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THE CROSS SECTION FOR HARD PROCESSES INVOLVING TWO QUARKS AND FOUR GLUONS

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ABSTRACT

The cross section for hard processes involving two quarks and four gluons is given in a form suitable for fast numerical calculations.

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Four-jet spectroscopy at present ($Sp\bar{p}S$ and TeV1) and future (SSC) hadron colliders holds great promise for testing QCD as well as for the discovery of new physics. Standard QCD interactions will cause significant backgrounds to many processes of interest e.g. pair production of electroweak bosons [1]. Hence it is very important that we have a detailed understanding of conventional QCD four-jet production. In particular, knowledge of cross sections for hard parton subprocesses is crucial for any reliable phenomenology of jet physics. In ref.[2] we took one step in this direction, by presenting the cross section for four gluon production by gluon-gluon fusion. In a similar spirit, in this paper we give the cross section for hard processes involving two quarks and four gluons, in the tree approximation of QCD. The final cross section, which is presented in a form suitable for fast numerical evaluation, is applicable for the following processes: quark-antiquark annihilation into four gluons, quark-gluon inelastic scattering into quark and three gluons, and gluon-gluon fusion into a quark-antiquark pair and two gluons. Guided by analyses of three-jet production (see e.g. ref.[1]), we expect that for moderate values of the transverse momentum these processes are at least as important as the purely gluonic process. Since the masses of light quarks can be neglected at high-energy hadron colliders, we consider here the case of massless quarks.

Our calculation makes use of techniques developed in ref.[3], based on the application of extended supersymmetry. We adopt the convention that all particles involved in a scattering process are incoming. An outgoing particle of momentum p and helicity s will be represented as an incoming antiparticle of momentum -p and helicity -s. Our convention is

that the left-handed quark is represented by a spin one-half Weyl particle q, whereas the right-handed quark by a spin one-half Weyl particle $\bar{r}$. The left- and right-handed antiquarks are represented by r and $\bar{q}$, respectively. Let $M(z^1_{s_1},..,z^n_{s_n})$ denote the amplitude for the process with the incoming particles $z^1,..,z^n$ of helicities $s_1,..,s_n$ and momenta $p_1,..,p_n$. The momenta satisfy the conservation equation, $\sum_{i=1}^{i=6} p_i = 0$. We find that all nonvanishing quark - antiquark - four gluon helicity amplitudes can be obtained by crossing and/or complex conjugation from two amplitudes, $M(g^1_+,q^2_+,g^3_+,g^4_+,\bar{q}^5_-,g^6_-)$ and $M(q^1_+,g^2_+,g^3_+,\bar{q}^4_-,g^5_-,g^6_-)$. These amplitudes can be expressed in terms of the amplitudes for processes involving spin zero massless scalar gluons ϕ, spin zero massless squarks σ, left-handed quarks q, left-handed antiquarks r and a smaller number of gluons, using supersymmetry relations (on-shell Ward-Takahashi identities). The first of two relations is very simple:

$$\left|M(g^1_+,q^2_+,g^3_+,g^4_+,\bar{q}^5_-,g^6_-)\right| = \frac{s_{56}}{\sqrt{|s_{23}s_{35}|}}\left|M(g^1_+,\sigma^2_+,\phi^3_+,g^4_+,\sigma^5_-,\phi^6_-)\right| \qquad (1)$$

where the Lorentz invariants s_{ij} are defined as usual, $s_{ij} = (p_i+p_j)^2$, and the scalar product is given by

$$p_ip_j = p^x_ip^x_j + p^y_ip^y_j + p^z_ip^z_j - E_iE_j \qquad (2)$$

($E_i=p^t_i$; all particles are on mass-shell, $p_ip_i=0$). The second supersymmetry relation is more complicated. However, it simplifies considerably in the c.m.s. of particles 1 and 4:

$$\left|M(q^1_+,g^2_+,g^3_+,\bar{q}^4_-,g^5_-,g^6_-)\right| = \frac{1}{s_{23}s_{56}}\Big|(s_{12}+s_{13}+s_{23})^2 M(q^1_+,\phi^2_+,\phi^3_+,\bar{q}^4_-,\phi^5_-,\phi^6_-)$$

