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Worldsheet Reparametrizations and the Tomonaga–Schwinger–Dirac Formalism for String Theory[f1]

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ABSTRACT

We develop a Tomonaga–Schwinger–Dirac type formulation of the first quantized free bosonic string. The wave functional depends not only on the curve along which the string lies in coordinate space, but also the curve along which it lies on the world sheet. In this formalism the Virasoro operators have natural geometric meanings. We speculate about how one might construct interacting field theories of strings on the basis of symmetries of an appended loop space.

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As a consequence of reparametrization invariance the momenta $P^\mu(\sigma,\tau)$ canonically conjugate to $x^\mu(\sigma,\tau)$ are constrained

$$\begin{aligned} P^\mu P_\mu + x^{\mu\prime} x_\mu{}' = 0 \\ x'^\mu P_\mu = 0. \end{aligned} \tag{1.5}$$

If we impose the standard communication relations between P^μ and x^μ, we find that these momentum constraints generate reparametrizations. Specifically, in terms of the Fourier transforms of the constraints* [1]

$$L_n = \int_{-\pi}^{+\pi} d\sigma\, e^{in\sigma}\{P^\mu + x'^\mu\}^2 \tag{1.5a}$$

$$\tilde{L}_n = \int_{-\pi}^{+\pi} d\sigma\, e^{-in\sigma}\{P^\mu - x'^\mu\}^2 , \tag{1.5b}$$

one has for the open string in the orthonormal gauge,

$$[L_n + \tilde{L}_n, x^\mu] = x^\mu(\tau + \cos n\sigma \quad ,\sigma) \tag{1.7a}$$

$$[L_n - \tilde{L}_n, x^\mu] = x^\mu(\tau,\sigma + \sin n\sigma) . \tag{1.7b}$$

The simple forms for (1.5) and (1.6) follow from the fact that in the orthonormal gauge $P^\mu = \dot{x}^\mu$ and the Nambu action (1.2) has the simple form

$$S = -\frac{1}{2}\int d\tau\, d\sigma (x'^\mu x_\mu{}' - \dot{x}^\mu \dot{x}_\mu) . \tag{1.8}$$

Note that in the classical theory τ and σ enters the formalism in an essentially symmetric manner. There is nothing to distinguish between τ and σ apart from the timelike character of τ ($\dot{x}^\mu \dot{x}_\mu < 0$) and the spacelike character of σ

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* These expressions are for the closed string with usual periodic boundary conditions. These are also valid for the open string if one extends the interval $0 \leqq \sigma \leqq \pi$ to $-\pi \leqq \sigma \leqq +\pi$ by imposing $x^\mu(-\sigma) = x^\mu(\sigma)$ in which case $\tilde{L}_n = L_{-n}$.

I. Introduction

The dynamical variables describing the bosonic string are the continuously infinite set of spacetime coordinates which tell how the string's two-dimensional worldsheet is embedded in spacetime:

$$x^\mu = x^\mu(\tau, \sigma) \,, \tag{1.1}$$

where τ and σ are, respectively, timelike and spacelike parameters. If the classical equations of motion of the string arise from the Nambu action,

$$S = -\frac{1}{2} \int d\tau \, d\sigma \{ (\dot{x}^\mu x_\mu{}')^2 - (\dot{x}^\mu \dot{x}_\mu)(x'^\mu x_\mu{}') \}^{1/2} \,, \tag{1.2}$$

the dynamics of the string will be invariant under arbitrary *reparametrization* of σ and τ

$$\begin{aligned} \tau \rightarrow \bar{\tau} &= \bar{\tau}(\tau, \sigma) \\ \sigma \rightarrow \bar{\sigma} &= \bar{\sigma}(\tau, \sigma) \end{aligned} \tag{1.3}$$

since the value of the action (1.2) is unchanged by the transformation (1.3).

Reparametrization is a fundamental symmetry of the first quantized theory as well. To quantize the theory we must first make a specific choice of parameters σ and τ; this is referred to as "choosing a gauge." The orthonormal gauge is, for example, a class of parameter choices for which the following constraints hold

$$\dot{x}^\mu \dot{x}_\mu + x'^\mu x_\mu{}' = 0 \tag{1.4a}$$

$$\dot{x}^\mu x_\mu{}' = 0 \,. \tag{1.4b}$$

($x'^{\mu}x_{\mu}' > 0$). In a canonical first quantization we begin to see a distinction arising between the two; however, reparametrizations of τ and σ still treat the two parameters "democratically."

When we pass to the second quantized formalism – that is "string field theory" – the democracy disappears. This can be seen to occur at the very first step of second quantization: one usually defines a Schrödinger picture wave functional $\psi[x^{\mu}(\sigma),\tau]$* which gives the probability amplitude for finding the string lying along the spacelike line $x^{\mu}(\sigma)$ at parameter time τ. Subsequently, in the field theory, ψ becomes a quantum operator. In the Schrödinger picture the action of the canonical momentum on the wave functional is represented by

$$P^{\mu} \to -i\frac{\delta}{\delta x^{\mu}(\sigma)}, \tag{1.9}$$

so that the Virasoro operators become

$$L_n = \int_{-\pi}^{+\pi} d\sigma\, e^{in\sigma}\{-i\frac{\delta}{\delta x^{\mu}(\sigma)} + x'^{\mu}\}^2 \tag{1.10a}$$

$$\tilde{L}_n = \int_{-\pi}^{+\pi} d\sigma\, e^{in\sigma}\{-i\frac{\delta}{\delta x^{\mu}(\sigma)} - x'^{\mu}\}^2 . \tag{1.10b}$$

As we have seen, the Virasoro operators have their origin as generators of infinitesimal reparametrizations of both τ and σ. The invariance of the theory under such reparametrization is expressed, in the Schrödinger picture, by imposing the following constraints on physical states $|\psi\rangle$:

$$L_n|\psi\rangle = 0 \quad n > 0$$

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* The dependence on τ can be removed by a fourier transform leading to a wave functional $\Phi[x(\sigma)]$ [2].

