expression for F, the partial differential equation (54) obeys the Fredholm condition and is therefore solvable. However, we do not have to construct a solution explicitly as we can obtain the necessary information in a different way: the function F is just another expression for the divergence of ξ, evaluated for the (unconstructed) minimizing χ:

$$\nabla\cdot\xi = \nabla\cdot(\psi\nabla A + \chi\nabla A \times \underline{e}_z) = F. \tag{59}$$

Substituting this result into the functional of equation (48), we can finally state the following form of Bernstein's principle:

A translational invariant equilibrium of the ideal MHD equations is stable against symmetry preserving perturbations, when the following functional is positive definite:

$$\delta^2 W = \int_V \frac{1}{2\mu_0}(\nabla\delta A_z)^2 - \frac{1}{2}\frac{\partial^2 p}{\partial A^2}(\delta A_z)^2 d^3r + \int_V \frac{1}{2}\gamma p\left(\frac{\frac{d}{dp}\int_{V'}\delta(p-p')\frac{\partial p'}{\partial A'}\delta A_z' d^3r'}{\int_{V'}\delta(p-p')d^3r'}\right)^2 d^3r. \tag{60}$$

From now on we restrict our discussion on those configurations which are equilibria of both ideal and nonideal MHD, i.e. on solutions of the Grad-Shafranov equation (30 resp. 43) with the choice (31) of the pressure function. Comparing the functional $\delta^2 W$ to the second variation of the free energy $\delta^2 I\!F$ which appears in the thermodynamic stability criterion (eq. 36), we see that the first terms of both functional are identical. However, there is a difference in the third term, which can be written

$$\delta^2 W\{\delta A_z\} - \delta^2 I\!F\{\delta A_z\} = \int_V \frac{1}{2}\gamma p\left(\frac{\frac{d}{dp}\int_{V'}\delta(p-p')\frac{\partial p'}{\partial A'}\delta A_z' d^3r'}{\int_{V'}\delta(p-p')d^3r'}\right)^2 d^3r - \frac{1}{2}\frac{(\int_V \frac{\partial p}{\partial A}\delta A_z d^3r)^2}{\int_V p d^3r}. \tag{61}$$

By introducing the following functions of the pressure p (which runs from 0 to ∞)

$$\alpha(p) = \sqrt{\frac{p}{\int_{V'}\delta(p-p')\,d^3r'}}\,\frac{d}{dp}\left(\frac{\partial p}{\partial A}\int_{V'}\delta(p-p')\delta A_z d^3r'\right) \tag{62}$$

$$\beta(p) = \sqrt{p\int_{V'}\delta(p-p')\,d^3r'}, \tag{63}$$

we can formulate the difference between $\delta^2 W$ and $\delta^2 F$ in a clearer form:

$$\begin{aligned}\delta^2 W\{\delta A_z\} - \delta^2 \mathbb{F}\{\delta A_z\} &= \frac{1}{2}\gamma \int_0^\infty \alpha^2 dp - \frac{1}{2}\frac{(\int_0^\infty \alpha\beta dp)^2}{\int_0^\infty \beta^2 dp} \\ &= \frac{1}{2}(\gamma - 1)\int_0^\infty \alpha^2 dp + \frac{1}{2}(\int_0^\infty \alpha^2 dp - \frac{(\int_0^\infty \alpha\beta dp)^2}{\int_0^\infty \beta^2 dp}).\end{aligned} \tag{64}$$

The first term of the last expression is positive because the adiabatic index γ is always greater or equal than unity; the second term is positive as well due to Schwarz's inequality:

$$\int_0^\infty \alpha^2 dp \int_0^\infty \beta^2 dp \geq (\int_0^\infty \alpha\beta dp)^2. \tag{65}$$

Therefore, for any given perturbation δA_z, the functional $\delta^2 W$ of Bernstein's energy principle always exceeds the functional $\delta^2 \mathbb{F}$ of the thermodynamic criterion. Hence, we can state:

For any given equilibrium of both the ideal MHD and the nonresistive MHD equations, i.e. for any solution of (30) and (31), stability according to the thermodynamic criterion also implies stability according to Bernstein's principle. However, stability according to the latter does *not* imply stability according to the former.