$$-2i\sqrt{-s_{14}}\,(s_{12}+s_{13}+s_{23})(p^x_5+p^x_6+ip^y_5+ip^y_6)\,M(\sigma^1_+,\phi^2_+,\phi^3_+,\sigma^4_-,\phi^5_-,\phi^6_-)$$

$$-s_{14}(p^x_5+p^x_6+ip^y_5+ip^y_6)^2\,M(\bar{r}^1_+,\phi^2_+,\phi^3_+,r^4_-,\phi^5_-,\phi^6_-)\Big| \qquad (3)$$

Particles 1 and 4 are chosen to move along the negative and positive z-axis, respectively.

We calculate the full cross section by first computing the amplitudes which occur on the r.h.s. of equations (1) and (3), and subsequently using those equations in order to obtain the desired quark - antiquark - four gluon amplitudes. These amplitudes are calculated in the c.m.s. of particles 1 and 4, and then reexpressed in terms of Lorentz invariants. Before presenting the result, let us adopt some convenient notation.

The squares of the absolute values of all helicity amplitudes will be generated from two generic functions, $B_0(p_1,p_2,p_3,p_4,p_5,p_6)$ and $B_2(p_1,p_2,p_3,p_4,p_5,p_6)$, defined as

$$B_0(p_1,p_2,p_3,p_4,p_5,p_6) = |M(q^1_+,g^2_+,g^3_+,\bar{q}^4_-,g^5_-,g^6_-)|^2$$

$$B_2(p_1,p_2,p_3,p_4,p_5,p_6) = |M(g^1_+,q^2_+,g^3_+,g^4_+,\bar{q}^5_-,g^6_-)|^2 \qquad \textbf{(4)}$$

where the r.h.s. implicitly contains the sum over the color indices of quarks and gluons, and the flavor indices of quarks. The square of the modulus of the invariant matrix element for quark - antiquark - four gluon process, averaged over initial colors, flavors and polarizations, and summed over final colors, flavors and polarizations, is given by

$$
\begin{aligned}
|M(q^1,\bar{q}^2,g^3,g^4,g^5,g^6)|^2 = 2^{1-G-Q}\,(N^2-1)^{-G}\,N^{-Q}\,N_f^{1-Q} \\
\cdot\{\, B_0(134256) + B_0(135246) + B_0(136245) + B_0(154236) \\
+ B_0(164235) + B_0(234156) \\
+ B_2(314526) + B_2(314625) + B_2(316524) + B_2(624513) \\
+ B_2(614523) + B_2(623514) + B_2(324615) + B_2(324516)\,\}
\end{aligned}
\tag{5}
$$

where

$$B_{\binom{0}{2}}(ijklmn) = B_{\binom{0}{2}}(p_i,p_j,p_k,p_l,p_m,p_n).$$

Here N is the number of colors ($N=3$ for QCD), N_f is the number of light flavors, Q is the number of initial quarks and antiquarks, and G is the number of initial gluons. For example, the cross section for the annihilation of a quark with momentum p_1 and an antiquark with momentum p_2 into four gluons with momenta p_3,p_4,p_5,p_6 is obtained from eq.(5) by setting $Q=2$, $G=0$, and replacing the momenta p_3,p_4,p_5,p_6 by $-p_3,-p_4,-p_5,-p_6$.

As the result of the computation of two hundred thirty two Feynman diagrams, we obtain

$$B_0(p_1,p_2,p_3,p_4,p_5,p_6) = \left(\mathcal{D}_0^\dagger,\ \mathcal{D}_{0\pi}^\dagger\right)\cdot\begin{pmatrix} K & K_\rho \\ K_\rho & K \end{pmatrix}\cdot\begin{pmatrix} \mathcal{D}_0 \\ \mathcal{D}_{0\pi} \end{pmatrix}$$

