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On A Generalization of BRS and Gauge Transformations

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Abstract

Generalized non–linear BRS and gauge transformations containing non–trivial Lie algebra cocycles and acting on differential p–form gauge fields ($p \geq 1$) are constructed in the context of free minimal differential algebras. The associated gauge algebra is analyzed and a method for constructing BRS gauge–invariant homogeneous polynomials of the gauge fields and their curvatures is given by quotienting the associated BRS algebra with an appropriate ideal.

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I. Introduction

BRS-transformations[1] are known to yield the relevant symmetries of the quantum action of gauge theories[2]. As such, the BRS symmetries have turned out to play a major role in the understanding of the occurrence of anomalies and of their geometrical interpretation[3-10]. Furthermore, they form the natural setting for the description of canonical systems with first class constraints[11-13]. In a recent paper[14], the BRS transformations were generalized to include terms with non-trivial Lie algebra p-cocycles, $p \geq 3$ using free minimal differential algebras[15], the natural setting allowing the extension of the notion of Lie algebras. Theories with a free differential algebra (FDA) as the underlying symmetry have been proposed in the context of supergravity theories, e.g., in refs. [16-17]. The FDA setting allows one to introduce in a natural way antisymmetric tensor gauge field theories. Such theories and their quantization have been extensively studied in the literature, c.f., e.g., refs. [18-21].

In ref. [14], a bigraded differential algebra is considered with the two differentials d and s, d will be identified for example with the differential operator on space-time and s will be identified with the differential operator of the BRS transformations.The BRS algebra interpolates between the FDA with differential s which defines the symmetry and the Weil algebra[22-23] with differential d, generated by the p-form gauge fields and their curvatures, the d cohomology therein being trivial. The connection between the FDA and the Weil algebra is provided by a generalization of the so-called "Russian Formula"[3-9],c.f. also refs. [24-25].

From a physical point of view, the BRS symmetry arises as a secondary symmetry, an effective invariance of the quantum action due to a gauge fixing procedure (the Faddeev-Popov prescription[27]). As such, the gauge transformations are the fundamental symmetries on the classical level.

In the present article, we study a generalization of BRS and gauge transformations associated with the construction given in ref.[14]. The occurrence of higher-order antisymmetric tensor gauge fields allows one to consider non-trivial Lie algebra cocycles in the definition of their infinitesimal gauge transformations. As a consequence, these transformations become non-linear in the gauge fields and their algebraic structure is much more involved. In particular, the commutator of two infinitesimal gauge transformations does not close in the usual way, leading to the presence of non-trivial, field-dependent, two-cocycles in the gauge algebra. We would like to stress that the generalization of gauge transformations as we propose here are enforced upon us by the necessity to have an algebraic structure in which every object plays a non-trivial role.

Besides the possible applications of these extended BRS and gauge structures to supergravity theories and to a better understanding of the group manifold approach to these theories[17], one of our other motivations for developing these concepts is related to the extension of the notion of integrability to dimensions higher than two[28]. In particular, the zero curvature condition will no longer have simple solutions of the type $A = dU.U^{-1}$ because of the occurrence of the higher Lie algebra p-cocycles, leading to nonlinear terms. As an other application, we believe that these generalized gauge structures could also be relevant in the context of string field theories.

This article is organized as follows. In Section II, we give the basic definitions and notations on minimal free differential algebras (FDA) and recall briefly some results of ref.[14] on the construction of the associated BRS algebra and introduce some new useful concepts for their analysis. In Section III, generalized gauge transformations are given, together with the study of their algebra. In Section IV, we rewrite these structures in a compact operator formulation. An example, using a Lie algebra 3-cocycle is given in Section V. Finally, we show, in Section VI, how to construct BRS-invariant quantities containing only the gauge fields and their curvatures by quotienting the BRS algebra with an appropriate ideal.

II. The BRS Algebra of a Minimal FDA

In order to make the article self-contained, we give in what follows some basic definitions and notations and recall some of the results of ref.[14]. Let $\mathcal{H}$ be a graded commutative free differential algebra (FDA) which is connected in degree zero, i.e., $\mathcal{H} = \bigoplus_{p\geq 0} \mathcal{H}_p = \mathcal{H}_0 \bigoplus \mathcal{H}_+$ with $\mathcal{H}_0 = \mathbf{K}$ being the ground field ($\mathbf{R}$ or $\mathbf{C}$) and $\mathcal{H}_+$ the part generated in positive degrees. We denote the differential of $\mathcal{H}$ by s.

It is known from ref.[15] that any FDA can be decomposed uniquely into a tensor product of a minimal algebra $\mathcal{M}$ and a contractible one $\mathcal{C}$ [3] and that there is a unique procedure to construct the most general minimal FDA $\mathcal{M}$. See also refs.[16] for the case of superalgebras.

A minimal FDA is obtained by extending the Maurer–Cartan forms in the dual $\mathcal{G}^*$ of a Lie (super)–algebra $\mathcal{G}$, identifying $\wedge\mathcal{G}^*$ with $\mathcal{M}_1$, the subalgebra generated in degree one and by adding new generators in degrees higher than one. Choosing a basis $\{\chi_1^\alpha, \alpha = 1, \cdots, N_1\}$ of $\mathcal{G}^*$, ($N_1 = \dim\, \mathcal{G}$) and representations $D^{(p)}$ of $\mathcal{G}$, i.e., $D^{(p)}(E_\alpha^{(1)})^i_j = d^{(p)i}_{\alpha j}$, $\{E_\alpha^{(1)}\}$ being the basis of $\mathcal{G}$ such that $[E_\alpha^{(1)}, E_\beta^{(1)}] = C^\gamma_{\alpha\beta} E_\gamma^{(1)}$ where $C^\gamma_{\alpha\beta}$ are the structure constants of $\mathcal{G}$. We introduce at each level (p), $p \geq 2$ a

[3] We recall that a contractible differential algebra $\mathcal{C}$ has the property that $s\mathcal{C}_p \subset \mathcal{C}_{p+1}$, whereas a minimal algebra $\mathcal{M}$ obeys $s\mathcal{M} \subset \mathcal{M}_+ \cdot \mathcal{M}_+$.[15]

set of new generators $\{\chi_p^i, i = 1, \cdots, N_p\}$ in degree p, ($N_p = \dim D^{(p)}$), and write the action of the differential s on χ_p^i by imposing the minimality condition:

$$s\chi_p^i = -d_{\alpha j}^{(p)i} \; \chi_1^\alpha \; \chi_p^\alpha + \Omega_{p+1}^i \; [\chi_1, \cdots, \chi_{p-1}] \tag{2.1}$$

in which Ω_{p+1}^i is a Chevalley $(p+1)$–cocycle of s with value in $D^{(p)}$. This cocycle condition is required in order to ensure the nilpotency of the differential operator s, i.e., $s^2 = 0$. By a theorem of Chevalley and Eilenberg[29], it is known that there is no non–trivial cocycle for semi–simple Lie algebras apart in the scalar representation. To overcome this restriction, one has then to work with non semi–simple Lie algebras.

As in ref.[14], we will consider the subspace η of the dual of $\mathcal{M}_+$, the part generated in positive degrees of a minimal FDA $\mathcal{M}$ defined by:

$$\eta = \{\omega \in (\mathcal{M}_+)^* \; / \; \omega(a_1 \cdot a_2) = 0 \;\; \forall a_1, a_2 \in \mathcal{M}_+\} \;\; .$$

Then, η has a canonical Lie algebra structure. We may choose a basis $\{E_i^{(p)}, i = 1, \cdots, N_p\}, p \geq 1$ of η with even or odd parity according to the degree, i.e., $sE^{(p)} = (-1)^{p+1}E^{(p)}s$. Let $\{\chi_p^i\}$ denote the generators of $\mathcal{M}_+$ as before, we can then introduce the elements of $\mathcal{M}_+ \bigotimes \eta$:

$$\chi = \sum_{p,i} \; \chi_p^i \; E_i^{(p)} \tag{2.2}$$

where we have a formal sum over forms of all degrees, at each level taking values in the corresponding representation $D^{(p)}$. These kind of superforms have been used by Quillen in another context[30]. Note that χ anticommutes with s. In order now to rewrite the relations (2.1) in a compact form, we introduce the multilinear maps:

$$C^{(p)} : \wedge^p \eta \longrightarrow \eta \; , \;\; p = 2, 3, \cdots$$

In terms of the superforms (2.2), the relations (2.1) read simply:

$$s\chi + \sum_{p\geq 2} \frac{1}{p!}\ C^{(p)}\ (\chi,\cdots,\chi) = 0 \tag{2.3}$$

which is a generalization of the Maurer–Cartan equations with $C^{(p)}$ extended to $\mathcal{M}_+ \otimes \eta$. The nilpotency of s implies the following cocycle conditions on the $C^{(p)}$:

$$\sum_{p,q\geq 2} \frac{1}{(p-1)!q!} C^{(p)}\left(\chi,\cdots,\chi, C^{(q)}(\chi,\cdots,\chi)\right) = 0\ . \tag{2.4}$$

