Fermi National Accelerator Laboratory

FERMILAB-Pub-88/26

On Estimating Mean Lifetimes in the Presence of Background*

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February 15, 1988

*Submitted to Phys. Rev. D

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

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Abstract

The exact sampling theory pertaining to the problem of estimating the mean lifetime of a particle, when the data sample contains contributions from several background sources, is developed in detail. Measurement errors are taken into account, on an event by event basis, by assuming the experimental resolution function to be a Gaussian distribution of known variance and zero mean. While the results presented are valid for samples of any size they are more suited to the analysis of data samples containing few events. In this sense they offer a method of analysis which is complementary to the method of maximum likelihood.

1 Introduction

We have witnessed, in the past few years, a resurgence of interest in the measurement of the mean lifetimes of particles. This is due, in part, to the relatively recent availability of high precision vertex detectors, and, in large measure, to the recognition that these measurements provide a valuable means of testing certain aspects of the standard model.

It is well-known that a weighted sum of unbiased lifetime measurements provides an unbiased estimate of the mean lifetime. A method of calculating exact confidence intervals for this sum was presented in a recent paper [1]. Those results apply when the background level is negligible. In general, however, a sample of measurements will contain contributions from background events which must be accounted for in order to ensure that the estimate of the mean lifetime is free from background induced bias. The usual method of analysis is that of maximum likelihood. (See, for example, Ref. [2].) It should be emphasized, however, that a rigorous basis for its use exists only for samples which are "sufficiently large" [3]. There are, and will continue to be, instances in which, because of the nature of the decay channel being measured, the data samples obtained will contain very few events; then there is no guarantee that the method of maximum likelihood is optimal: lifetime estimates will, in general, be biased and "standard error" intervals will not necessarily be standard, that is, have probability content equal to 0.683.

Moreover, it is common practice to form "world" averages of estimates from different experiments. Clearly, that task is made easier by having estimates with as little bias as possible, and having "errors" whose interpretation is statistically unambiguous. Therefore, if a rigorous unbiased method of analysis exists for a particular problem, at the very least, that method ought to be explored even if, as is ordinarily the case, the variance of an unbiased estimate is larger than that of one which is biased. The purpose of this paper is to present such an analysis using well-accepted methods of mathematical statistics. It is hoped, that the formulas presented here offer a practical solution to the estimation of particle lifetimes in the presence of background, and in particular when an analysis must be based on very few events.

In Sec. 2 the sampling theory for the problem is presented and in Sec. 3 we show how the results are to be used in practice. Concluding remarks are made in Sec. 4.

2 Theory

2.1 The general case

In the notation of Ref. [1]

$$t = \sum_{n=1}^{N} c_n t_n, \tag{1}$$

is the weighted average of N lifetime measurements, with $\sum_{n=1}^{N} c_n = 1$. The experimental procedure leading to the sample $\{t_n\}$ will be unbiased if the mean values $< t_n >$ are linearly related to the mean lifetime to be estimated. In this case, a suitable linear function of the estimator in Eq. (1) will yield unbiased estimates of the mean lifetime. The estimator t is very simple; the whole difficulty lies in calculating the associated "errors". To do so we need the probability density function (PDF) of t. It is given by

$$P(t|\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} F(\omega) d\omega, \tag{2}$$

where

$$F(\omega) = \prod_{n=1}^{N} \int_{-\infty}^{\infty} e^{-i\omega c_n t_n} P(t_n) dt_n. \tag{3}$$

The function $P(t_n)$ describes the parent population from which the singular sample t_n is drawn; $P(t_n)$ will be a weighted sum of the PDF's for the signal and the various sources of background. If there are M different sub-populations contributing to the parent population we can write

$$P(t_n) = \sum_{m=1}^{M} f_m p(t_n|\tau_m), \tag{4}$$

with

$$\sum_{m=1}^{M} f_m = 1, \tag{5}$$

and where f_m is the weight of sub-population m and τ_m is the corresponding mean lifetime, or a linear function thereof; the functions $p(t_n|\tau_m)$ are the normalized densities which describe the sub-populations. The M weights and the $M-1$ mean lifetimes of the background sub-populations are assumed known. (Usually, these parameters are known with some uncertainty, in which case their variation within reasonable bounds will determine the *systematic* uncertainty in the estimate.) We shall suppose that the mean lifetime to be estimated is $\tau \equiv \tau_1$.