$$B_2(p_1,p_2,p_3,p_4,p_5,p_6) = \left(\mathcal{D}_2^\dagger,\ \mathcal{D}_{2\tau}^\dagger\right)\cdot\begin{pmatrix} K & K_\rho \\ K_\rho & K \end{pmatrix}\cdot\begin{pmatrix} \mathcal{D}_2 \\ \mathcal{D}_{2\tau} \end{pmatrix} \tag{6}$$

where $\mathcal{D}$, $\mathcal{D}_\pi$ and $\mathcal{D}_\tau$ are 16-component complex vector functions of the momenta p_1,p_2,p_3,p_4,p_5 and p_6, and K and K_ρ are constant, symmetric 16×16 matrices. The vectors $\mathcal{D}_\pi$ and $\mathcal{D}_\tau$ are obtained from the vectors

$\mathscr{D}$ by the permutations $(p_2 \leftrightarrow p_3)$ and $(p_1 \leftrightarrow p_4)$, respectively, of the momentum variables in $\mathscr{D}$. The individual components of the vectors $\mathscr{D}$ represent the sums of all contributions proportional to the appropriately chosen sixteen basis color factors. The matrices K, which are the suitable sums over the color indices of products of the color bases, contain four independent structures:

$$K = g^8(N^2-1)/16N^3 \cdot \{ N^6 K^{(6)} + N^4 K^{(4)} + N^2 K^{(2)} + K^{(0)} \}. \tag{7}$$

Here g denotes the gauge coupling constant. The matrices $K^{(6)}$, $K^{(4)}$, $K^{(2)}$ and $K^{(0)}$ are given in Table I. The vectors $\mathscr{D}$ are related to the twenty two diagrams $D^F(I=1\text{-}22)$ involving left-handed quarks and four scalar gluons, twenty four diagrams $D^A(I=1\text{-}24)$ involving left-handed antiquarks and four scalar gluons, twenty two diagrams $D^S(I=1\text{-}22)$ involving squarks and four scalar gluons, and forty eight diagrams $D^G(I=1=48)$ involving squarks, two gluons and two scalar gluons, in the following way:

$$\begin{aligned} \mathcal{D}_0 = \frac{1}{\sqrt{|s_{15}s_{45}|}\; s_{23}s_{56}} \Big\{ & t_{123}^2\, C^F \cdot D^F \\ & + 2t_{123} H(p_5+p_6, p_5) \cdot C^S \cdot D^S \\ & + H(p_5+p_6, p_5) \cdot C^A \cdot D^A \Big\} \end{aligned}$$

$$\mathcal{D}_2 = \frac{s_{56}}{2\sqrt{|s_{23}s_{35}|}\; s_{14}}\; C^G \cdot D^G \tag{8}$$

where the constant matrices $C^F(16\times22)$, $C^A(16\times24)$, $C^S(16\times22)$ and $C^G(16\times48)$ are given in Table II. The Lorentz invariants s_{ij} and t_{ijk} are defined as $s_{ij} = (p_i+p_j)^2$, $t_{ijk} = (p_i+p_j+p_k)^2$ and the complex function H is given by

$$H(p_i,p_j) = 2\cdot\{ (p_1p_4)(p_ip_j) - (p_1p_i)(p_jp_4) - (p_1p_j)(p_ip_4) + i\varepsilon_{\mu\nu\rho\lambda}p_1^\mu p_i^\nu p_j^\rho p_4^\lambda \} \tag{9}$$

where ε is the totally antisymmetric tensor, $\varepsilon_{xyzt}= 1$. For the future use, we define one more function,

$$F(p_i,p_j) = \{(p_1p_4)(p_ip_j) + (p_1p_i)(p_jp_4) - (p_1p_j)(p_ip_4)\}/(p_1p_4). \tag{10}$$

Note that when evaluating B_0 and B_2 at crossed configurations of the momenta, care must be taken with the implicit dependence of the functions H and F on the momenta p_1 and p_4.