The Weil algebra $W(\mathcal{M})$ associated with $\mathcal{M}$ is obtained by introducing for each generator χ^i_p of $\mathcal{M}_+$ a connection $A^i_{(p)}$ and a curvature $F^i_{(p)}$ defined by the action of another differential d such that the d–cohomology is trivial. These connection and curvature having respectively the degree (p) and $(p+1)$ with respect to the differential d. Thus, we associate to each superform (2.2) the elements of $W(\mathcal{M}) \otimes \eta$:

$$\begin{aligned} \mathcal{A} &= \sum_{i,p\geq 1} A^i_{(p)}\ E^{(p)}_i \\ \mathcal{F} &= \sum_{i,p\geq 1} F^i_{(p)}\ E^{(p)}_i\ . \end{aligned} \tag{2.5}$$

The Weil algebra $W(\mathcal{M})$ being the graded commutative differential algebra generated by $\{A^i_{(p)}\}$ and $\{F^i_{(p)}\}$. The action of its differential d on the generalized connection $\mathcal{A}$ and curvature $\mathcal{F}$ is given in compact notations by:

$$\begin{aligned} d\mathcal{A} &= \mathcal{F} - C^{(0)}_{\mathcal{A}} \\ d\mathcal{F} &= -C^{(1)}_{\mathcal{A}}(\mathcal{F}) \end{aligned} \tag{2.6}$$

where we have introduced the $\mathcal{A}$–dependent multilinear maps $C^{(p)}_{\mathcal{A}}$ acting on variables

ν_i as:

$$C^{(p)}_{\mathcal{A}}(\nu_1,\cdots,\nu_p)=\sum_{\substack{q\geq p\\ q\geq 2}}\frac{1}{(q-p)!}\;C^{(q)}\;(\mathcal{A},\cdots,\mathcal{A},\nu_1,\cdots,\nu_p)\quad. \tag{2.7}$$

We note that as a consequence of the cocycle conditions (2.6), the $C^{(p)}_{\mathcal{A}}$ obey the following relations (c.f. Appendix A):

$$\sum_{r=0}^{q}\sum_{\pi}(-1)^{\pi}(-1)^{r(k+1)}C^{(r+1)}_{\mathcal{A}}\left(\nu_{\pi_1},\cdots,\nu_{\pi_r},C^{(q-r)}_{\mathcal{A}}\left(\nu_{\pi_{r+1}},\cdots,\nu_{\nu_q}\right)\right)=0 \tag{2.8}$$

where the sum is over all the permutations $\pi\in S_q/S_r\times S_{q-r}$ (S_n being the symmetric group of n elements). Introducing the operator (extension of the usual covariant derivative)

$$d_{\mathcal{A}}\cdot=d\cdot+C^{(1)}_{\mathcal{A}}(\cdot) \tag{2.9}$$

we obtain that eq. (2.6) provides a generalization of the Bianchi identity, i.e., $d_{\mathcal{A}}\mathcal{F}=0$.

The main result of ref.[14] is the construction of the BRS algebra $U(\mathcal{M})$ associated with a minimal FDA $\mathcal{M}$, i.e., a bigraded differential algebra $U(\mathcal{M})=\bigoplus_{p,q}U_{p,q}$ such that $\bigoplus_{p\geq 1}U_{0,q}$ is isomorphic to $\mathcal{M}_+$ and $\bigoplus_{p\geq 1}U_{p,0}$ is isomorphic to the Weil algebra $W(\mathcal{M})$. This construction amounts to an extension of the so-called "Russian Formula" [3–9]. The prescription is that one identifies the generators $A^i_{(p)}$ and $F^i_{(p)}$ of $W(\mathcal{M})$ in (2.5) with the generators ${A^i_{(p)}}^{p,0}$ and ${F^i_{(p)}}^{p+1,0}$ of $U(\mathcal{M})$ (the superscript indices are respectively the d and s degrees of each generator) and one performs the translation $d\longrightarrow d+s$ together with:

$$\begin{aligned}A^i_{(p)}&\longrightarrow\;{A^i_{(p)}}^{p,0}\;+\;{A^i_{(p)}}^{p-1,1}+\cdots+{A^i_{(p)}}^{0,p}\\ F^i_{(p)}&\longrightarrow\;{F^i_{(p)}}^{p+1,0}\;+\;{F^i_{(p)}}^{p,1}+\cdots+{F^i_{(p)}}^{2,p-1}\end{aligned} \tag{2.10}$$

with $p \geq 1$, where ${A^i_{(p)}}^{k,l}$ and ${F^i_{(p)}}^{k,l}$ are the generators of $U(\mathcal{M})$ together with $d{A^i_{(p)}}^{k,l}$ and $d{F^i_{(p)}}^{k,l}$ or related generators, such that $s{A^i_{(p)}}^{k,l} \in U_{k,l+1}$, $d{A^i_{(p)}}^{k,l} \in U_{k+1,l}$, etc. Then one identifies the elements of the same bidegree. The "Russian Formula" means that eqs. (2.6) are invariant under the translations $d \to d+s$ and (2.10). In ref.[14], the resulting BRS algebra $U(\mathcal{M})$ is explicitly constructed and it is shown that its d and $(d+s)$-cohomologies are trivial, see also ref.[7]. For our purpose, we recall here the relevant formulae. By using the superforms elements of $U(\mathcal{M}) \bigotimes \eta$:

$$\begin{aligned}\mathcal{A}_p &= \sum_{q\geq 0} {A^i_{(p+q)}}^{q,p} \quad E_i^{(p+q)} \\ \mathcal{F}_p &= \sum_{q\geq 1} {F^i_{(p+q)}}^{q+1,p} \quad E_i^{(p+q)}\end{aligned} \tag{2.11}$$

with $p+q \geq 1$, $p \geq 0$ and $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{F}_0 = \mathcal{F}$, we obtain the following compact formulae for the action of the differential s on the generators of the BRS algebra:

$$s\mathcal{A}_p + d_{\mathcal{A}_0}\mathcal{A}_{p+1} + \sum_{\substack{q\geq 2\\ i_n\geq 1}} \sum_{\substack{i_1<i_2\cdots<i_n\\ \sum_k i_k=p+1}} \Gamma_q(\{i\})\, C^{(q)}_{\mathcal{A}_0}(\mathcal{A}_{i_1}\cdots\mathcal{A}_{i_1},\cdots,\mathcal{A}_{i_n}\cdots\mathcal{A}_{i_n}) = \mathcal{F}_{p+1} \tag{2.12}$$

$\Gamma_q(\{i\}) = \frac{1}{n_{i_1}!}\cdots\frac{1}{n_{i_n}!}$ where n_{i_k} is the number of $\mathcal{A}_r$ with $r = i_k$ in each $C^{(q)}_{\mathcal{A}_0}$. In the same way we obtain:

$$s\mathcal{F}_p + d_{\mathcal{A}_0}\mathcal{F}_{p+1} + \sum_{\substack{q\geq 2\\ i_0\geq 0, i_1\geq 1}} \sum_{\substack{i_1<i_2\cdots<i_n\\ i_0+\sum i_n=p+1}} \Gamma_q(\{i\}) C^{(q)}_{\mathcal{A}_0}(\mathcal{A}_{i_1}\cdots\mathcal{A}_{i_1},\cdots,\mathcal{A}_{i_n}\cdots\mathcal{A}_{i_n},\mathcal{F}_{i_0}) = 0\ . \tag{2.13}$$

These formulae are, in fact, obtained by projecting out each constant ghost degree in the following translated equations of eqs. (2.6):

$$\begin{aligned}(d+s)\,\tilde{\mathcal{A}} &= \tilde{\mathcal{F}} - C^{(0)}_{\tilde{\mathcal{A}}} \\ (d+s)\,\tilde{\mathcal{F}} + C^{(1)}_{\tilde{\mathcal{A}}}(\mathcal{F}) &= 0\end{aligned} \tag{2.14}$$

where $\tilde{\mathcal{A}} = \sum_{p\geq 0} \mathcal{A}_p$ and $\tilde{\mathcal{F}} = \sum_{p\geq 0} \mathcal{F}_p$.