and for closed strings, in addition

$$\tilde{L}_n|\psi\rangle = 0 \quad n > 0 .$$

This ensures that the matrix elements of all L_n and $\tilde{L}_n$ between physical states vanish. In the Schrödinger picture, "half" of the Virasoro generators still have simple physical interpretations as reparametrization generators. These are

$$(L_n - \tilde{L}_n) + (L_{-n} - \tilde{L}_{-n}) = \int_{-\pi}^{+\pi} d\sigma \cos n\sigma \, x'^{\mu} \frac{\delta}{\delta x^{\mu}(\sigma)} \tag{1.11a}$$

$$i[(L_n - \tilde{L}_n) - (L_{-n} - \tilde{L}_{-n})] = \int_{-\pi}^{+\pi} d\sigma \sin n\sigma \, x'^{\mu} \frac{\delta}{\delta x^{\mu}(\sigma)} . \tag{1.11b}$$

(The first combination, of course, vanishes for the open string.) These correspond to reparametrizations of σ of the form

$$\sigma \to \sigma + \varepsilon \begin{Bmatrix} \sin n\sigma \\ \cos n\sigma \end{Bmatrix} . \tag{1.12}$$

However, the remaining Virasoro generators

$$(L_n + \tilde{L}_n) + (L_{-n} + \tilde{L}_{-n}) = \int d\sigma \cos n\sigma \left(-\frac{\delta^2}{\delta x^2} + x'^2\right)$$

$$(L_n + \tilde{L}_n) - (L_{-n} + \tilde{L}_{-n}) = \int d\sigma \sin n\sigma \left(-\frac{\delta^2}{\delta x^2} + x'^2\right) , \tag{1.13}$$

do not have a simple geometric interpretation.

The reason for this is clear. In our canonical quantization in the Schrödinger picture, τ plays a special role. The state of a string is described by picking a constant$-\tau$ slice on the world sheet and specifying the wave functional $\psi[x^{\mu}(\sigma), \tau]$. Given the initial data on any such constant$-\tau$ line, the wave functional at any other constant$-\tau$ line may be obtained by integrating the

Schrodinger equation. The restriction to straight lines of constant τ - i.e. the same τ for each value of σ - renders the notion of a σ-dependent reparametrization of τ meaningless. And that is what the "other half" of the Virasoro generators implement in the Heisenberg picture.

This asymmetric treatment of σ and τ is totally contrary to the spirit of Nambu's original formulation of string theory as the dynamics of the world sheet. It would be ideal to have a formulation in which σ and τ are treated symmetrically. In such a formalism duality would be manifest. Nambu himself has taken some steps towards such a formulation.[3]

In this paper we strive towards a similar goal, but adopt a rather different approach. We introduce a new Schrödinger picture for the string which is a direct logical extension of that used by Feynman [4] in describing a Klein-Gordon particle. In the latter, the wave function $\psi[x^\mu, \tau]$ is a function of the particle coordinate as well as the parameter τ which denotes the position on its world line. A natural extension to strings would be a wave functional $\psi[x^\mu(\xi), \sigma(\xi), \tau(\xi)]$. This depends on not only the position of the string in space-time, but also on the world sheet. The latter curve is described by the two functions $\sigma(\xi)$ and $\tau(\xi)$ and hence may be of arbitrary shape - provided it is everywhere spacelike. Now, arbitrary deformations of this curve capture the notion of σ and τ reparametrizations. That is why in this formalism the relationship of the "other half" of the Virasoro generators to τ reparametrizations is maintained. We construct this formalism in Section II following Kuchar's presentation [5] of the Tomonaga-Schwinger-Dirac method [6].

The potential importance of such a formulation becomes clear when we pass from the first quantized Schrödinger picture to the second quantized string field theory in which ψ is itself a dynamical variable. Consider the conventional string field $\Phi[x^\mu(\sigma)]$. This scalar functional is equivalent to an infinite number of ordinary tensorial fields in space-time [7,8,9]. It has recently been shown that the linearized gauge transformations of the massless fields are contained in "chordal" gauge transformations generated by Virasoro operators [2,9-11]. Thus for the open string this chordal transformations are

$$\Phi[x(\sigma)] \to \Phi[x(\sigma)] + \sum_{n>0} \varepsilon_{-n} L_{-n} \Omega_{-n}[x(\sigma)] , \tag{1.14}$$

where Ω_n's are arbitrary scalar functionals. In particular the analog of (1.14) for closed strings contains the linearized gauge transformations of the graviton field. Now, gravitational gauge transformations are usually thought of as arising from general coordinate transformations; however no such geometrical principle seems to be operating here. Finding the geometrical principle which leads to these transformations is important for obtaining the nonlinear transformations of the interacting theory. This can only be made easier by formulating the theory in such a way that all the gauge generators have explicit geometric meanings to begin with.

Another way of seeing this is to note that while the expressions (1.11) for $(L_n - \tilde{L}_n) \pm (L_{-n} - \tilde{L}_{-n})$ does not involve the space-time metric, the expression (1.13) for $(L_n + \tilde{L}_n) \pm (L_{-n} + \tilde{L}_{-n})$ requires a flat metric $\eta_{\mu\nu}$ to contract space-time tensor indices. It is widely believed that in string theory, the string field itself contains all information about space-time geometry [10]. Such a theory, together with its chordal transformations, should not depend a priori on a preferred background metric-flat or otherwise. A string field theory based on the formulation described here would satisfy this requirement. In Section III we discuss one approach towards a geometric string field theory inspired by the recent work of Bardakci [14].