In the absence of measurement errors, each term in $P(t_n)$ is a pure exponential. However, in experiments for which the results given here are most useful, namely colliding beam experiments, the measurement errors usually cannot be neglected; $P(t_n)$ should then be smeared out by the experimental resolution function. We shall take this function to be a Gaussian distribution of zero mean and known variance σ_n^2. When smeared, each term in $P(t_n)$ assumes the form [4]

$$p(t_n|\tau_m) = \frac{1}{2\tau_m}\exp(\sigma_n^2/2\tau_m^2)\exp(-t_n/\tau_m)\mathrm{erfc}(\frac{\sigma_n\tau_m^{-1} - \sigma_n^{-1}t_n}{\sqrt{2}}), \tag{6}$$

where erfc(x) is the complementary error function:

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \epsilon^{-y^2}dy.$$

Note, that by allowing σ to depend on n we are allowing for the possibility that the resolution function depends on the nature of the event; ideally, the resolution function would be event independent.

Substituting Eq. (4) into Eq. (3) yields

$$F(\omega) = \prod_{n=1}^{N}\sum_{m=1}^{M} f_m \int_{-\infty}^{\infty} \epsilon^{-ic_n t_n\omega} p(t_n|\tau_m)dt_n, \tag{7}$$

which, upon performing the integrals, using the result

$$\begin{aligned}\int_{-\infty}^{t} \epsilon^{-az}\mathrm{erfc}\left(\frac{\sigma b - \sigma^{-1}z}{\sqrt{2}}\right)dz &= \frac{1}{a}\epsilon^{\sigma^2(a^2/2-ab)}\mathrm{erfc}\left(\frac{\sigma b - \sigma a - \sigma^{-1}t}{\sqrt{2}}\right)\\ &- \frac{1}{a}\epsilon^{-at}\mathrm{erfc}\left(\frac{\sigma b - \sigma^{-1}t}{\sqrt{2}}\right),\end{aligned} \tag{8}$$

becomes

$$F(\omega) = E(\omega)G(\omega). \tag{9}$$

where

$$E(\omega) = \prod_{n=1}^{N} \sum_{m=1}^{M} \frac{\alpha_{nm} f_m}{i(\omega - i\alpha_{nm})}, \tag{10}$$

$$G(\omega) = e^{-\sigma^2\omega^2/2}, \tag{11}$$

$$\sigma^2 \equiv \sum_{n=1}^{N} c_n^2 \sigma_n^2, \tag{12}$$

and

$$\alpha_{nm} \equiv \frac{1}{c_n \tau_m}. \tag{13}$$

The PDF, $P(t|\tau)$, can be expressed as the convolution

$$P(t|\tau) = \int_0^\infty e(x) g(t-x) dx, \tag{14}$$

of the functions

$$e(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} E(\omega) d\omega, \tag{15}$$

and

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} G(\omega) d\omega, = \frac{e^{-t^2/2\sigma^2}}{\sigma\sqrt{2\pi}}. \tag{16}$$

We observe that, in general, no two of the matrix elements α_{nm} will have *precisely* the same value whence the function $E(\omega)$, regarded as a function of a complex variable, will have simple poles at $\omega = i\alpha_{nm}$; Eq. (15) can therefore be readily evaluated by contour integration. The result is

$$e(t) = \sum_{k=1}^{N} \sum_{j=1}^{M} e^{-\alpha_{kj} t} \alpha_{kj} W_{kj}, \tag{17}$$

where we have defined:

$$W_{kj} \equiv f_j \prod_{n\neq k}^{N} \sum_{m=1}^{M} \frac{\alpha_{nm} f_m}{(\alpha_{nm} - \alpha_{kj})}. \tag{18}$$

We note that the function $e(t)$ is just the PDF of t when there are no measurement errors. From this function one can derive the interesting identities