Using the cocycle conditions (2.9) for $C^{(q)}_{\mathcal{A}_0}$, it is easy to show that s and d extend to two anticommuting differentials of $U(\mathcal{M})$, i.e., such that $s^2 = d^2 = sd + ds = 0$. It has to be noted that in general, due to the occurrence of higher–order Lie algebra cocycles, $\mathcal{F}_p \neq 0$ for $p \geq 1$, contrary to the usual case where the curvature is not translated. This means that in our more general situation, it is necessary, in order to keep the algebra $U(\mathcal{M})$ free of algebraic relations among the generators, to have also ghost fields associat edto $\mathcal{F}$. In particular, the r.h.s. of eq.(2.12) is non–zero even for $p = 0$ which gives the action of s on the generalized connection $\mathcal{A}_0$.

III. The Generalized Gauge Transformations

When dealing with a BRS algebra in the usual case, i.e., the case of a Lie algebra instead of a minimal FDA, the infinitesimal gauge transformations can be recovered from the BRS transformations by "replacing" the ghost fields by the infinitesimal gauge parameters. It is a priori not at all clear that a similar prescription works in the present case of a minimal FDA. In fact, the appearance of ghost fields of degree higher than 2 suggests the existence of objects which at the same time play the role of gauge parameters as well as of fields, i.e., objects that are transformed under the gauge transformations, leading to a residual invariance of the classical action after a gauge fixing procedure and hence to the phenomenon of ghosts of ghosts[20–21]. Nevertheless, it is possible to proceed almost as usual, the classical sector of the theory being the Weil algebra $W(\mathcal{M})$. Thus, we introduce generalized gauge parameters ϵ and η, i.e.,

$$\epsilon = \sum_{\substack{p,i \\ p\geq 1}} \epsilon^i_{(p)} E^{(p)}_i \ , \quad \eta = \sum_{\substack{p,i \\ p\geq 2}} \eta^i_{(p)} E^{(p)}_i \tag{3.1}$$

replacing the ghost fields $\mathcal{A}_1$ and $\mathcal{F}_1$ respectively, where $\epsilon^i_{(p)}$ and $\eta^i_{(p)}$ are respectively $(p-1)$ and (p)–forms with respect to the differential d and carrying no ghost degree. We then define following eqs. (2.12, 2.13), the infinitesimal gauge transformations on $\mathcal{A}$ and $\mathcal{F}$ by a derivation $\delta(\epsilon,\eta)$ depending on ϵ and η acting as follows:

$$\begin{aligned} \delta(\epsilon,\eta)\mathcal{A} &= \eta + d_{\mathcal{A}}\epsilon \\ \delta(\epsilon,\eta)\mathcal{F} &= d_{\mathcal{A}}\eta + C^{(2)}_{\mathcal{A}}(\mathcal{F},\epsilon) \quad . \end{aligned} \tag{3.2}$$

Furthemore, we impose that $\delta(\epsilon,\eta)$ does not act on the field independent gauge parameters. Successive applications of gauge transformations (3.2) lead to the following closure relations:

$$\begin{aligned} [\delta(\epsilon,\eta), \delta(\epsilon',\eta')]\mathcal{A} &= \delta(\epsilon_{\mathcal{A}},\eta_{\mathcal{A}})\mathcal{A} \\ [\delta(\epsilon,\eta), \delta(\epsilon',\eta')]\mathcal{F} &= \delta(\epsilon_{\mathcal{A}},\eta_{\mathcal{A}})\mathcal{F} \end{aligned} \tag{3.3}$$

in which

$$\begin{aligned} \epsilon_{\mathcal{A}} &= C^{(2)}_{\mathcal{A}}(\epsilon,\epsilon') \\ \eta_{\mathcal{A}} &= C^{(2)}_{\mathcal{A}}(\eta,\epsilon') - C^{(2)}_{\mathcal{A}}(\eta',\epsilon) + C^{(3)}_{\mathcal{A}}(\mathcal{F},\epsilon,\epsilon') \quad . \end{aligned} \tag{3.4}$$

As the closure relations (3.3) involve field–dependent parameters $\epsilon_{\mathcal{A}}$ and $\eta_{\mathcal{A}}$, i.e. new parameters that depend on $\mathcal{A}$ and $\mathcal{F}$, the gauge algebra is not closed in the usual sense. In fact, the transformation on the r.h.s. of (3.3) defines a new type of gauge transformation depending now on two sets of parameters (ϵ,η) and (ϵ',η'). Indeed, due to the presence of non–trivial Lie algebra cocycles $C^{(p)}, p \geq 3$, it is impossible to define a set of two parameters $(\tilde{\epsilon},\tilde{\eta})$, as in (3.1), functions of only (ϵ,η) and (ϵ',η') (and not of the fields $\mathcal{A}$, $\mathcal{F}$) such that the r.h.s. of (3.3) be given in terms of gauge transformations $\delta(\tilde{\epsilon},\tilde{\eta})$, even different from those given by (3.2), but depending as parameters only on $(\tilde{\epsilon},\tilde{\eta})$. Following this remark, we can give another interpretation of the closure relation (3.3). Let us at first rewrite the r.h.s. of (3.3) in two parts, one being a transformation depending on parameters $(\bar{\epsilon},\bar{\eta})$, defined as in (3.1), which are functions of (ϵ,η) and (ϵ',η') only:

$$[\delta\,(\epsilon,\eta)\,,\delta\,(\epsilon',\eta')]\,\mathcal{A}=\delta\,(\bar{\epsilon},\bar{\eta})\,\mathcal{A}+\Omega\,(\epsilon,\eta;\epsilon',\eta')\,\mathcal{A} \tag{3.5}$$

and a similar formula for the curvature $\mathcal{F}$, where $\delta(\bar{\epsilon},\bar{\eta})$ is given by (3.2) with parameters:

$$\begin{aligned}\bar{\epsilon} &= C^{(2)}(\epsilon,\epsilon')\\ \bar{\eta} &= C^{(2)}(\eta,\epsilon')-C^{(2)}(\eta',\epsilon)\end{aligned} \tag{3.6}$$

and $\Omega(\epsilon,\eta;\epsilon',\eta')$ is a gauge transformation acting on the fields $\mathcal{A}$ and $\mathcal{F}$ depending intrinsically on two sets of parameters whose explicit expression is given from eqs. (3.2 - 3.4). Then, using the fact that the commutator in (3.3) and hence in (3.5) obey the Jacobi identity as can be explicitly shown, the transformation $\Omega(\epsilon,\eta;\epsilon',\eta')\mathcal{A}$ in (3.5) is to be interpreted as a non-trivial field dependent two cocycle (living in a non-trivial representation) of the algebra of gauge transformations $\delta(\epsilon,\eta)$ given by eqs. (3.2 - 3.6).

It has to be stressed that this new effect occurs only in the presence of non-trivial Lie algebra cocycles $C^{(p)}, p\geq 3$. Let us, in fact, consider for one moment the particular case where all the cocycles $C^{(p)}, p\geq 3$, are set equal to zero, but where we still have the generalized gauge field $\mathcal{A}$ and curvature $\mathcal{F}$, each containing their complete tower of higher-order antisymmetric tensor fields. The parameters ϵ and η are still given by (3.1). Then, the gauge transformations (3.2) linearize in terms of

$\mathcal{A}$ and $\mathcal{F}$ and the algebra of gauge transformations closes since in that case $\Omega \equiv 0$ in (3.5). Moreover, the constraint $\eta = 0$ defines a sub-algebra of gauge transformations since $\bar{\eta} = 0$ if $\eta = \eta' = 0$ in (3.6), which is not true in the general situation due to the last term in eq. (3.4). Let us note that this particular case where $C^{(p)} \equiv 0$ for $p \geq 3$, is already a generalization of the usual gauge theories where $\mathcal{A}$ contains only a one-form gauge field. Furthermore, in this case the infinitesimal gauge transformations form a Lie algebra and can be exponentiated to give finite gauge transformations forming a group.