II. Tomonaga–Schwinger–Dirac Formulation of Orthonormal Gauge String Theory*

We begin with the classical bosonic string theory in orthonormal gauge, i.e. with the action (1.7) and the gauge conditions (1.4). Denoting

$$\sigma^0 = -\sigma_0 = \tau \; ; \quad \sigma^1 = \sigma_1 = \sigma \; , \tag{2.1}$$

the action reads

$$S = -\frac{1}{2}\int d^2\sigma \frac{\partial x^\mu}{\partial \sigma^i}\frac{\partial x_\mu}{\partial \sigma_i} \; , \tag{2.2}$$

while the orthonormal gauge constraints are

$$\left(\frac{\partial x^\mu}{\partial \sigma^0}\right)^2 + \left(\frac{\partial x^\mu}{\partial \sigma^1}\right)^2 = 0$$

$$\frac{\partial x^\mu}{\partial \sigma^0}\cdot\frac{\partial x_\mu}{\partial \sigma^1} = 0 \; . \tag{2.3}$$

We introduce a new set of arbitrary curvilinear coordinates on the world sheet, (ξ^α)

$$\xi^\alpha = \xi^\alpha(\sigma^\alpha) \qquad \alpha = 0, 1 \; , \tag{2.4}$$

ξ^0 and ξ^1 are, respectively, timelike and spacelike

$$\frac{\partial x^\mu}{\partial \xi^0}\cdot\frac{\partial x_\mu}{\partial \xi_0} < 0 \; ; \quad \frac{\partial x^\mu}{\partial \xi^1}\cdot\frac{\partial x_\mu}{\partial \xi_1} > 0 \; . \tag{2.5}$$

The transformation (2.4) must be nondegenerate, i.e.

$$J = |\det(\frac{\partial \sigma^i}{\partial \xi^\alpha})| \neq 0 \; . \tag{2.6}$$

* A short account of this section is contained in Ref. [13].

We shall refer to σ^i as the "flat" parameters, and the ξ^α as the curved parameters. Using (2.4) and (2.6) the action may be rewritten as

$$S = \int d^2\xi \mathbf{L} \tag{2.7}$$

where

$$\mathbf{L} = -\frac{1}{2} J\, g^{\alpha\beta} \frac{\partial x^\mu}{\partial \xi^\alpha} \cdot \frac{\partial x_\mu}{\partial \xi^\beta}$$

$$g^{\alpha\beta} \equiv \frac{\partial \xi^\alpha}{\partial \sigma^i} \cdot \frac{\partial \xi^\beta}{\partial \sigma_i} . \tag{2.8}$$

Equations (2.7) and (2.8) present us with a dynamical system with dynamical variables $x^\mu(\xi)$ and a timelike parameter ξ^0, the σ^i's acting as fixed non-dynamical functions of the ξ^α's. We therefore define new canonical momenta as

$$\pi_\mu \equiv \frac{\partial \mathbf{L}}{\partial(\partial x^\mu / \partial \xi^0)}$$

$$= -J \frac{\partial \xi^0}{\partial \sigma^i} \frac{\partial \xi^\alpha}{\partial \sigma_i} \frac{\partial x_\mu}{\partial \xi^\alpha} \tag{2.9}$$

The action may now be written as

$$S = \int d^2\xi (\pi_\mu \dot{x}^\mu - \mathbf{h}) \tag{2.10}$$

where $\dot{x}^\mu \equiv \partial x^\mu / \partial \xi^0$ and $\mathbf{h}$ is the curved hamiltonian

$$\mathbf{h} = J \frac{\partial \xi^0}{\partial \sigma^i} T^i_j \dot{\sigma}^j , \tag{2.11}$$

where T^i_j is the flat energy momentum tensor

$$T^i_j = \frac{\partial x^\mu}{\partial \sigma^i} \cdot \frac{\partial x_\mu}{\partial \sigma_j} - \frac{1}{2} \delta^i_j \left| \frac{\partial x^\mu}{\partial \sigma^k} \cdot \frac{\partial x_\mu}{\partial \sigma_k} \right| . \tag{2.12}$$

The key observation to be made at this point is that **h** is *linear* in the "kinematical velocities" $\dot{\sigma}^j$.* If we define "kinematical momenta" P_i

$$P_i \equiv -J \frac{\partial \xi^0}{\partial \sigma^j} T_i^j , \tag{2.13}$$

the action becomes

$$S = \int d^2\xi (\pi_\mu \dot{x}^\mu + P_i \dot{\sigma}^i) . \tag{2.14}$$

Formally (2.14) is the action for a system with (a) dynamical variables $x^\mu(\xi^a)$ and $\sigma^i(\xi^a)$; and (b) a vanishing Hamiltonian so that x^μ and σ^i are independent of the time ξ^0. In fact, we are perfectly justified in regarding (2.14) in this manner – provided the momenta conjugate to the σ^i's are not independent, but constrained by equations (2.13).

The meaning of these constraints becomes transparent if we decompose (2.13) into components normal and tangent to the spacelike lines of constant ξ^0 on the world sheet (using a Lorentzian metric). The tangent to the curve ξ^0 = constant is given by $\partial\sigma^i/\partial\xi^1$ so that the unit normal n^i satisfies

$$n^i n_i = -1 \tag{2.15}$$

and

$$n_i \sigma'^i = 0 , \tag{2.16}$$

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* Consider the first two factors on the right-hand-side of (2.11). One has $J\frac{\partial \xi^0}{\partial \sigma^0} = \left[\frac{\partial \sigma^0}{\partial \xi^0} \cdot \frac{\partial \sigma^1}{\partial \xi^1} - \frac{\partial \sigma^0}{\partial \xi^1} \cdot \frac{\partial \sigma^1}{\partial \xi^0}\right] \frac{\partial \xi^0}{\partial \sigma^1} = \frac{\partial \sigma^1}{\partial \xi^1}$ since $\frac{\partial \sigma^0}{\partial \xi^a} \cdot \frac{\partial \xi^a}{\partial \sigma^0} = 1$ and $\frac{\partial \sigma^0}{\partial \xi^a} \cdot \frac{\partial \xi^a}{\partial \sigma^1} = 0$. Similarly $J\frac{\partial \xi^0}{\partial \sigma^1} = -\frac{\partial \sigma^0}{\partial \xi^1}$. Hence $J\frac{\partial \xi^0}{\partial \sigma^i}$ is independent of $\dot{\sigma}^i$. A calculation similar to that leading to eq. (2.22) shows that T_j^i is also independent of $\dot{\sigma}^i$.

where primes denote partial differential with respect to ξ^1. The components of n^i are therefore

$$n^0 = (\sigma'^i \sigma_i')^{-1/2} \sigma'^1$$

$$n^1 = (\sigma'^i \sigma_i')^{-1/2} \sigma'^0 \, . \tag{2.17}$$

The constraints (2.13) may be thus replaced by

$$\mathbf{P} = 0 \, , \quad \mathbf{H} = 0 \, , \tag{2.18}$$

where

$$\mathbf{P} = \sigma'^i \Big(P_i - J \frac{\partial \xi^0}{\partial \sigma^j} T_i^j\Big) \tag{2.19a}$$

$$\mathbf{H} = n^i \Big(P_i - J \frac{\partial \xi^0}{\partial \sigma^j} T_i^j\Big) \, , \tag{2.19b}$$

are referred to, respectively, as the supermomentum and the superhamiltonian.