The diagrams D^F are listed below:

$$D^F(1) = \frac{4}{s_{12}\, s_{46}\, t_{125}} \Big\{ F(p_5,p_6)\cdot H(p_2,p_5) - F(p_2,p_6)\cdot H(p_5,p_5) + \big[F(p_2,p_5) + s_{12} \big]\cdot H(p_6,p_5) \Big\} ,$$

$$D^F(2) = \frac{4}{s_{12}\, s_{45}\, t_{126}} \Big\{ F(p_6,p_5)\cdot H(p_2,p_5) - F(p_2,p_5)\cdot H(p_6,p_5) + \big[F(p_2,p_6) + s_{12} \big]\cdot H(p_5,p_5) \Big\} ,$$

$$D^F(3) = \frac{-4}{s_{12}\, s_{36}\, s_{45}} \Big\{ \big[F(p_2,p_6) + \tfrac{s_{12}}{2} \big]\cdot H(p_5,p_5) + \big[F(p_6,p_5) + \tfrac{s_{45}}{2} \big]\cdot H(p_2,p_5) - F(p_2,p_5)\cdot H(p_6,p_5) \Big\} ,$$

$$D^F(4) = \frac{-4}{s_{12}\, s_{35}\, s_{46}} \Big\{ \big[F(p_2,p_5) + \tfrac{s_{12}}{2} \big]\cdot H(p_6,p_5) + \big[F(p_5,p_6) + \tfrac{s_{46}}{2} \big]\cdot H(p_2,p_5) - F(p_2,p_6)\cdot H(p_5,p_5) \Big\} ,$$

$$D^F(5) = \frac{4}{s_{36}\, s_{45}\, t_{136}} \Big\{ F(p_6,p_5)\cdot H(p_3,p_5) - F(p_3,p_5)\cdot H(p_6,p_5) - \big[F(p_6,p_3) - \tfrac{s_{36}}{2} - \tfrac{s_{13}}{2} + \tfrac{s_{16}}{2} \big]\cdot H(p_5,p_5) \Big\} ,$$

$$D^F(6) = \frac{4}{s_{35}\, s_{46}\, t_{135}} \Big\{ F(p_5,p_6)\cdot H(p_3,p_5) - F(p_3,p_6)\cdot H(p_5,p_5) - \Big[F(p_5,p_3) - \frac{s_{35}}{2} - \frac{s_{13}}{2} + \frac{s_{15}}{2}\Big]\cdot H(p_6,p_5)\Big\} ,$$

$$D^F(7) = \frac{4}{s_{25}\, s_{46}\, t_{125}} \Big\{ F(p_5,p_6)\cdot H(p_2,p_5) - F(p_2,p_6)\cdot H(p_5,p_5) - \Big[F(p_5,p_2) - \frac{s_{25}}{2} - \frac{s_{12}}{2} + \frac{s_{15}}{2}\Big]\cdot H(p_6,p_5)\Big\} ,$$

$$D^F(8) = \frac{4}{s_{12}\, s_{36}\, t_{125}} \Big\{ F(p_2,p_6)\cdot H(p_3,p_5) - F(p_2,p_3)\cdot H(p_6,p_5) - \Big[F(p_3,p_6) + \frac{s_{46}}{2} - \frac{s_{34}}{2} - \frac{s_{36}}{2}\Big]\cdot H(p_2,p_5)\Big\} ,$$

$$D^F(9) = \frac{4}{s_{12}\, s_{35}\, t_{126}} \Big\{ F(p_2,p_5)\cdot H(p_3,p_5) - F(p_2,p_3)\cdot H(p_5,p_5) - \Big[F(p_3,p_5) + \frac{s_{45}}{2} - \frac{s_{34}}{2} - \frac{s_{35}}{2}\Big]\cdot H(p_2,p_5)\Big\} ,$$

$$D^F(10) = \frac{2}{s_{36}\, s_{45}\, t_{145}} \Big[s_{26} + s_{36} - s_{23}\Big]\cdot H(p_5,p_5) ,$$

$$D^F(11) = \frac{2}{s_{25}\, s_{46}\, t_{146}} \Big[s_{25} + s_{35} - s_{23}\Big]\cdot H(p_6,p_5) ,$$

$$D^F(12) = \frac{2}{s_{12}\, s_{36}\, t_{124}} \Big[s_{56} - s_{35} - s_{36}\Big]\cdot H(p_2,p_5) ,$$