$$\begin{aligned}
\sum_{k=1}^{N}\sum_{j=1}^{M} \alpha_{kj}^{n} W_{kj} &= \delta_{nN}(-1)^{N+1} \prod_{k=1}^{N}\sum_{j=1}^{M} \alpha_{kj} f_j, \quad n > 0, \\
\sum_{k=1}^{N}\sum_{j=1}^{M} W_{kj} &= 1\ , \\
\sum_{k=1}^{N}\sum_{j=1}^{M} \frac{W_{kj}}{\alpha_{kj}} &= \sum_{k=1}^{N}\sum_{j=1}^{M} \frac{f_j}{\alpha_{kj}}, \\
\sum_{k=1}^{N}\sum_{j=1}^{M} \frac{W_{kj}}{\alpha_{kj}^2} &= \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{M}\sum_{n=1}^{N}\sum_{m=1}^{M} \frac{(1-\delta_{kn}) f_j f_m}{\alpha_{kj}\alpha_{nm}} + \sum_{k=1}^{N}\sum_{j=1}^{M} \frac{f_j}{\alpha_{kj}^2},
\end{aligned} \tag{19}$$

which are a generalization of the those given in Ref. [1]. A proof of the first identity is presented in Appendix A; the other three follow most readily from the moments of $e(t)$.

We now complete our derivation of the densities pertaining to the estimator t. From Eq. (14) we obtain

$$P(t|\tau) = \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{M} e^{X_{kj}} \mathrm{erfc}(Y_{kj}) \alpha_{kj} W_{kj}. \tag{20}$$

where

$$X_{kj} \equiv -\alpha_{kj} t + \sigma^2 \alpha_{kj}^2 / 2, \tag{21}$$

and

$$Y_{kj} \equiv \frac{\sigma\alpha_{kj} - \sigma^{-1} t}{\sqrt{2}}. \tag{22}$$

The cumulative distribution function (CDF), $C(t|\tau)$, is defined by

$$C(t|\tau) = \int_{-\infty}^{t} P(z|\tau)dz. \tag{23}$$

Applying Eq. (8) to the above yields:

$$\begin{aligned} C(t|\tau) &= 1 - \frac{1}{2}\mathrm{erfc}\left(\frac{t}{\sigma\sqrt{2}}\right) \\ &- \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{M} e^{X_{kj}}\mathrm{erfc}(Y_{kj})W_{kj}. \end{aligned} \tag{24}$$

Finally, the moment generating function (MGF), $M(\beta)$, defined by

$$M(\beta) = \int_{-\infty}^{\infty} e^{\beta t}P(t|\tau)dt, \tag{25}$$

can be expressed as:

$$M(\beta) = \sum_{k=1}^{N}\sum_{j=1}^{M} \frac{\alpha_{kj}e^{\sigma^2\beta^2/2}}{\alpha_{kj} - \beta} W_{kj}. \tag{26}$$

The moments M_n can be explicitly evaluated, using $M(\beta)$; we find:

$$M_n = n!\sum_{k=1}^{N}\sum_{j=1}^{M} \frac{W_{kj}}{\alpha_{kj}^{n}} \sum_{r=0}^{[n/2]} \frac{(\sigma\alpha_{kj})^{2r}}{2^r r!}. \tag{27}$$

where $[n/2]$ stands for the integer part of $n/2$. In particular, the first two moments $M_1 \equiv < t >$ and $M_2 \equiv < t^2 >$ of $P(t|\tau)$ are:

$$\begin{aligned} < t > &= \sum_{k=1}^{N}\sum_{j=1}^{M} \frac{W_{kj}}{\alpha_{kj}} \\ &= \sum_{j=1}^{M} f_j\tau_j, \end{aligned} \tag{28}$$

and

$$\begin{aligned} < t^2 > &= \sigma^2 + 2\sum_{k=1}^{N}\sum_{j=1}^{M} \frac{W_{kj}}{\alpha_{kj}^2} \\ &= \sigma^2 + \left(2\sum_{j=1}^{M} f_j\tau_j^2 - < t >^2\right)\sum_{k=1}^{N} c_k^2 + < t >^2 . \end{aligned} \tag{29}$$

By definition, the variance of $P(t|\tau)$ is $V[t] = <t^2> - <t>^2$; therefore,

$$V[t] = \sum_{k=1}^{N} c_k^2 V[t_k], \tag{30}$$

with

$$V[t_k] = \sigma_k^2 + 2\sum_{j=1}^{M} f_j \tau_j^2 - <t>^2 . \tag{31}$$

2.2 Special cases

Hitherto, we have made no assumptions regarding the magnitude of the mean lifetimes of the background sub-populations. However, in many experiments the background is principally from particles with mean lifetimes very much shorter than the lifetime being measured. Let us suppose that this source of background corresponds to sub-population M and is characterized, to a good approximation, by setting $\tau_M = 0$. To analyze this case it is convenient to express W_{kj} as

$$W_{kj} = f_j \prod_{n\neq k}^{N} \left(\frac{c_k f_j}{c_k - c_n} - c_k \tau_j \sum_{m\neq j}^{M} \frac{f_m}{c_k \tau_j - c_n \tau_m} \right) . \tag{32}$$