Using (2.9) and (2.13) the supermomentum (2.19a) reduces to

$$\mathbf{P} = \sigma'^i P_i + x'^\mu \pi_\mu \, , \tag{2.20}$$

while (2.17) and (2.19b) yields

$$\mathbf{H} = (\sigma'^i \sigma_i')^{-1/2} \{\sigma'^1 P_0 + \sigma'^0 P_1 - \sigma'^1 J \frac{\partial \xi^0}{\partial \sigma^i} T_0^i - \sigma'^0 J \frac{\partial \xi^0}{\partial \sigma^i} T_1^i\} \, . \tag{2.21}$$

Using the chain rule T_j^i becomes

$$T_j^i = \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x_\mu}{\partial \xi^\beta} \left[\frac{\partial \xi^\alpha}{\partial \sigma_i} \cdot \frac{\partial \xi^\beta}{\partial \sigma^j} - \frac{1}{2} \delta_j^i \frac{\partial \xi^\alpha}{\partial \sigma^k} \cdot \frac{\partial \xi^\beta}{\partial \sigma_k} \right] ,$$

so that

$$\sigma'^1 J \frac{\partial \xi^0}{\partial \sigma^i} T\delta + \sigma^0{}' J \frac{\partial \xi^0}{\partial \sigma^i} T\} = \frac{J}{2} \dot{x}^\mu \dot{x}_\mu \frac{\partial \xi^0}{\partial \sigma^i} \cdot \frac{\partial \xi^0}{\partial \sigma_i} \left[\frac{\partial \xi^0}{\partial \sigma^0} \sigma^1{}' + \frac{\partial \xi^0}{\partial \sigma^1} \sigma^0{}' \right]$$

$$+ J \dot{x}^\mu \dot{x}_\mu{}' \frac{\partial \xi^0}{\partial \sigma^i} \frac{\partial \xi^0}{\partial \sigma_i} \left[\frac{\partial \xi^1}{\partial \sigma^0} \sigma^1{}' + \frac{\partial \xi^1}{\partial \sigma^1} \sigma^0{}' \right]$$

$$+ J x'^\mu x_\mu{}' \left[\frac{\partial \xi^0}{\partial \sigma^i} \cdot \frac{\partial \xi^1}{\partial \sigma_i} \left[\frac{\partial \xi^1}{\partial \sigma^0} \sigma^1{}' + \frac{\partial \xi^1}{\partial \sigma^1} \sigma^0{}' \right] \right.$$

$$\left. - \frac{1}{2} \frac{\partial \xi^1}{\partial \sigma^i} \frac{\partial \xi^1}{\partial \sigma_i} \left[\frac{\partial \xi^0}{\partial \sigma^0} \sigma'^1 + \frac{\partial \xi^0}{\partial \sigma'} \sigma'^0 \right] \right] . \tag{2.22}$$

To simplify this expression, note that $\partial \xi^0 / \partial \sigma^i$ is normal to the tangent vector σ'^i

$$\frac{\partial \xi^0}{\partial \sigma^i} \cdot \frac{\partial \sigma^i}{\partial \xi^1} = 0 . \tag{2.23}$$

Thus the unit normal may be written as

$$n^i = \left[\frac{\partial \xi^0}{\partial \sigma^i} \cdot \frac{\partial \xi^0}{\partial \sigma_i} \right]^{-1/2} \cdot \frac{\partial \xi^0}{\partial \sigma_i} . \tag{2.24}$$

Comparing this with (2.17) we find

$$\frac{\partial \sigma^1}{\partial \xi^1} = a \frac{\partial \xi^0}{\partial \sigma^0} \tag{2.5a}$$

$$\frac{\partial \sigma^0}{\partial \xi^1} = -a \frac{\partial \xi^0}{\partial \sigma^1} \tag{2.25b}$$

where

$$a = (\sigma'^i \sigma'_i)^{1/2} \left[\frac{\partial \xi^0}{\partial \sigma^i} \cdot \frac{\partial \xi^0}{\partial \sigma_i} \right]^{-1/2} . \tag{2.26}$$

From (2.25) and (2.6b) one has

$$J = a\,. \tag{2.27}$$

Using (2.25–2.27) in (2.22) one has for H

$$\mathrm{H} = (\sigma_i{}'\sigma'^i)^{-1/2}\{\sigma'^1 P_0 + \sigma'^0 P_1 + \frac{1}{2}(\pi_\mu \pi^\mu + x'^\mu x_\mu{}')\}\,. \tag{2.28}$$

Similarly using (2.13) and (2.25–2.27) the orthonormal gauge conditions (2.3) become

$$\frac{1}{\sigma^{0\prime} \pm \sigma^{1\prime}}(P_1 \pm P_0) = 0\,. \tag{2.29}$$

Since the Hamiltonian of our system vanishes the dynamics is determined entirely by the supermomentum and superhamiltonian constraints and the orthonormal gauge constraints (2.29).