Physically, the increased stability in ideal MHD can be explained by the adiabatic behavior of the plasma (if present, i.e. if $\gamma = 5/3$), and by the "frozen-in" field line effect. This fact is obvious for the first term in equation (64), which vanishes only in the isothermal case $\gamma = 1$. To see it for the the second term, let us go back to the formula (33) of the second chapter from which we can obtain the free energy $\delta^2 \mathbb{F}$ as a function of the perturbations of the vector potential and the particle density:

$$\delta \mathbb{F}\{\delta A_z, \delta n\} = \int_V \frac{1}{2\mu_0}(\nabla \delta A_z)^2 + \frac{1}{2}p(\frac{\delta n}{n})^2 - \frac{1}{2} we \delta n \delta A_z \; d^3 r. \tag{66}$$

From this expression we received the final form of $\delta^2 \mathbb{F}\{\delta A_z\}$ by minimizing with respect to δn under the constraint of fixed particle number $\int_V \delta n \, d^3 r = 0$ (the constraint of the nonresistive model). However, if we had used the constraint of ideal MHD, we would have to write:

$$\delta n = -\nabla\cdot(n\xi) = -\xi \cdot \nabla A \frac{\partial n}{\partial A} - n\nabla\cdot\xi = \frac{\partial n}{\partial A}\delta A_z - n\nabla\cdot(-\frac{\delta A_z}{(\nabla A)^2}\nabla A + \chi \nabla A \times \underline{e}_z), \tag{67}$$

where we were only free to vary χ, the component of ξ perpendicular to ∇A. Inserting this expression into (66) gives rise to exactly the same variational problem we discussed above (eq. 49 - 59), and we can read off the minimizing δn under ideal MHD constraints:

$$\delta n^*\{\delta A_z\} = \frac{\partial n}{\partial A}\delta A_z - nF. \tag{68}$$

Clearly, the test functions of form (67) obey $\int_V \delta n d^3r = 0$, which means they are a subset of those allowed in the variation under the nonresistive constraints. Restricting the class of test functions of a variational problem always raises the minimum; hence we can conclude:

$$\delta^2 \mathbb{F}\{\delta A_z\} \equiv \delta^2 \mathbb{F}\{\delta A_z, \delta n\{\delta A_z\}\} \leq \delta^2 \mathbb{F}\{\delta A_z, \delta n^*\{\delta A_z\}\}. \tag{69}$$

As a matter of a fact, we have

$$\delta^2 \mathbb{F}\{\delta A_z, \delta n^*\{\delta A_z\}\} = \int_V \frac{1}{2\mu_0}(\nabla \delta A_z)^2 - \frac{1}{2}\frac{\partial^2 p}{\partial A^2}(\delta A_z)^2 + F^2 \, d^3r \equiv \delta^2 W\{\delta A_z\} \tag{70}$$

(for the case $\gamma = 1$). This result closes the chain of our arguments and shows that it is indeed the constraint of frozen-in field lines that raises the functional $\delta W\{\delta A_z\}$ over $\delta \mathbb{F}\{\delta A_z\}$ and hence increases the stability properties of ideal MHD.

IV. Discussion

Our investigations have shown that it is possible to rederive the statistical mechanics stability criterion for current carrying plasmas within the framework of a one-fluid description, employing a nonideal magnetohydrodynamic theory which takes into account all kinds of dissipational effects except the resistivity, and that this criterion has a definite relation to the ideal MHD energy principle by Bernstein et al. In this chapter we summarize our results and discuss their physical interpretation.