We observe that in $P(t|\tau)$ and $C(t|\tau)$ the functions $\exp(X_{kM})\mathrm{erfc}(Y_{kM})$ go to zero exponentially as $\tau_M \to 0$; therefore, we need only consider terms with $j < M$, hence

$$W_{kj} = f_j \prod_{n\neq k}^{N} \left(\frac{c_k f_j}{c_k - c_n} - f_M - c_k \tau_j \sum_{m\neq j}^{M-1} \frac{f_m}{c_k \tau_j - c_n \tau_m} \right) . \tag{33}$$

For the simple, but important, case in which all background sources are due to short-lived particles $M = 2$ and, therefore, only the first two terms within the parentheses will remain in Eq. (33), while the expressions for the

PDF and the CDF will contain only the sum over k; for example, writing $\alpha_k \equiv \alpha_{k1} = (c_k\tau)^{-1}$, and letting f be the background fraction, the CDF would be:

$$
\begin{aligned}
C(t|\tau) &= 1 - \frac{1}{2}\mathrm{erfc}(\frac{t}{\sigma\sqrt{2}}) \\
&- \frac{1-f}{2}\sum_{k=1}^{N} e^{-\alpha_k t+\sigma^2\alpha_k^2/2}\mathrm{erfc}(\frac{\sigma\alpha_k - \sigma^{-1}t}{\sqrt{2}}) \\
&\times \prod_{n\neq k}^{N}[(1-f)\frac{c_k}{c_k - c_n} + f]. \qquad (34)
\end{aligned}
$$

The other special case we shall consider is that in which the weights c_k are all equal. Equal weights would be appropriate when the variances of the individual measurements are of equal, or very similar, magnitude. Below we derive an expression for $C(t|\tau)$ by taking the limit of Eq. (24) as $c_k \to 1/N$.

Let us write

$$\alpha_{kj} = (1+\epsilon_k)\lambda_j \qquad (35)$$

where $\lambda_j = N/\tau_j$ and the ϵ_k are numbers in the neighborhood of zero. It is convenient to define the quantities

$$
\begin{aligned}
X_j &\equiv -\lambda_j t + \sigma^2\lambda_j^2/2, \\
Y_j &\equiv (\sigma\lambda_j - \sigma^{-1}t)/\sqrt{2}, \\
Z_j &\equiv \sigma\lambda_j/\sqrt{2}, \qquad (36)
\end{aligned}
$$

in terms of which X_{kj} and Y_{kj} may be written as

$$X_{kj} = X_j + \epsilon_k Z_j(2Y_j + \epsilon_k Z_j), \qquad (37)$$

and

$$Y_{kj} = Y_j + \epsilon_k Z_j. \qquad (38)$$

We now expand the function

$$e^{X_{kj}}\ \mathrm{erfc}(Y_{kj})\ W_{kj}$$

in powers of ϵ_k. The result is

$$e^{X_j} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{\mathrm{erfc}^{(n)}(Y_j)}{n!\ l!} \sum_{i=0}^{l} \binom{l}{i}\ (2Y_j)^{l-i}\ (\epsilon_k Z_j)^{n+l+i}\ W_{kj}, \tag{39}$$

where

$$\begin{aligned}
\mathrm{erfc}^{(0)}(x) &\equiv \mathrm{erfc}(x), \\
\mathrm{erfc}^{(n)}(x) &\equiv \frac{d^n}{dx^n}\mathrm{erfc}(x), \\
&= \frac{2}{\sqrt{\pi}}(-1)^n e^{-x^2} H_{n-1}(x),
\end{aligned}$$

and $H_n(x)$ are the Hermite polynomials [5].