Canonical quantization now proceeds by imposing the standard commutation relations between the coordinates $x^\mu(\xi)$ and $\sigma^i(\xi)$ and their momenta $\pi^\mu(\xi)$ and $P^i(\xi)$. In the Schrödinger picture, dynamical variables do not depend on time ξ^0 and the momenta are represented by

$$\pi_\mu(\xi) \to -i\frac{\delta}{\delta x^\mu(\xi)}\,;\quad P_i(\xi) \to -i\frac{\delta}{\delta\sigma^i(\xi)}\,.$$

(Henceforth ξ shall denote ξ^1.) The state of the system is described by the wave functional $\tilde{\psi}[x^\mu(\xi), \sigma^i(\xi), \xi^0]$ – however $\tilde{\psi}$ is actually independent of ξ^0 since the Hamiltonian vanishes. The quantum dynamics are thus purely governed by the constraints which are imposed on the wave functional. The quantum supermomentum and superhamiltonian constraints become*

$$\{\sigma'^i(\xi)\frac{\delta}{\delta\sigma^i(\xi)} + x'^\mu(\xi)\frac{\delta}{\delta x^\mu(\xi)}\}\tilde{\psi}[x(\xi), \sigma^i(\xi)] = 0 \tag{2.31a}$$

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* The usual factor ordering ambiguity does not appear in this context. See Ref. [5], p. 265.

$$\{-i\varepsilon^{ij}\sigma_i(\xi)\frac{\delta}{\delta\sigma^j(\xi)} + \frac{1}{2}[-\frac{\delta}{\delta x^\mu(\xi)}\frac{\delta}{\delta x_\mu(\xi)} + x'^\mu(\xi)x_\mu'(\xi)]\}\tilde{\psi} = 0\,,$$

(2.31b)

where we have removed an overall factor from (2.31b). We can equally well multiply the orthonormal gauge constraints (2.29) by the factor $i(\sigma'^0 \pm \sigma'^1)^2$ obtaining the quantum constraints

$$[\sigma'^0(\xi) \pm \sigma'^1(\xi)]\left[\frac{\delta}{\delta\sigma^1(\xi)} \pm \frac{\delta}{\delta\sigma^0(\xi)}\right]\tilde{\psi} = 0\,. \tag{2.32}$$

Equivalently, we can use the fourier transformed versions of (2.32)

$$Q_n\tilde{\psi} = 0\,, \quad \tilde{Q}_n\tilde{\psi} = 0 \tag{2.33}$$

where

$$Q_n = \frac{1}{2}\int_{-\pi}^{+\pi} d\xi\, e^{in\xi}(\eta^{ij} - \varepsilon^{ij})\sigma_i'(\xi)\frac{\delta}{\delta\sigma^j(\xi)}$$

$$\tilde{Q}_n = \frac{1}{2}\int_{-\pi}^{+\pi} d\xi\, e^{in\xi}(\eta^{ij} + \varepsilon^{ij})\sigma_i'(\xi)\frac{\delta}{\delta\sigma^j(\xi)} \quad (n = 0, \pm 1, \pm 2, \cdots)\,, \tag{2.34}$$

(where ε^{ij} is the antisymmetric tensor $\varepsilon^{01} = -\varepsilon^{10} = 1$). Actually in analogy with (1.10) we should impose the constraints (2.33) only for $n > 0$.

The commutators of Q_n's may be easily worked out. We find

$$[Q_m, Q_n] = (m - n)Q_{m+n}$$

$$[\tilde{Q}_m, \tilde{Q}_n] = (m - n)Q_{m+n}$$

$$[Q_m, \tilde{Q}_n] = 0 \,, \tag{2.35}$$

which is the same algebra as that obeyed by the Virasoro operators L_n and $\tilde{L}_n$, except for the central charge. (The absence of the central charge is not surprising since in the usual theory the *normal ordered* Virasoro operators obey the algebra with central charge. The operators Q_n and $\tilde{Q}_n$ are not normal ordered – in fact we have not introduced a vacuum state with respect to which one may normal order.) Furthermore the operators (2.34) manifestly generate shifts of the σ^i's. We have thus constructed a formalism in which both σ and τ reparametrizations are represented in a simple geometric manner.*

What does the wave functional $\tilde{\psi}[x(\xi), \sigma^i(\xi)]$ mean? $\sigma^i(\xi)$ denotes a spacelike line on the world sheet while $x^\mu(\xi)$ denotes a curve in space–time. If ξ parametrizes the position along the string, $\tilde{\psi}$ describes the probability amplitude that the string occupies a curve $x^\mu(\xi)$ in coordinate space as well as a curve $\sigma^i(\xi)$ in parameter space (i.e., the world sheet). The lines $\sigma^i(\xi)$ are of course arbitrary, so long as they are spacelike. Since the σ^i are coordinates on the world sheet, changes in the functions $\sigma^i(\xi)$ (i.e., changes in the curves on the world sheet) incorporate the notion of reparametrizations on the world sheet.

The equations (2.31) are of the general form

$$a\tilde{\psi} = \mathbf{A}\tilde{\psi} \,,$$

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* In fact, in this formalism it is possible to think of reparametrizations of σ and τ which do not involve any transformation of the $x^\mu(\xi)$'s.

where α acts on variables in $\tilde{\psi}$ which specify the string in parameter space while A acts on the real dynamical variables $x^{\mu}(\xi)$. These operators may be shown to obey the consistency conditions derived by Dirac [6].

Such a description is, of course, highly redundant. But that is what we are trying to achieve – introduce enough redundancy so as to make the symmetries of the theory manifest.

Some of this redundancy may be removed without loss of generality – but at the cost of treating σ and τ asymmetrically. Note the description of our string in coordinate space and on world sheet depended on the parameter ξ. Under an arbitrary change of the parameter ξ

$$\xi \rightarrow \xi + \delta\xi \,, \tag{2.36}$$

the wave functional changes by

$$\delta\tilde{\psi} = \int_{-\pi}^{+\pi} d\xi \, \delta\xi \, \mathbf{P}(\xi)\tilde{\psi} \,. \tag{2.37}$$

The supermomentum constraint $\mathbf{P}\tilde{\psi} = 0$ thus means that $\tilde{\psi}$ is insensitive to changes in our choice of ξ.