Returning now to the general case, i.e., where there is at least one non-zero cocycle $C^{(p)}, p \geq 3$, we wish to make a general remark on the algebraic structure of the gauge algebra: due to the extended closure relations (3.3) or (3.5), the original gauge transformations (3.2) do no more form a basis of the gauge algebra and are only generating the complete algebra through successive commutators. The algebra will contain, in general, gauge transformations of order n depending on a set of parameters $\{(\epsilon_1, \eta_1); \cdots; (\epsilon_n, \eta_n)\}$ the first non-trivial example being given by the transformation of order two $\Omega(\epsilon, \eta; \epsilon', \eta')$ in (3.5).

The next problem we wish to deal with concerns the construction of d and s invariant field dependent quantities. Another non-trivial effect of the presence of the non-trivial Lie algebra cocycles $C^{(p)}, p \geq 3$ is in order at this point. Indeed, the transformation law of the curvature $\mathcal{F}$ is not given by a standard commutator of ϵ and $\mathcal{F}$, but rather contains a $d_{\mathcal{A}}\eta$ term (coming from the ghost associated to the higher curvatures) and a field dependent bilinear product between ϵ and $\mathcal{F}$. In the next section, we reformulate this problem in an operator language that will enable us to analyze in a more convenient way the construction of invariants under the BRS transformations.

IV. The Operator Formalism

In order to analyze the construction of d and s invariants, we rewrite the structures given in the last sections in term of operators acting on arbitrary elements ν of $U(\mathcal{M})\otimes\eta$. Actually, the idea is to use a generalization of the usual adjoint mapping, but here acting on $U(\mathcal{M})\otimes\eta$ and defined through the bilinear map $C^{(2)}_{\mathcal{A}}$ given in eq.(2.7). Hence, to any homogeneous element ζ of $U(\mathcal{M})\otimes\eta$ different from $\mathcal{A}$, we associate an $\mathcal{A}$–dependent linear operator $\hat{\zeta}$ acting on any element ν of $U(\mathcal{M})\otimes\eta$ as follows:

$$\hat{\zeta}\,(\nu) = C^{(2)}_{\mathcal{A}}(\zeta,\nu)\ . \tag{4.1}$$

More generally, to any set of all different homogeneous elements $\zeta_1,\cdots,\zeta_p$ of $U(\mathcal{M})\otimes\eta$ all different from $\mathcal{A}$, we define the linear operator:

$$[\zeta_1\wedge\cdots\wedge\zeta_p]\,\hat{}\,(\nu) = C^{(p+1)}_{\mathcal{A}}(\zeta_1,\cdots,\zeta_p,\nu)\ . \tag{4.2}$$

Furthermore, we define the gauge field and curvature operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{F}}$ in the following way:

$$\hat{\mathcal{A}}(\nu)\ = C^{(1)}_{\mathcal{A}}(\nu) \tag{4.3a}$$

$$\hat{\mathcal{F}}(\nu)\ = C^{(2)}_{\mathcal{A}}(\mathcal{F},\nu) \tag{4.3b}$$

where eq.(4.3b) is obtained by taking ζ to be equal to $\mathcal{F}$ in eq.(4.1). Then eqs.(2.6) can be rewritten in the following form (c.f. AppendixB):

$$d\hat{\mathcal{A}} = \hat{\mathcal{F}} - \hat{\mathcal{A}}.\hat{\mathcal{A}} \tag{4.3}$$

and for the operator Bianchi identity:

$$d_{\hat{\mathcal{A}}}.\hat{\mathcal{F}} = d\hat{\mathcal{F}} + \hat{\mathcal{A}}.\hat{\mathcal{F}} - \hat{\mathcal{F}}.\hat{\mathcal{A}} = 0\ . \tag{4.4}$$

where we have used the operator covariant derivative $d_{\hat{\mathcal{A}}}$ defined by:

$$d_{\hat{\mathcal{A}}}.\hat{\zeta} = d\hat{\zeta} + \hat{\mathcal{A}}.\hat{\zeta} - (-1)^{|\zeta|}\hat{\zeta}.\hat{\mathcal{A}} \tag{4.5}$$

where the dot products in (4.4) are the usual operator products defined by successive application of operators and $|\zeta|$ stands for the total degree of the associated homogeneous element ζ in $U(\mathcal{M})\otimes\eta$. We wish to stress at this point that, in this operator language, the relations between the gauge field operator $\hat{\mathcal{A}}$ and the curvature operator $\hat{\mathcal{F}}$ in (4.5, 4.6) become usual, even if the definitions of these objects contain all the tower of higher–order antisymmetric tensor fields and of the Lie algebra cocycles $C^{(p)}, p \geq 3$. The BRS transformations (2.12,2.13) take now the form:

$$\begin{aligned} s\hat{\mathcal{A}} &= (\hat{\mathcal{F}}_1 - [\mathcal{F}\wedge\mathcal{A}_1]\,\hat{}\,) - d_{\hat{\mathcal{A}}}.\hat{\mathcal{A}}_1 \\ s\hat{\mathcal{F}} &= -d_{\hat{\mathcal{A}}}.\left(\hat{\mathcal{F}}_1 - [\mathcal{F}\wedge\mathcal{A}_1]\,\hat{}\,\right) + \hat{\mathcal{F}}.\hat{\mathcal{A}}_1 - \hat{\mathcal{A}}_1.\hat{\mathcal{F}} \end{aligned} \tag{4.6}$$

where $\hat{\mathcal{A}}_1$ and $\hat{\mathcal{F}}_1$ are the operators associated with $\mathcal{A}_1$ and $\mathcal{F}_1$ defined by eq.(2.11). Here again, besides the "translation" term $\hat{\gamma} = (\hat{\mathcal{F}}_1 - [\mathcal{F}\wedge\mathcal{A}_1]\,\hat{}\,)$ in (4.7) we recover for the BRS transformations of $\hat{\mathcal{A}}$ and $\hat{\mathcal{F}}$ almost the usual formula given by the ghost operator $\hat{\mathcal{A}}_1$. We have to remark, however, that the field $\mathcal{A}$ enters through eq.(4.1) in the definition of $\hat{\mathcal{A}}_1$, leading back, as it should be, to the generalized closure relations (3.3 - 3.5), but here in a more convenient formulation.

From eqs.(4.5, 4.6), it is clear that there is a possibility to construct d–invariants by considering an operator–trace of polynomials of $\hat{\mathcal{F}}$. Here, the operators we have constructed have to be considered as endomorphisms acting on a vector space. The trace is then defined as usual (we give an example in the following) and denoted by $tr(\ldots)$. We have

$$d\ \ tr\left(\hat{\mathcal{F}}.\hat{\mathcal{F}}.\cdots.\hat{\mathcal{F}}\right) = 0 \tag{4.7}$$

and one can construct the corresponding Chern-Simons term[26] by using the usual homotopy formulae since the d-cohomology of the algebra generated by the operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{F}}$ is trivial. As a simple example, we have

$$d \ tr\left(\hat{\mathcal{A}}.\hat{\mathcal{F}} - \frac{1}{3}\hat{\mathcal{A}}.\hat{\mathcal{A}}.\hat{\mathcal{A}}\right) = tr \ \hat{\mathcal{F}}^2 \quad . \tag{4.8}$$

Hence, by identifying in (4.9) the terms of the same form degree (with respect to d) we obtain generalized Chern-Simons terms. Note that we also have:

$$s \ tr(\underbrace{\hat{\mathcal{F}}.\hat{\mathcal{F}}\cdots\hat{\mathcal{F}}}_{n-\text{times}}) = -n \ d \ tr(\hat{\gamma}. \ \underbrace{\hat{\mathcal{F}}\cdots\hat{\mathcal{F}}}_{(n-1)-\text{times}}) \quad . \tag{4.9}$$