Let us write Eq. (39) in terms of the new index $r = n + l + i$. The condition $i \geq 0 \Rightarrow \max[l] = r - n$; the condition $i \leq l \Rightarrow \min[l] =$ integer part of $(r - n - 1)/2$, while the bounds on $l \Rightarrow n \leq r$. With these bounds on l we recognise the sum over l to be proportional to a modified Hermite polynomial, $h_n(x) \equiv i^{-n} H_n(ix)$; in fact, the sum equals $h_{r-n}(Y_j)/(r-n)!$. Equation (39) then becomes:

$$e^{X_j} \sum_{r=0}^{\infty} \frac{(-\epsilon_k)^r W_{kj}}{r!} (-Z_j)^r \sum_{n=0}^{r} \binom{r}{n} \mathrm{erfc}^{(n)}(Y_j) h_{r-n}(Y_j). \tag{40}$$

Consider the sum over k of the expression in Eq. (40); the only term which depends on ϵ_k is the sum:

$$V_{jr} \equiv \sum_{k=1}^{N} (-\epsilon_k)^r W_{kj}$$

$$= \sum_{k=1}^{N}(-\epsilon_k)^r [\prod_{n\neq k}^{N}\frac{1}{\epsilon_n-\epsilon_k}]$$
$$\times \; f_j \prod_{n\neq k}^{N}[f_j+\epsilon_n f_j+(\epsilon_n-\epsilon_k)\sum_{m\neq j}^{M}\frac{\alpha_{nm}f_m}{\alpha_{nm}-\alpha_{kj}}]. \tag{41}$$

In view of the identity

$$\sum_{k=1}^{N}(-\epsilon_k)^r \prod_{n\neq k}^{N}\frac{1}{\epsilon_n-\epsilon_k}=\delta_{rN-1}, \quad r\leq N-1, \tag{42}$$

which is a special case of the first identity listed in Eq. (19), the sum V_{jr} involves subtle cancellations, consequently, the limit $\epsilon_k \to 0$ must be analyzed with some care. This analysis is presented in Appendix B, together with the general result for V_{jr}. Note, also, that the sum in Eq. (42) is $O(\epsilon)$ for $r > N-1$ which implies that, for these values of r, $V_{jr} \to 0$ as $\epsilon_n \to 0$.

Finally, collecting together all the pieces, we arrive at the expression

$$\begin{aligned} C(t;\tau) &= 1-\frac{1}{2}\mathrm{erfc}(\frac{t}{\sigma\sqrt{2}}) \\ &- \frac{1}{2}\sum_{j=1}^{M}e^{X_j}\sum_{r=0}^{N-1}\frac{V_{jr}}{r!}(-Z_j)^r\sum_{n=0}^{r}\binom{r}{n}\mathrm{erfc}^{(n)}(Y_j)h_{r-n}(Y_j). \end{aligned} \tag{43}$$

for the CDF in the limit of equal weights, that is, for the estimator:

$$t=\frac{1}{N}\sum_{n=1}^{N}t_n.$$

It is worth remarking that a slight generalisation of Eq. (43) can also be used to calculate $C(t;\tau)$ when the ϵ_n are not necessarily close to zero. In this case the sum over r would extend to infinity and the coefficients V_{jr} would have to be calculated using Eq. (41). Owing to the rapid increase in the magnitude of W_{kj} with sample size, it might be easier to calculate $C(t|\tau)$ using the expanded form in lieu of the closed expression given in Eq. (24).

3 Application to lifetime measurements

The estimator t yields an unbiased estimate of $< t >$, the mean of the density $P(t|\tau)$; we want, however, an unbiased estimate of τ. A suitable estimator, t', follows immediately from Eq. (28): namely,

$$t' = (t - \sum_{j=2}^{M} f_j \tau_j)/f_1, \tag{44}$$

which is evidently an unbiased estimator for $\tau \equiv \tau_1$. The variance pertaining to t' is

$$V[t'] = V[t]/f_1^2. \tag{45}$$

As noted in Ref. [1], a complete estimate of the mean lifetime should specify both an estimate of the variance, which is needed to compute world averages, and a confidence interval (which by convention is normally a central interval at 68.3 % confidence level). Exact (central) confidence intervals $[\underline{\tau}(t'), \overline{\tau}(t')]$ can be obtained from the equations

$$\begin{aligned} C(t(t')|\overline{\tau}) &= (1-\beta)/2, && (46) \\ C(t(t')|\underline{\tau}) &= (1+\beta)/2, && (47) \end{aligned}$$

where here β is the desired level of confidence. The above equations can be solved by a straightforward application of the Newton-Raphson method. The intervals so obtained are exact in the sense that the statement $\mathrm{Prob}(t' \in [\underline{\tau}(t'), \overline{\tau}(t')]) = \beta$ is exactly true.