Therefore, without any loss of information we may choose

$$\xi = \sigma \,. \tag{2.38}$$

The wave functional is now $\tilde{\psi}[x(\sigma), \tau(\sigma), \sigma]$. However, the supermomentum constraint relates explicit dependence on σ to dependence on $x(\sigma)$ and $\tau(\sigma)$

$$[-i\frac{\delta}{\delta\sigma} - i\tau'(\sigma)\frac{\delta}{\delta\tau(\sigma)} - ix^{\mu}(\sigma)\frac{\delta}{\delta x^{\mu}(\sigma)}]\tilde{\psi} = 0 \tag{2.39}$$

(where $'$ now denotes differentiation with respect to σ). One may now use (2.39) to remove any explicit dependence on σ. Thus plugging (2.39) into the superhamiltonian equation (2.31b) one obtains an equation for the functional

$$\psi[x^\mu(\sigma,\tau(\sigma)] \, ,$$

which reads

$$i\frac{\delta\psi}{\delta\tau(\sigma)} = \left\{\frac{1}{2(1-\tau')^2}\left[-i\,\frac{\delta}{\delta x^\mu(\sigma)} - \tau' x^{\mu\prime}\right]^2 + \frac{1}{2}\,x'^\mu x_\mu{}'\right\}\psi \, . \tag{2.40}$$

The constraints (2.29) acting on the wave functional ψ become (after removing a nonzero factor)

$$\left[\frac{\delta}{\delta\tau(\sigma)} \pm \left[\tau'(\sigma)\frac{\delta}{\delta\tau(\sigma)} + x'^\mu(\sigma)\frac{\delta}{\delta x^\mu(\sigma)}\right]\right]\psi = 0 \, . \tag{2.41}$$

The fourier transforms of the operators within the brackets

$$R_n = \int_{-\pi}^{+\pi} d\sigma\, e^{in\sigma}\left\{\frac{\delta}{\delta\tau(\sigma)} - \left[\tau'(\sigma)\frac{\delta}{\delta\tau(\sigma)} + x'^\mu(\sigma)\frac{\delta}{\delta x^\mu(\sigma)}\right]\right\}$$

$$\tilde{R}_n = \int_{-\pi}^{+\pi} d\sigma\, e^{in\sigma}\left\{\frac{\delta}{\delta\tau(\sigma)} + \left[\tau'(\sigma)\frac{\delta}{\delta\tau(\sigma)} + x'^\mu(\sigma)\frac{\delta}{\delta x^\mu(\sigma)}\right]\right\} , \tag{2.42}$$

once again form a Virasoro algebra without central charge. Just as in conventional quantization we shall not impose the full equations (2.41) but only

$$R_n\psi = 0 \qquad n > 0 \, , \tag{2.43a}$$

for open strings and in addition

$$\tilde{R}_n\psi = 0 \qquad n > 0 \, , \tag{2.43b}$$

for closed strings.

Even though σ and τ have been treated asymmetrically the R_n's generate σ and τ reparametrizations. In fact the combinations

$$(R_n - \tilde{R}_n) \pm (R_{-n} - \tilde{R}_{-n})$$

generate reparametrizations of σ, while

$$(R_n + \tilde{R}_n) \pm (R_{-n} + \tilde{R}_{-n})$$

generate σ dependent reparametrizations of τ. In this framework the latter are simply deformations of the spacelike curve $\tau(\sigma)$ which describe the string on the world sheet.

The wave functional $\psi[x(\sigma), \tau(\sigma)]$ denotes the probability amplitude that the string lies along $x^\mu(\sigma)$ in coordinates space *and* along $\tau(\sigma)$ on the world sheet. Given the functional on any such spacelike curve on the world sheet, equation (2.40) may be integrated to obtain the functional on any other curve – provided, of course, the constraints (2.43) are simultaneously satisfied.

The standard formalism may be now recovered by restricting to "flat" spacelike curves

$$\tau(\sigma) = \tau_0 \ \text{(constant)}\,.$$

The wave functional now becomes a functional of $x(\sigma)$ and a function of τ_0, and the relevant question is to study dependence on τ_0. This may be done by noting that

$$\frac{\partial \psi}{\partial \tau_0} = \int d\sigma \, \frac{\delta \tau(\sigma)}{\delta \tau_0} \frac{\delta \psi}{\delta \tau(\sigma)}\,, \tag{2.44}$$

and using $\delta\psi/\delta\tau(\sigma)$ from (2.40). In this case $\tau'(\sigma) = 0$ and one obtains

$$\frac{\partial\psi}{\partial\tau_0} = \frac{1}{2}\int d\sigma\left[-\frac{\delta^2}{\delta x^2} + \frac{1}{2}x'^2\right]\psi\,. \tag{2.45}$$

Since the operator on R.H.S. does not involve τ_0 one may perform a fourier transform

$$\psi[x(\sigma),\tau_0] = \int_0^{+\infty} dm^2 e^{im^2\tau_0}\Phi_{m^2}[x(\sigma)]\,, \tag{2.45}$$

to obtain

$$\frac{1}{2}\int d\sigma[-\frac{\delta^2}{\delta x^2} + \frac{1}{2}x'^2]\Phi = L_0\Phi = m^2\Phi\,, \tag{2.46}$$

and the usual string is obtained for $m^2 = 1$.

To close this section, let us discuss how one might construct a second quantized field theory in our framework. It is natural to take the superhamiltonian equation as the mass shell condition and the rest as constraints. It is not clear at this moment how the $\tau-\sigma$ symmetric formalism can be elevated to a field theory. In the formalism based on $\psi[x(\sigma),\tau(\sigma)]$, the latter may be regarded as a quantum field constrainted by

$$R_n\psi = 0 \qquad n > 0$$

and having an action

$$\int \mathrm{D}x^\mu(\sigma)\mathrm{D}\tau(\sigma)d\sigma\,\psi\,[i\frac{\delta}{\delta\tau(\sigma)} - O]\psi\,, \tag{2.47}$$

where

$$O = \frac{1}{2(1-\tau')^2}(-i\frac{\delta}{\delta x^\mu(\sigma)} - \tau'\,x'^\mu) + \frac{1}{2}x'^2\,. \tag{2.48}$$

Just as a field theory of scalar particles based on a wave function $\psi[x^\mu,\tau]$ actually corresponds to an infinite number of usual Klein-Gordon particles labeled by their mass, the string field theory (2.47) is expected to contain an infinite number of conventional string fields. In the free theory this is, of course, not problematic, since there is no interaction. To extract out the usual string field theory one may write (2.47) as a sum over paths of constant $\tau(\sigma)$ plus all the other paths. The first term then yields a class of fields $\Phi_{m_i}[x(\sigma)]$ in the way outlined above. One may hope that analogous to the theory of relativistic particles, interaction terms do not mix up the various string fields and the above method of extraction still works.