Let us now give a further generalization of the above picture, namely, by defining a new differential operator $\mathcal{D}$ such that $\mathcal{D} = d + s$. The complete BRS algebra was obtained from the eqs.(2.14), i.e.:

$$\mathcal{D}\tilde{\mathcal{A}} = \tilde{\mathcal{F}} - C^{(0)}_{\tilde{\mathcal{A}}} \tag{4.10}$$

$$\mathcal{D}\tilde{\mathcal{F}} + C^{(1)}_{\tilde{\mathcal{A}}}\left(\tilde{\mathcal{F}}\right) = 0 \tag{4.11}$$

where $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{F}}$ contain all the fields and their associated ghosts. Thus, we can define new operators $\hat{\tilde{\zeta}}$ as in eq. (4.1), but now using $\tilde{\mathcal{A}}$ and $C^{(p)}_{\tilde{\mathcal{A}}}$ instead of $\mathcal{A}$ and $C^{(p)}_{\mathcal{A}}$. All the preceding relations remain true, but now with $\tilde{\mathcal{A}}, \tilde{\mathcal{F}}$ and $\mathcal{D}$ replacing $\mathcal{A}, \mathcal{F}$ and d. In particular, we have:

$$\mathcal{D}tr\left(\hat{\tilde{\mathcal{F}}}^n\right) = 0 \tag{4.12}$$

and since the $\mathcal{D}$ cohomology is trivial[14] we obtain:

$$I^{(n)} = tr\left(\hat{\tilde{\mathcal{F}}}^n\right) = \mathcal{D}J^{(n)} \tag{4.13}$$

where the $J^{(n)}$ can be obtained by standard homotopy formulae[3-8] written in terms of the operators $\hat{\hat{\mathcal{A}}}$ and $\hat{\hat{\mathcal{F}}}$ which verify:

$$\mathcal{D}\,\hat{\hat{\mathcal{A}}} = \hat{\hat{\mathcal{F}}} - \hat{\hat{\mathcal{A}}}\cdot\hat{\hat{\mathcal{A}}} \tag{4.14}$$

and:

$$\mathcal{D}_{\hat{\hat{\mathcal{A}}}}\cdot\hat{\hat{\mathcal{F}}} = \mathcal{D}\,\hat{\hat{\mathcal{F}}} + \hat{\hat{\mathcal{A}}}\cdot\hat{\hat{\mathcal{F}}} - \hat{\hat{\mathcal{F}}}\cdot\hat{\hat{\mathcal{A}}} = 0 \ . \tag{4.15}$$

Now we can expand $I^{(n)}$ and $J^{(n)}$ with respect to the ghost degrees, namely:

$$I^{(n)} = \sum_{j=0}^{N} I_j^{(n)} \tag{4.16}$$

$$J^{(n)} = \sum_{j=0}^{M} J_j^{(n)} \tag{4.17}$$

where j stands for the ghost degree and M and N are two finite integers such that $M \geq N$,and $D \geq M$ where D is the dimension of the differentiable manifold with differential operator d (with the example of space time), D being supposed arbitrary but finite here. Then, eqs.(4.13,4.14) give the following 'descent' relations:

$$\begin{aligned} dI_0^{(n)} &= 0 \\ dI_1^{(n)} + sI_0^{(n)} &= 0 \\ &\vdots \\ dI_k^{(n)} + sI_{k-1}^{(n)} &= 0 \\ &\vdots \\ sI_N^{(n)} &= 0 \end{aligned} \tag{4.18}$$

together with:

$$dJ_0^{(n)} = I_0^{(n)}$$

$$\begin{aligned} dJ_1^{(n)} + sJ_0^{(n)} &= I_1^{(n)} \\ &\vdots \\ dJ_k^{(n)} + sJ_{k-1}^{(n)} &= I_k^{(n)} \\ &\vdots \\ sJ_M^{(n)} &= 0 \end{aligned} \tag{4.19}$$

with noting that $I_k^{(n)} = 0$ for $k \geq N$. All these quantities are polynomials in the gauge fields, their curvatures and their associated ghosts, $I_0^{(n)}$ and $J_0^{(n)}$ being polynomials of only the physical fields $\hat{\mathcal{A}}$ and $\hat{\mathcal{F}}$. We note also that there always exist among the $I_j^{(n)}$ an object $I_N^{(n)}$ of maximal (finite) ghost degree which is an s–invariant and similarly there exist among the $J_j^{(n)}$ an object of maximal (finite) ghost degree $J_M^{(n)}$ s-invariant. Furthermore, we can obtain other s-invariant quantities from equations (4.18,4.19) by projecting out each constant form degree (with respect to d) for a given ghost degree. These s-invariants can be trivial.

V. An Example With a 3–Cocycle

Before going to the construction of s–invariants containing, for example, only the gauge fields and their curvatures along the above line, we wish to give here a simple example of the structures we have described in Section IV.

Let us consider the following FDA generated in degree one by $\{\chi_1^a, a = 1, \cdots, N\}$ and $\{\chi_1^i, i = 1, \cdots, n\}$ and in degree two by $\{\chi_2^a, a = 1, \cdots N\}$ and $\{\chi_2^i, i = 1, \cdots, n\}$. The action of s being defined by:

$$\begin{aligned} s\chi_1^a &= -\frac{1}{2}\, C_{bc}^a\ \chi_1^b\ \chi_1^c \\ s\chi_1^i &= -D_{aj}^i\ \chi_1^a\ \chi_1^j \end{aligned}$$

$$
\begin{aligned}
s\chi_2^a &= -C^a_{bc}\ \chi_1^b\ \chi_2^c + \frac{1}{6}\ P^a_{ijk}\ \chi_1^i\ \chi_i^j\ \chi_1^k \\
s\chi_2^i &= -D^i_{aj}\left(\chi_1^a\ \chi_2^j - \chi_2^a\ \chi_1^j\right)
\end{aligned}
\tag{5.1}
$$

The nilpotency of the differential s is equivalent to the following constraints on the coefficients C^a_{bc}, D^i_{aj} and P^a_{ijk}:

$$
\begin{aligned}
C^a_{bc}\, C^b_{de} &+ \mathrm{cycl}\,(cde) = 0 \\
D^i_{aj}\, D^j_{bk} &- D^i_{bj}\ D^j_{ak} - C^c_{ab}\ D^i_{ck} = 0 \\
C^a_{bc}\, P^c_{ijk} &- 3P^a_{eij}\ D^e_{bk} + \mathrm{cycl}\,(ijk) = 0 \\
D^i_{aj}\, P^a_{klm} &+ \mathrm{perm}\,(jklm)\,.\mathrm{sign}\,(\mathrm{perm}) = 0\ .
\end{aligned}
\tag{5.2}
$$

The first relation is the Jacobi identity for the coefficients C^c_{ab} thus defining a Lie algebra $\mathcal{G}_0$, the second one is the condition that the matrices $(D_a)^i_j \equiv D^i_{aj}$ form a representation of $\mathcal{G}_0$. Hence, the two first relations define also a non semi-simple Lie algebra $\mathcal{G}$, having $\mathcal{G}_0$ as a sub-algebra.

The two last relations in (5.2) imply that P^a_{ijk} is a 3-cocycle of $\mathcal{G}$ in the adjoint representation of $\mathcal{G}_0$. The existence of this cocycle is not excluded since it does not violate the Chevalley-Eilenberg theorem[29]. In order to illustrate the operator formalism of section IV, we introduce the following matrices $\hat{\chi}_1$ and $\hat{\chi}_2$:

$$
\begin{aligned}
(\hat{\chi}_1)^a_c &= C^a_{bc}\ \chi_1^b \qquad (\hat{\chi}_1)^i_j = D^i_{aj}\chi_1^a \qquad (\hat{\chi}_1)^i_a = -D^i_{aj}\ \chi_1^j \\
(\hat{\chi}_2)^a_c &= -C^a_{bc}\ \chi_2^b \quad (\hat{\chi}_2)^i_j = -D^i_{aj}\ \chi_2^a \\
(\hat{\chi}_2)^i_a &= D^i_{aj}\ \chi_2^j \quad (\hat{\chi}_2)^a_i = -\frac{1}{2}P^a_{ijk}\ \chi_1^j\ \chi_1^k\ .
\end{aligned}
\tag{5.3}
$$