We turn now to the construction of an unbiased estimator for the variance. To this end we could try to generalize the estimator given previously [1]; however, here we shall proceed somewhat differently. It is evident that t^2 yields an unbiased estimate of $< t^2 >$, therefore, if we can find an unbiased estimator for $< t >^2$ our problem is solved. In fact such an estimator can always be found for all samples with $N \geq 2$.

Consider the double sum

$$\sum_{k=1}^{N} \sum_{n=1}^{N} A_{kn} c_k c_n t_k t_n.$$

The coefficients A_{kn} must be chosen so that the expectation value of the above is equal to $< t >^2$. We note that for $k \neq n$ t_k and t_n are independent; therefore, in this case $< t_k t_n >=< t_k >< t_n >$ which is just $< t >^2$. The terms with $k = n$ must be excluded in order to exclude contributions from the second moment; hence. $A_{nm} \propto 1 - \delta_{kn}$. Finally, the constant of proportionality must be chosen so that the coefficient of $< t >^2$ is unity. Therefore,

$$V[t] \approx t^2 - \frac{\sum_{k=1}^{N} \sum_{n=1}^{N} (1 - \delta_{kn}) c_k c_n t_k t_n}{\sum_{k=1}^{N} \sum_{n=1}^{N} (1 - \delta_{kn}) c_k c_n}. \tag{48}$$

provides an unbiased estimate of the variance of t and therefore of t'.

Ideally, the weights c_k should be set $\propto 1/V[t_k]$ this being the optimal choice in that it minimizes the variance of t. In practice the terms in $V[t_k]$ involving the unknown parameters τ_1^2 and $< t >^2$ will either have to be replaced with estimates or simply dropped altogether.

4 Conclusions

If we have an unbiased sample of lifetime measurements (in the sense defined in Sec. 2), and perhaps containing contributions from background events, a linear function of the weighted average of these measurements is the simplest unbiased estimate of the mean lifetime. The results presented herein offer a rigorous alternative to lifetime analyses based on the principle of maximum likelihood. We have shown how in principle, for any sample size, exact confidence intervals and an unbiased estimate of the variance can be calculated, provided that the background contributions are known in the mean. It may, however, become increasingly difficult to calculate $C(t|\tau)$ with sufficient precision as the sample size increases. This is because of the rapid increase in the magnitude of W_{kj} with N. Therefore, insofar as the method developed here can be applied more easily to (arbitrarily) small samples than to large ones it may be regarded as being complementary to the method of maximum likelihood, which is known to be satisfactory for large samples.

5 Acknowledgement

This work was in supported in part by the U.S. Department of Energy under contract DE-AC02-76CH03000.

6 Appendix: A

Consider the function

$$g(t,\delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\prod_{n=1}^{N}\sum_{m=1}^{M}\frac{\alpha_{nm}f_m}{i(\omega - i\delta/t - i\alpha_{nm})}d\omega, \tag{49}$$

$$= e^{-\delta}\sum_{k=1}^{N}\sum_{j=1}^{M} e^{-\alpha_{kj}t}\alpha_{kj}W_{kj}$$

$$= -e^{-\delta}\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=1}^{N}\sum_{j=1}^{M}(-\alpha_{kj})^{n+1}W_{kj}. \tag{50}$$

where δ is any positive real number. Note, $e(t) = g(t,0)$. Changing to the variable $z = \omega t$ in Eq. (49) yields

$$g(t,\delta) = \frac{t^{N-1}}{2\pi}\int_{-\infty}^{\infty} e^{iz}\prod_{n=1}^{N}\sum_{m=1}^{M}\frac{\alpha_{nm}f_m}{i(z - i\delta - i\alpha_{nm}t)}dz. \tag{51}$$

We seek the limit of $g(t,\delta)$ as $t \to 0$ with δ held fixed. (A non-zero value of δ serves merely to avoid a pole on the real axis, thereby simplifying the calculation). The product term in Eq. (51) can be expressed as

$$\frac{1}{i^N(z - i\delta)^N}\prod_{n=1}^{N}\sum_{m=1}^{M}\alpha_{nm}f_m\left(1 - \frac{i\alpha_{nm}t}{z - i\delta}\right)^{-1},$$

which in the limit $t \to 0$ becomes

$$\frac{1}{i^N(z - i\delta)^N}\prod_{n=1}^{N}\sum_{m=1}^{M}\alpha_{nm}f_m + O(t).$$