Starting point of our considerations was the formulation of the so-called nonresistive magnetohydrodynamics, a plasma model which neglects the influence of the resistivity in Ohm's law but allows for other dissipational effects like viscosity and diffusion. We argued that this description is valid in the presence of microscopic field fluctuations with a spectrum dominant in directions perpendicular to the current density, a situation given e.g. in the tail of the earth's magnetosphere.

In the first part of our work (section II) we used this plasma model, specialized to the case of translational invariant modes and provided with appropriate boundary conditions, to rederive the statistical mechanics stability criterion for current carrying plasmas which so far had only been obtained for a multi-species plasma description. (Actually, the form we found corresponds to the quasi-neutral two-fluid case discussed by Kiessling[22]). As the MHD equations can be derived from the multi-particle description by invoking additional assumptions, our results basically state that these assumptions are compatible with the validity of the thermodynamic stability analysis. Clearly, this is a conclusion which is essential for estimating the validity domain of the thermodynamic approach.

The second part of this paper (section III) was devoted to comparing the rederived thermodynamic stability criterion with the energy principle by Bernstein et al. which is based on the more restrictive plasma description of ideal magnetohydrodynamics. After sketching the assumptions invoked in Bernstein's principle and choosing appropriate boundary conditions, we specialized it likewise to the case of translationally invariant motion and reduced it to a form directly comparable to the thermodynamic criterion, i.e. to a variational principle in the z-component of the vector potential perturbation only. Applying Schwarz's inequality, we finally arrived at the desired relation $\delta^2 W\{\delta A_z\} \geq \delta^2 I\!F\{\delta A_z\}$, showing that for any given δA_z the functional of Bernsteins principle is always larger than the corresponding functional of the entropy principle, and hence the latter is always more "pessimistic" than the former: stability according to the thermodynamic criterion also implies stability according to Bernsteins's principle, but not vice versa.

A physical explanation for this definite relation can be drawn from the fact that the plasma motion in the nonideal model has much more degrees of freedom than it has in ideal MHD. In fact, as the form of the dissipative terms is not totally specified, all dynamical modes in the nonideal model are allowed as long they are compatible with the balance equations of particle number, momentum and energy and with the second law of thermodynamics, whereas the ideal MHD motions are severely restricted by the "frozen-in field line" effect and by the requirement of adiabatic behavior.

As a consequence, for any given equilibrium, the set of perturbations allowed in ideal magnetohydrodynamics is is considerably smaller than the one allowed in the model of nonresistive MHD. More strictly spoken, the former is always a true subset of the latter; and correspondingly, the class of test functions for Bernstein's energy principle is also a true subset of those employed in the statistical mechanics version. (Note that this connection was not a-priori clear in the multi-species formulation of the thermodynamic stability criterion; therefore, the first part of our work was necessary for our conclusions.) From the point of view we have reached now, it becomes obvious that we had to arrive at our conclusion $\delta^2 \mathbb{F}\{\delta A_z\} \leq \delta^2 W\{\delta A_z\}$. The minimization process in a larger space of test functions necessarily leads to a lower minimum, and we have indeed explicitly shown that one can obtain Bernstein's principle by imposing additional constraints on the variation if $\delta^2 \mathbb{F}$.

As a closing remark, let us emphasize that there is an important difference in the interpretation of the results of both criteria in the case of instability. Whereas in the ideal MHD case the growth rate of the instable mode can directly be inferred from Bernstein's energy principle (resp. from the solution of the associated eigenvalue problem), no such analysis is possible in the nonresistive model: $\delta^2 \mathbb{F} \leq 0$ only indicates that free energy is available; the way in which the system takes advantage of this free energy is not determined (as long as the dissipation process remains unspecified). Therefore the situation may arise that the system is thermodynamically unstable but, due to the possibly long time constants of the unstable mode, still appears as stable. This, however, is a feature of all thermodynamic stability theories. Further investigations, leading to an estimation of the time scales of the instability, can only be carried through if the dissipation processes are specified in a more detailed way.

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