One may write down a gauge invariant theory by introducing a projector analogous to Ref. [2] and [9–11]. The resulting theory would have a chordal invariance of the form

$$\psi[x(\sigma),\tau(\sigma)] \rightarrow \psi[x(\sigma),\tau(\sigma)] + \sum_{n>0} \varepsilon_{-n} R_{-n} \Omega_n[x(\sigma),\tau(\sigma)] \,. \tag{2.49}$$

The advantage is that, as shown above, R_n has simple geometric meanings.

Some Speculations about Field Theories

In the previous sections, we found that one possible natural arena for string field theory is the appended loop space $\{x^\mu(\sigma),\tau(\sigma)\}$. Acting on functionals living in this space, the Virasoro operators have simple geometric meanings; viz. they generate σ and τ reparametrizations of the form

$$\sigma \rightarrow \overline{\sigma} = \sigma + \delta\sigma(\sigma)$$

$$\tau(\sigma) \rightarrow \hat{\tau}(\sigma) = \tau(\sigma) + \delta\tau(\sigma) \,. \tag{3.1}$$

It is tempting to speculate that the dynamics of strings field theory is intimately related to the differential geometry of "appended" loop space. As yet, we do not know whether this is true. In this section we briefly describe the first steps toward developing a calculus of objects that transform as tensors under σ and $\tau(\sigma)$ reparametrization. Our work is an extension of that of Bardakci [14] who constructed a tensor calculus for σ-reparametrizations alone.

The transformation in (3.1) may be called "global" reparametrizations since the transformation functions do not depend on the particular string on which they act. A "local" reparametrization would be

$$\sigma \to \overline{\sigma} = g(x^\mu(\sigma), \tau(\sigma), \sigma) \tag{3.2a}$$

$$\tau(\sigma) \to \hat{\tau}(\sigma) = h(x^\mu(\sigma), \tau(\sigma), \sigma)\,. \tag{3.2b}$$

It is natural to consider transformations which depend on both $x^\mu(\sigma)$ and $\tau(\sigma)$ since in our formalism the string field depends on both.

Under a σ reparametrization(3.2a) the string coordinates transform as

$$x^\mu(\sigma) \to \overline{x}^\mu(\overline{\sigma}) = x^\mu(\sigma)$$

$$\tau(\sigma) \to \overline{\tau}(\overline{\sigma}) = \tau(\sigma)\,. \tag{3.3}$$

Consider general objects in this space: these are functionals of $x^\mu(\sigma)$ and $\tau(\sigma)$ and, in addition, functions of σ

$$\phi^{(n,m)}[x^\mu(\sigma), \tau(\sigma), \sigma]\,. \tag{3.4}$$

The superscript (n,m) means that ϕ is a vector of weight n under σ transformations and of weight m under τ transformation. Actually $\phi^{(n,m)}$ should also carry m additional indices $\sigma_1 \cdots \sigma_m$. Let U denote the action of σ transformations (3.2a). Then

$$U\phi^{n,m}[x(\sigma),\tau(\sigma),\sigma]U^{-1} = \left|\frac{\partial g}{\partial \sigma}\right|^{n} \phi^{n,m}[\overline{x}(\overline{\sigma}),\overline{\tau}(\overline{\sigma}),\overline{\sigma}] . \tag{3.5}$$

Similarly if V implements the transformation (3.2b)

$$V\phi^{n,m}_{\sigma_1 \cdots \sigma_n}[x(\sigma),\tau(\sigma),\sigma]V^{-1} = \int d\sigma_1' \cdots d\sigma_\mu' \frac{\delta\hat{\tau}(\sigma_1')}{\delta\tau(\sigma_1)}$$

$$\cdots \frac{\delta\hat{\tau}(\sigma_n')}{\delta\tau(\sigma_n)} \phi^{n,m}_{\sigma_1' \cdots \sigma_n'}[x,\hat{\tau},\sigma] . \tag{3.6}$$

These transformations may be called tangent space transformations. One can also define coordinate space transformations analogous to those defined in Ref. [14], except one now has an extra coordinate $\tau(\sigma)$. We shall not consider these here, though the full machinery is necessary to actually write down a string field theory.

Consider now tangent space tensors of weight (0,0). We shall use the notation

$$\phi^{(0,0)}[x(\sigma),\tau(\sigma),\sigma] \equiv \phi_\sigma(x,\tau) . \tag{3.7}$$

Under the local transformations of (3.2) the derivatives of ϕ transform inhomogeneously under U:

$$U\frac{\delta\phi_\sigma(x,\tau)}{\delta x^\mu(s)}U^{-1} = \frac{\delta\phi_{\overline{\sigma}}(\overline{x},\overline{\tau})}{\delta\overline{x}^\mu(\overline{s})} + \frac{\delta g_\sigma(\overline{x})}{\delta\overline{x}^\mu(\overline{s})}\frac{\partial\phi_{\overline{\sigma}}(\overline{x},\overline{\tau})}{\partial\overline{\sigma}}$$

$$U\frac{\delta\phi_\sigma(x,\tau)}{\delta\tau^\mu(s)}U^{-1} = \frac{\delta\phi_{\overline{\sigma}}(\overline{x},\tau)}{\delta\overline{\tau}^\mu(\overline{s})} + \frac{\delta g_\sigma(\overline{x},\overline{\tau})}{\delta\overline{\tau}(\overline{s})}\frac{\partial\phi_{\overline{\sigma}}(\overline{x},\overline{\tau})}{\partial\overline{\sigma}} \tag{3.8}$$

Under V, however, there are no such inhomogeneous terms

$$V\frac{\delta\phi_\sigma(x,\tau)}{\delta x^\mu(s)}V^{-1} = \frac{\delta\phi_\sigma(x,\hat{\tau})}{\delta x^\mu(s)}$$

$$V\frac{\delta\phi_\sigma(x,\tau)}{\delta\tau(s)}V^{-1} = \int d\sigma' \left|\frac{\delta\hat{\tau}(\sigma')}{\delta\tau(s)}\right| \frac{\delta\phi_\sigma(x,\hat{\tau})}{\delta\hat{\tau}(\sigma')} . \tag{3.9}$$