Therefore, in this limit $g(t,\delta)$ reduces to

$$g(t,\delta) = \left(\prod_{n=1}^{N}\sum_{m=1}^{M}\alpha_{nm}f_m\right)\frac{t^{N-1}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{iz}}{i^N(z - i\delta)^N}dz$$

$$= \frac{t^{N-1}}{(N-1)!}e^{-\delta}\prod_{n=1}^{N}\sum_{m=1}^{M}\alpha_{nm}f_m + O(t^N). \tag{52}$$

A comparison of coefficients in Eqs. (50) and (52) leads to the first identity in Eq. (19).

7 Appendix: B

The f_j-dependent term in Eq. (41) can be written in the equivalent form:

$$\frac{1}{1+\epsilon_k}\prod_{n=1}^{N}[f_j+\epsilon_n f_j+(\epsilon_n-\epsilon_k)\sum_{m\neq j}^{M}\frac{\alpha_{nm}f_m}{\alpha_{nm}-\alpha_{kj}}]. \tag{53}$$

Let $D^{(q)}$ be the qth derivative of the product term with respect to $-\epsilon_k$, evaluated at $\epsilon_k=0$; then, Eq. (53) can be expressed as:

$$\begin{aligned}&\frac{1}{1+\epsilon_k}\sum_{i=0}^{\infty}\frac{D^{(i)}}{i!}(-\epsilon_k)^i\\ =&\sum_{i=0}^{\infty}(-\epsilon_k)^i\sum_{q=0}^{i}\frac{D^{(q)}}{q!},\end{aligned} \tag{54}$$

whence,

$$V_{jr}=\sum_{i=0}^{\infty}\sum_{q=0}^{i}\sum_{k=1}^{N}\frac{D^{(q)}}{q!}(-\epsilon_k)^{r+i}\prod_{n\neq k}^{N}\frac{1}{\epsilon_n-\epsilon_k}. \tag{55}$$

We first observe that the derivatives, although they depend on ϵ_n, are independent of the index k; therefore. they can be brought outside the summation sign in Eq. (55) in which case, in accordance with Eq. (42),

$$V_{jr}=\sum_{q=0}^{N-1-r}\frac{D^{(q)}}{q!},\quad r\leq N-1. \tag{56}$$

For $r>N-1$, $V_{jr}\to 0$ as $\epsilon_n\to 0$.

Secondly, we note that the product term in Eq. (53) can be written as two terms: one is $\propto\sum_{n=1}^{N}\epsilon_n$, while the other is a function of $-\epsilon_k$ only; but since, ultimately, $\epsilon_n\to 0$, the derivatives will be determined solely by the latter term, which is:

$$[f_j+\sum_{m\neq j}^{M}\frac{\lambda_m f_m(-\epsilon_k)}{\lambda_m-\lambda_j+\lambda_j(-\epsilon_k)}]^N. \tag{57}$$

A recursive application of Leibniz' theorem to the above yields:

$$\frac{D^{(q)}}{q!} = \sum_{\{l_i\}} A_{l_1} A_{l_2} \ldots A_{l_N}, \tag{58}$$

where

$$\begin{aligned} A_0 &= f_j, \\ A_n &= -\frac{1}{\lambda_j} \sum_{m \neq j}^{M} \lambda_m f_m \left(\frac{-\lambda_j}{\lambda_m - \lambda_j}\right)^n, \quad n \geq 1, \end{aligned} \tag{59}$$

and where the sum is over all N-part compositions of q, that is, over the ordered set of indices: $\{l_1, l_2, \ldots : \ l_i \geq 0; \ \sum_{i=1}^{N} l_i = q\}$. A practical realisation of this sum is:

$$\begin{aligned} \frac{D^{(q)}}{q!} = \sum_{l_1=0}^{q} A_{l_1} \sum_{l_2=0}^{q-l_1} A_{l_2} \sum_{l_3=0}^{q-l_1-l_2} A_{l_3} \ldots \\ \ldots \sum_{l_{N-1}=0}^{q-l_1 \ldots -l_{N-2}} A_{l_{N-1}} A_{l_N}, \end{aligned} \tag{60}$$

where.

$$l_N = q - l_1 - l_2 \ldots - l_{N-1}.$$

See. also. Ref. [6] for a simple algorithm for generating compositions.

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