Then using (5.1) and (5.2) we can compute the action of the differential s on these quantities as follows (the dot (.) is now the usual matrix product):

$$
\begin{aligned}
s\hat{\chi}_1 &= -\hat{\chi}_1 \cdot \hat{\chi}_1 \\
s\hat{\chi}_2 &= -\hat{\chi}_1 \cdot \hat{\chi}_2 + \hat{\chi}_2 \cdot \hat{\chi}_1 \ .
\end{aligned}
\tag{5.4}
$$

Let us note that these transformations look as usual ones in this operator formulation, even if $\hat{\chi}_2$ contains the 3–cocycle in its definition. It is now easy to see, for example, that the quantity:

$$
\begin{aligned}
tr\,(\hat{\chi}_2 \cdot \hat{\chi}_2) &= C^a_{db}\ C^b_{ea} \quad \chi^d_2 \cdot \chi^e_2 \\
&+\ D^i_{dj}\ D^j_{ei}\ \chi^d_2\ \chi^e_2 \\
&-\ P^a_{ijk}\ D^i_{al}\ \chi^j_1\ \chi^k_1\ \chi^l_2
\end{aligned}
\tag{5.5}
$$

is a non trivial s–invariant. Note that the two first terms are just given by the usual trace of two objects defined in the adjoint representation of $\mathcal{G}_0$, while the last one is constructed with the 3–cocycle P^a_{ijk}. Using eq. (5.4), it is easy to construct other s–invariants.

The associated Weil algebra is generated, according to Section II, by the connections A^a_1 and A^i_1 in degree one, by A^a_2 and A^i_2 in degree two and by their respective curvatures F^a_2 and F^i_2 in degree two, F^a_3 and F^i_3 in degree three, with respect to the differential d, its action being given by:

$$
\begin{aligned}
d\,A^a_1 &= F^a_2 - \frac{1}{2} C^a_{bc}\ A^b_1\ A^c_1 \\
d\,A^i_1 &= F^i_2 - D^i_{aj}\ A^a_1\ A^j_1 \\
d\,A^a_2 &= F^a_3 - C^a_{bc}\ A^b_1\ A^c_2 + \frac{1}{6}\ P^a_{ijk}\ A^i_1\ A^j_1\ A^k_1 \\
d\,A^i_2 &= F^i_3 - D^i_{aj}\ (A^a_1\ A^j_2 - A^a_2\ A^j_1)
\end{aligned}
\tag{5.6}
$$

together with the Bianchi identities:

$$\begin{aligned}
d\,F_2^a &= C^a_{bc}\ F_2^b\ A_1^c \\
d\,F_2^i &= D^i_{aj}\ (F_2^a\ A_1^j - A_1^a\ F_2^j) \\
d\,F_3^a &= C^a_{bc}\ (F_2^b\ A_2^c - A_1^b\ F_3^c) - \frac{1}{2}\,P^a_{ijk}\ F_2^i\ A_1^j\ A_1^k \\
d\,F_3^i &= D^i_{aj}\ (F_2^a\ A_2^j - A_1^a\ F_3^j - F_3^a\ A_1^j - A_2^a\ F_2^j)\,.
\end{aligned} \tag{5.7}$$

As before, we can construct the matrices $\hat{A}_1, \hat{A}_2$ and $\hat{F}_2, \hat{F}_3$, representing the operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{F}}$. The matrices $\hat{A}_1$ and $\hat{A}_2$ are given by formulae (5.3) when replacing χ_1 and χ_2 respectively by A_1 and A_2 and we define the matrices $\hat{F}_2$ and $\hat{F}_3$ by:

$$\begin{aligned}
\left(\hat{F}_2\right)^a_c &= C^a_{bc}\ F_2^b, \qquad \left(\hat{F}_2\right)^i_a = -D^i_{aj}\ F_2^j \\
\left(\hat{F}_2\right)^i_j &= D^i_{aj}\ F_2^a \\
\left(\hat{F}_3\right)^a_c &= -C^a_{bc}\ F_3^b \qquad \left(\hat{F}_3\right)^i_a = D^i_{aj}\ F_3^j \\
\left(\hat{F}_3\right)^i_j &= -D^i_{aj}\ F_3^a \qquad \left(\hat{F}_3\right)^a_i = -P^a_{ijk}\ F_2^j\ A_1^k\,.
\end{aligned} \tag{5.8}$$

The action of d can then be given in a compact form:

$$\begin{aligned}
d\,\hat{A}_1 &= \hat{F}_2 - \hat{A}_1.\hat{A}_1 \\
d\,\hat{A}_2 &= \hat{F}_3 - \hat{A}_1.\ \hat{A}_2 + \hat{A}_2.\ \hat{A}_1
\end{aligned} \tag{5.9}$$

and for the operator Bianchi identities:

$$\begin{aligned}
d\,\hat{F}_2 &+ \hat{A}_1.\hat{F}_2 - \hat{F}_2.\hat{A}_1 = 0 \\
d\,\hat{F}_3 &+ \hat{A}_1.\hat{F}_3 + \hat{F}_3.\hat{A}_1 + \hat{A}_2.\hat{F}_2 - \hat{F}_2.\hat{A}_2 = 0\ .
\end{aligned} \tag{5.10}$$

Eqs.(5.9, 5.10) stand for the explicit realization of the general relations (4.5), (4.6) in the case of a non-trivial 3-cocycle. It is now easy to write the Chern-Simons term associated to $tr(\hat{\mathcal{F}}^2)$ in degree five with respect to d as follows:

$$J = \frac{1}{2}\, tr\left(\hat{A}_2\, \hat{F}_2 + \hat{A}_1\, \hat{F}_3 - \hat{A}_1\, \hat{A}_1\, \hat{A}_2\right) \tag{5.11}$$

and we have by explicit computation, $dJ = I$, with

$$I = tr\left(\hat{F}_2.\hat{F}_3\right) \quad . \tag{5.12}$$

Hence, I is just the degree five of the Pontryagin density in (4.9). Note that, as it should be, I is a d-invariant element of the Weil algebra and that it contains an unusual killing form (defining the trace) $P^a_{ijk} D^i_{al} A^k + (l \leftrightarrow j)$ which, coming from $C^{(2)}_A$, is field dependent and is constructed with the 3-cocycle P^a_{ijk}.

VI. The BRS Invariants

We now can show how to obtain s invariant quantities from the preceding constructions. We have first to note that due to the translation term η in (3.2) the space of gauge and BRS transformations is, in general, too large to allow for the construction of s-invariant objects containing the physical fields only. For example, the Chern-Simons like terms constructed in the operator formulation, i.e., (4.9):

$$J = tr\left(\hat{\mathcal{A}}.\hat{\mathcal{F}} - \frac{1}{3}\, \hat{\mathcal{A}}.\hat{\mathcal{A}}.\hat{\mathcal{A}}\right) \tag{6.1}$$

such that $dJ = tr(\hat{\mathcal{F}}.\hat{\mathcal{F}})$ is not invariant under the BRS transformations (2.12, 2.13) due to the translation term $\hat{\gamma}$ in (4.7).

The main idea is, in fact, to quotient the original BRS algebra by an appropriate ideal in order to define another bigraded algebra Q in which there exist objects constructed with only generators of the Weil algebra $W(\mathcal{M})$ (i.e. the gauge fields and their curvatures) invariant under the BRS transformations. This is also equivalent to working in the BRS algebra $U(\mathcal{M})$ which in this case is no more free since one imposes algebraic relations among the generators leading to the construction of an ideal. Such ideal can be constructed, for example, by considering any element K of the s-cohomology of the BRS algebra, i.e. $sK = 0$, of homogeneous degree constructed from the ghosts and the fields which generate the minimal subalgebra of the BRS algebra with respect to the differential s (c.f. ref.[14] for more details). We consider now the subalgebra $\mathcal{T}$ of the BRS algebra generated by elements of the form $(K.p + dK.q)$ in which p and q are elements of the BRS algebra. Then it is easy to show that this subspace $\mathcal{T}$ is an ideal of the BRS algebra. Furthermore, we can show that $\mathcal{T}$ is stable by the action of d and s. Hence, the quotiented algebra $Q = \mathcal{U}(\mathcal{M})/\mathcal{T}$ is well defined and the two differentials d and s can be defined uniquely on it. The algebra Q inherits a bigraduation from the BRS algebra and is also a bigraded differential algebra since $\mathcal{T}$ is a bigraded ideal. This prescription also works for any "homogeneous Chevalley cocycle of the BRS algebra", i.e. a set $\{K^i, i = 1, \cdots, n\}$ of elements K^i taking values in a representation of dimension n of the original Lie (super)-algebra and such that $\nabla^i_j K^j = 0$ where ∇^i_j is the covariant s operator in this representation.