Thus, one needs to define covariant derivatives and connections to define objects that transform as tensors under U. Let us define these as follows:

$$D_{\mu s}\phi_\sigma \equiv \frac{\delta\phi_\sigma}{\delta x^\mu(s)} - w^{(1)}_{\mu s,\sigma}\frac{\partial\phi_\sigma}{\partial\sigma}$$

$$\nabla_s\phi_\sigma \equiv \frac{\delta\phi_\sigma}{\delta\tau(s)} - w^{(2)}_{s,\sigma}\frac{\partial\phi_\sigma}{\partial\sigma} . \tag{3.10}$$

It may be easily checked that these covariant derivatives transform homogeneously, provided the connections $w^{(1)}$ and $w^{(2)}$ transform as follows

$$Uw^{(1)}_{\mu s,\sigma}U^{-1} = \left[\frac{\partial g_\sigma}{\partial\sigma}\right]^{-1}\left\{w^{(1)}_{\mu\bar{s},\bar{\sigma}} + \frac{\delta g_\sigma}{\delta\bar{x}^\mu(\bar{s})}\right\}$$

$$Uw^{(2)}_{s,\sigma}U^{-1} = \left[\frac{\partial g_\sigma}{\partial\sigma}\right]^{-1}\left\{w^{(2)}_{\bar{s},\bar{\sigma}} + \frac{\delta g_\sigma}{\delta\bar{\tau}(\bar{s})}\right\} . \tag{3.11}$$

Furthermore, the connections must be invariant under τ reparametrizations (i.e., under V transformations) in order to maintain tensorial property of the covariant derivatives (3.10) under V

$$V\,w^{(1)}_{\mu s,\sigma}V^{-1} = w^{(1)}_{\mu s,\sigma}$$

$$V\,w^{(2)}_{s,\sigma}V^{1-} = w^{(2)}_{s,\sigma} . \tag{3.12}$$

Under linearized transformations $w^{(1)}$ and $w^{(2)}$ transform as

$$w^{(1)}_{\mu s,\sigma} \to w'^{(1)}_{\mu s,\sigma} + \frac{\delta g_\sigma}{\delta x^\mu(s)}$$

$$w^{(2)}_{s,\sigma} \to w^{(2)}_{s,\sigma} + \frac{\delta g_\sigma}{\delta \tau(s)} \cdot \tag{3.13}$$

Looking back at the linear chordal transformations of the free theory it is suggestive to identify the string field as

$$\psi[x(\sigma),\tau(\sigma)] \equiv \int d\sigma \{(1-\tau_\sigma')w^{(2)}_{\sigma,\sigma} + x'^{\mu}_{\sigma} w^{(1)}_{\mu\sigma,\sigma}\} \,. \tag{3.14}$$

Thus, at least in the linearized limit, chordal transformations are nothing but local reparametrizations of σ, the transformation function $g[x(\sigma),\tau(\sigma),\sigma]$ taking the role of the functionals $\Omega_n[x(\sigma),\tau(\sigma)]$. (In fact the Ω_n's are fourier components of the g's.)

It may be noted that in a geometry of loop space defined by $x^\mu(\sigma)$ alone, one would have

$$\Phi[x(\sigma)] = \int d\sigma \, x'^{\mu}_{\sigma} w^{(1)}_{\mu\sigma,\sigma}$$

as in the work of Bardakci. In that case the linearized transformations of $w^{(1)}_{\mu,\sigma,\sigma}$ lead to

$$\delta\Phi[x(\sigma)] = \int d\sigma \, x'^{\mu} \frac{\delta g}{\delta x^\mu(\sigma)}$$

which are *not* symmetries of the gauge-invariant theory. (For open strings $\delta\Phi \sim (L_{+n} - L_{-n})g$ and *not* $L_{-n}g$ as required.) In our formalism, however, the identification (3.14) ensures that local reparametrizations of σ lead to the full chordal transformations of the string field in the linearized limit.

It is tempting to conjecture that the identifications (3.14) also holds in a suitable constructed interacting theory. Then the full nonlinear transformations of w would reduce to nonlinear gauge and general coordinate invariance of the massless modes of the string. At present, it is not clear whether this is indeed true. Note that τ reparametrizations do not play any role in chordal invariance. This might seem a bit strange. On the other hand, the nature of τ reparametrizations is rather different from σ reparametrizations and one should not expect that the two play similar roles. One might have a more symmetric situation in a field theory based on the fields $\tilde{\psi}[x(\xi), \sigma(\xi), \tau(\xi)]$ of Section II.

In any case, the geometry in appended loop space has to be developed further to see whether a string field based on (3.14) can lead to the correct field theory. We have discussed tangent space transformations so far. One has to construct, in addition, the calculus of coordinate space tensors. So far as U transformations are concerned, it is sufficient to simply repeat the work of Bardakci with an additional coordinate $x^{d+1}(\sigma) = \tau(\sigma)$. The structure of V-transformations is, however, very different. We hope to report on the full geometry in a later communication.

Finally let us mention some speculations as to how a field theory may be constructed. Once the tensor calculus of coordinate space is developed one can define a vielbein which relates coordinate space to tangent space. An analog of zero torsion condition would relate the coordinate space affine connections to the spin connections defined above – and geometric objects like the curvature tensor may be expressed in terms of the latter. Hopefully constraints consistent with τ and σ reparametrizations may be imposed which reduce the number of independent components of the curvature and which allow the spin connection to be expressed in terms of the string field ψ. This would allow one to write an invariant action in terms of ψ.

The field $\psi[x(\sigma),\tau(\sigma)]$ has, of course, many more modes than required. However, once a field theory in terms of ψ has been written down it is straightforward to write down the theory in terms of the conventional fields $\Phi[x(\sigma)]$ by choosing constant $\tau(\sigma)$ paths in the functional integral defining the action. What the formalism based on $\psi[x(\sigma),\tau(\sigma)]$ might buy us is a natural geometric way of constructing interacting theories.

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