It is now easy, at least formally, to see how these procedures work in the general case by using the operator formalism developed in Section IV. As can be seen from eq. (4.20), $J_0^{(n)}$ which contains only the physical field is not an s-invariant modulo d due to the presence of a nontrivial $I_1^{(n)}$ term which, in general, is not an s-invariant.

However, as we have noted already from eq. (4.19), there always exists an object $I_N^{(n)} \neq 0$ which is an s–invariant of the BRS algebra. Therefore, we can obtain a quotiented BRS algebra by imposing in a consistent way the s–invariant constraints $I_N^{(n)} = 0$ and $dI_N^{(n)} = 0$. In this new BRS algebra, we obtain that $I_{N-1}^{(n)}$ is an s–invariant. Hence, we can construct a nested series of BRS algebras, the last one being defined (consistently) by $I_j^{(n)} = 0$ for $j \geq 1$. In this new BRS algebra we have then $dJ_0^{(n)} = I_0^{(n)}, sI_0^{(n)} = 0$ and $dJ_1^{(n)} + sJ_0^{(n)} = 0$. Hence, $J_0^{(n)}$ is an s–invariant modulo d, which can be used for constructing non-trivial s–invariant quantities. Of course, such a set of consistent but implicit constraints have to be carefully analyzed in each case, since they are giving algebraic relations among the generators of the BRS algebra. Their role is, in fact, to put restrictions on the ghost fields of the BRS algebra and, in particular, to avoid arbitrary values of the ghosts $\mathcal{F}_p$ in (2.12, 2.13) since otherwise they could be used o completely gauge away the $\mathcal{A}_p$ and hence $\mathcal{A}_0$, with the exception of its one d form part. Note also that in the new BRS algebra, since $I_j^{(n)} = 0$ for $j \geq 1$, eq. (4.20) define a series of descent equations which generalize the usual one[3-9]. We now show two examples which use this procedure.

Example 1:

We consider the dual of a Lie algebra $\mathcal{G}$, generated by $\{\chi^\alpha\}$ in degree one and we add a generator of degree two β in the scalar representation of $\mathcal{G}$, the action of the differential s being given by:

$$\begin{aligned} s\chi^\alpha &= -\frac{1}{2}\ C^\alpha_{\beta\gamma}\ \chi^\beta\ \chi^\gamma \\ s\beta &= -\frac{1}{3}\ tr\chi^3\,. \end{aligned} \tag{6.2}$$

It is easy to see that $tr\chi^3$ is a 3–cocycle of $\mathcal{G}$ in the scalar representation. The associated Weil algebra is generated by connections and curvatures A^α and F^α in degrees one and two, and B and H in degrees two and three, with respect to the differential d. Its action is given by:

$$\begin{aligned} dA^\alpha &= F^\alpha - \frac{1}{2} C^\alpha_{\beta\gamma} \; A^\beta \, A^\gamma \\ dF^\alpha &= C^\alpha_{\beta\gamma} \; F^\beta \, A^\gamma \\ dB &= H - \frac{1}{3} \, tr \; A^3 \\ dH &= tr \; FA^2 \, . \end{aligned} \tag{6.3}$$

The associated BRS algebra can be defined by making the following translations: $d \to d + s, A \to A + \chi, F \to F, B \to B + b + \beta$ and $H \to H + h$. It is a bigraded algebra such that the action of the differentials d and s on some of its generators is given by (in addition to (6.2) and (6.3)):

$$\begin{aligned} sA^\alpha + d\chi^\alpha &= C^\alpha_{\beta\gamma} \; A^\beta \; \chi^\gamma \\ sF^\alpha &= C^\alpha_{\beta\gamma} \; F^\beta \, \chi^\gamma \\ sB + db &= h - tr \; \chi A^2 \\ sb + d\beta &= -tr \; \chi^2 A \\ sH + dh &= 2tr \;\; F\chi A \\ sh &= tr \; F\chi^2 \, . \end{aligned} \tag{6.4}$$

From eqs.(6.4), it can be easily seen that $(h - trF\chi)$ is an s-invariant. So we can obtain another algebra by quotienting this one with the ideal generated by the elements $(h - tr \;\; F\chi)$ and $d(h - tr \;\; F\chi)$. This new algebra is stable by d and s.

Its particularity is that one recovers the usual Chern–Simons term $tr(AF - \frac{1}{3}A^3)$ in the equation for dB. This can be shown easily by imposing in eqs.(6.3, 6.4) that $h = tr\ F\chi$ and by defining $H' = H - tr\ AF$. The d and s actions write as follows:

$$\begin{aligned} dB &= H' + tr\ \left(AF - \frac{1}{3}\,A^3\right) \\ sB + db &= tr\,\chi\left(F - A^2\right) \\ sb + d\beta &= -tr\ \chi^2 A \\ s\beta &= -\frac{1}{3}\ tr\ \chi^3 \end{aligned} \tag{6.5}$$

and:

$$\begin{aligned} dH' &= -tr\ F^2 \\ sH' &= 0\ . \end{aligned}$$

Notice that we have obtained an invariant curvature H' which can be used to construct an invariant action. This quotiented algebra is the usual one used in theories containing a two–form antisymmetric tensor field B coupled via a Chern–Simons term to one–form gauge potentials, see e.g.[26,31].

Example 2:

The same prescription can also be used in the generic example of Section V where we have constructed the Weil algebra and given d–invariants. The associated BRS algebra can be constructed by making the usual translations $d \to d + s, A_1^\alpha \to A_1^\alpha + \chi_1^\alpha, A_2^\alpha \to A_2^\alpha + \Gamma_1^2 + \chi_2^\alpha, F_2^\alpha \to F_2^\alpha, F_3^\alpha \to F_3^\alpha + \theta_1^\alpha$ with $\alpha = a, i$ and $a = 1, \cdots, N,\ \ i = 1, \cdots, n$. Here the subscripts of χ, Γ and θ stand for the ghost degree (degree with respect to s), the total s plus d degree being kept constant in each translation.

The minimal subalgebra of this BRS algebra with respect to the differential s is generated by the elements $\chi_1^a, \chi_2^a, F_2^a$ and θ_1^a. The action of the differential s can be given in the language by introducing the following matrix $\hat{\theta}_1$:

$$
\begin{aligned}
\left(\hat{\theta}_1\right)^a_b &= C^a_{bc}\ \theta^c_1 & \left(\hat{\theta}_1\right)^i_j = -D^i_{aj}\ \theta^a_1 \\
\left(\hat{\theta}_1\right)^a_i &= -P^a_{ijk}\ F^j_2\ \chi^k_1 & \left(\hat{\theta}_1\right)^i_a = D^i_{aj}\ \theta^j_1
\end{aligned}
\tag{6.6}
$$

and using the already defined operators $\hat{\chi}_1, \hat{\chi}_2$ and $\hat{F}_2$ of Section V:

$$
\begin{aligned}
s\hat{\chi}_1 &= -\hat{\chi}_1.\hat{\chi}_1 \\
s\hat{\chi}_2 &= -\hat{\chi}_1.\hat{\chi}_2 + \hat{\chi}_2.\hat{\chi}_1 \\
s\hat{F}_2 &= \hat{F}_2 \cdot \hat{\chi}_1 - \hat{\chi}_1 \cdot \hat{F}_2 \\
s\hat{\theta}_1 &= \hat{F}_2.\hat{\chi}_2 - \hat{\chi}_2.\hat{F}_2 - \hat{\theta}_1.\hat{\chi}_1 - \hat{\chi}_1.\hat{\theta}_1 \,.
\end{aligned}
\tag{6.7}
$$

Then it is easy to show that the quantity

$$
I^{(1)} = tr\left(\hat{\theta}_1.\hat{F}_2\right)
\tag{6.8}
$$

is a non trivial s-invariant constructed with only the generators of this minimal BRS subalgebra and corresponds to the Chern-Simons terms I and J in eqs.(5.11, 5.12). Hence, we can now obtain the quotient of the original BRS algebra by the ideal generated by elements of the form $I^{(1)}.p + dI^{(1)}.q$, p and q being any element of the BRS algebra. In particular, this quotiented algebra contains an object s-invariant modulo d built only with generators of the Weil algebra. It is the Chern-Simons term

J given in Section V. One has:

$$sJ = -dJ^{(1)} + I^{(1)} \tag{6.9}$$

where:

$$J^{(1)} = \frac{1}{2}\, tr\left(\hat{\Gamma}_1 \hat{F}_2 + \hat{\chi}_1 \hat{F}_3 + \hat{A}_1 \hat{\theta}_1 - 2\hat{\chi}_1 \hat{A}_1 \hat{A}_2 - \hat{A}_1 \hat{A}_1 . \hat{\Gamma}_1\right) \tag{6.10}$$

and $\hat{\Gamma}_1$ is given by:

$$\begin{aligned} \left(\hat{\Gamma}_1\right)^a_b &= C^a_{bc}\ \Gamma^c_1, \quad \left(\hat{\Gamma}_1\right) = -D^i_{aj}\ \Gamma^a_1 \\ \left(\hat{\Gamma}_1\right)^i_a &= D^i_{aj}\ \Gamma^j_1, \quad \left(\hat{\Gamma}_1\right)^a_i = -P^a_{ijk}\ A^j_1\ \chi^n_1 \end{aligned} \tag{6.11}$$

with

$$s\hat{F}_3 = -d\hat{\theta}_1 - \hat{A}_1 \cdot \hat{\theta}_1 - \hat{\theta}_1 \cdot \hat{A}_1 - \hat{F}_3 \cdot \hat{\chi}_1 - \hat{\chi}_1 \cdot \hat{F}_3 - \hat{\Gamma}_1 \cdot \hat{F}_2 + \hat{F}_2 \cdot \hat{\Gamma}_1 \tag{6.12}$$

$J^{(1)}$ is a three–form with ghost degree one. All these formulae are obtained from the relation $dJ = I$ and $dI = 0$ of Section V by making the translations $d \to d + s, J \to J + J^{(1)} + \cdots, I \to I + I^{(1)}$. Then taking into account $I^{(1)} \equiv 0$ which defines the ideal. We have, in the quotiented algebra:

$$sJ = -dJ^{(1)}. \tag{6.13}$$

J is a four–form containing only the gauge fields and their curvatures which can then be used in constructing an s–invariant action.

VII. Conclusions

In this article, we have shown how to obtain generalized non–linear BRS and gauge transformations from higher order Lie algebra cocycles and acting on p–form gauge fields ($p \geq 1$), using the concept of free minimal differential algebras. These algebraic structures are of interest in the context of supergravity theories, and for theories based on the group manifold approach. Along this line, we have generated algebras of gauge transformations containing a non–trivial field dependent two–cocycle. We have developed an operator formalism through a generalized adjoint mapping, i.e., associating to n objects of a non semi simple Lie algebra, an operator acting in its adjoint representation defined through an $(n+1)$ cocycle. This formalism allowed us to obtain a convenient treatment of the non–linearities of our gauge and BRS transformations and to find a way for constructing associated s–invariant objects. This was achieved by defining a quotiented BRS algebra by an appropriate ideal. Resulting Chern–Simons like terms can be obtained in any space time dimensions , as in the example of Section VI and which can be used to construct s–invariant actions. The quantization of such theories, that exhibit the ghost for ghost mechanism, will also require the incorporation of the corresponding anti–ghost fields, the algebraic structure of which has not been discussed here. Finally, let us note that we have obtained a generalization of the so–called 'descent equations' that will be useful in the discussion of possible consistent anomalies for these theories at the quantum level.

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Appendix A

We start from eq. (2.4) rewritten for $\mathcal{A}$:

$$\sum_{p,q\geq 2} \frac{1}{(p-1)!q!} C^{(p)}\left(\mathcal{A},\cdots,\mathcal{A},C^{(q)}(\mathcal{A},\cdots,\mathcal{A})\right) . \tag{A.1}$$

Then we make a translation $\mathcal{A} \to \mathcal{A} + \sum_{i=1}^{q} \theta^i \nu_i = \hat{\mathcal{A}}$ where $\hat{\mathcal{A}}$ is an homogenous quantity of odd degree, element of $U(\mathcal{M}) \otimes \eta$, the ν_i are element s of even degree of $U(\mathcal{M}) \otimes \eta$, the odd degrees being contained in the θ^i which are anticommuting variables. It is then easy to write the translated of relation (A.1) and to extract the term containing all the $\nu_i, i = 1, \cdots q$, only once each, which has to be zero. Taking into account the fact that the variables θ^i anticommute among themselves and with $\mathcal{A}$, and that we have the relation $C^{(p)}\theta^i = (-1)^{p+1}\theta^i C^{(p)}$, we obtain after some algebra eq. (2.8), the $C^{(p)}_{\mathcal{A}}$ being defined by eq. (2.7). We give now some examples that we use in the article:

$$q = 0 \quad C^{(1)}_{\mathcal{A}}\left(C^{(0)}_{\mathcal{A}}\right) = 0 \tag{A.2}$$

$$q = 1 \quad C^{(1)}_{\mathcal{A}}\left(C^{(1)}_{\mathcal{A}}(\nu_1)\right) - C^{(2)}_{\mathcal{A}}\left(C^{(0)}_{\mathcal{A}}, \nu_1\right) = 0 \tag{A.3}$$

$$\begin{aligned} q = 2 \quad & C^{(1)}_{\mathcal{A}}\left(C^{(2)}_{\mathcal{A}}(\nu_1,\nu_2)\right) - C^{(2)}_{\mathcal{A}}\left(\nu_1, C^{(1)}_{\mathcal{A}}(\nu_2)\right) \\ & - C^{(2)}_{\mathcal{A}}\left(C^{(1)}_{\mathcal{A}}(\nu_1), \nu_2\right) + C^{(3)}_{\mathcal{A}}\left(C^{(0)}_{\mathcal{A}}, \nu_1, \nu_2\right) = 0 . \end{aligned} \tag{A.4}$$

We wish to remark that the relation (A.4) is valid also for an odd variable ν_2 with ν_1 being an even element of $U(\mathcal{M}) \otimes \eta$. We give also the relation for variables W_i all odd:

$$\sum_{r=0}^{q} \sum_{\pi} C^{(r+1)}_{\mathcal{A}}\left(W_{\pi_1},\cdots,W_{\pi_r}, C^{(q-r)}\left(W_{\pi_{r+1}},\cdots,W_{\pi_q}\right)\right) = 0 . \tag{A.5}$$

Appendix B

Using the fact that $dC^{(p)} = (-1)^p C^{(p)} d$ we obtain from eq.(2.7) with the differential d not acting on the elements ν_i:

$$\begin{aligned} \left(dC_{\mathcal{A}}^{(p)}\right)(\nu_1,\cdots,\nu_p) &= \sum_{\substack{q\geq p \\ q\geq 2}} \frac{(-1)^{p+1}}{(q-p-1)!} C^{(q)}(\mathcal{A},\cdots,\mathcal{A},d\mathcal{A},\nu_1,\cdots,\nu_p) \\ &= (-1)^{p+1} C_{\mathcal{A}}^{(p+1)}(d\mathcal{A},\nu_1,\cdots,\nu_p) \quad . \end{aligned} \tag{B.1}$$

In particular, for $p = 1$, we obtain, using also eq. (A.3):

$$\begin{aligned} \left(dC_{\mathcal{A}}^{(1)}\right)(\nu) &= C_{\mathcal{A}}^{(2)}(d\mathcal{A},\nu) = C_{\mathcal{A}}^{(2)}\left(\mathcal{F} - C_{\mathcal{A}}^{(0)},\nu\right) \\ &= C_{\mathcal{A}}^{(2)}(\mathcal{F},\nu) - C_{\mathcal{A}}^{(1)}\left(C_{\mathcal{A}}^{(1)}(\nu)\right) \quad . \end{aligned} \tag{B.2}$$

Hence,

$$d\hat{\mathcal{A}} = \hat{\mathcal{F}} - \hat{\mathcal{A}}.\hat{\mathcal{A}} \tag{B.3}$$

We obtain in the same way eq. (4.6).

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