$$D^F(13) = \frac{2}{s_{12}\, s_{35}\, t_{124}} \left[s_{56} - s_{36} - s_{35} \right] \cdot H(p_2, p_5) \quad ,$$

$$\begin{aligned} D^F(14) = \frac{4}{s_{25}\, s_{36}\, t_{125}} \Big\{ & \left[F(p_5, p_6) + \frac{t_{125}}{4} \right] \cdot H(p_2, p_5) \\ & - \left[F(p_2, p_6) + \frac{t_{125}}{4} \right] \cdot H(p_5, p_5) \\ & - \left[F(p_5, p_2) + \frac{s_{15}}{2} - \frac{s_{12}}{2} - \frac{s_{25}}{2} \right] \cdot H(p_6, p_5) \Big\} \quad , \end{aligned}$$

$$\begin{aligned} D^F(15) = \frac{4}{s_{25}\, s_{36}\, t_{136}} \Big\{ & \left[F(p_6, p_5) + \frac{t_{136}}{4} \right] \cdot H(p_3, p_5) \\ & - \left[F(p_3, p_5) + \frac{t_{136}}{4} \right] \cdot H(p_6, p_5) \\ & - \left[F(p_6, p_3) + \frac{s_{16}}{2} - \frac{s_{13}}{2} - \frac{s_{36}}{2} \right] \cdot H(p_5, p_5) \Big\} \quad , \end{aligned}$$

$$\begin{aligned} D^F(16) = \frac{1}{s_{14}\, s_{25}\, s_{36}} \Big\{ & \left[s_{23} - s_{26} + s_{35} - s_{56} \right] \cdot H(p_5 - p_2, p_5) \\ & + \left[s_{23} - s_{35} + s_{26} - s_{56} \right] \cdot H(p_3 - p_6, p_5) \\ & + \left[s_{23} - s_{26} - s_{35} + s_{56} \right] \cdot H(p_2 + p_5, p_5) \Big\} \quad , \end{aligned}$$

$$D^F(17) = \frac{2}{s_{14}\, s_{36}\, t_{124}} \left[s_{56} - s_{35} - s_{36} \right] \cdot H(p_2, p_5) \quad ,$$

$$D^F(18) = \frac{2}{s_{14}\, s_{35}\, t_{124}} \left[s_{56} - s_{35} - s_{36} \right] \cdot H(p_2, p_5) \quad ,$$

$$D^F(19) = \frac{-2}{s_{14}\, s_{36}\, t_{145}} \left[s_{23} - s_{26} - s_{36} \right] \cdot H(p_5, p_5) \quad ,$$

$$D^F(20) = \frac{-2}{s_{14}\, s_{25}\, t_{146}} \left[s_{23} - s_{35} - s_{25} \right] \cdot H(p_6, p_5) \quad ,$$

$$D^F(21) = \frac{1}{s_{14}\, s_{36}} \; H(p_3 - p_6, p_5) \quad ,$$

$$D^F(22) = \frac{1}{s_{14}\, s_{25}} \; H(p_2 - p_5, p_5) \quad . \qquad \textbf{(11)}$$

The diagrams D^A are listed below:

$$D^A(1) = \frac{4}{s_{16}\, s_{24}\, t_{136}} \Big\{ F(p_3, p_2)\cdot H(p_5+p_6, p_6) - F(p_6, p_2)\cdot H(p_5+p_6, p_3) + \big[F(p_6, p_3) + s_{16}\big]\cdot H(p_5+p_6, p_2) \Big\} ,$$

$$D^A(2) = \frac{4}{s_{16}\, s_{34}\, t_{126}} \Big\{ F(p_2, p_3)\cdot H(p_5+p_6, p_6) - F(p_6, p_3)\cdot H(p_5+p_6, p_2) + \big[F(p_6, p_2) + s_{16}\big]\cdot H(p_5+p_6, p_3) \Big\} ,$$

$$D^A(3) = \frac{4}{s_{15}\, s_{24}\, t_{135}} \Big\{ F(p_3, p_2)\cdot H(p_5+p_6, p_5) - F(p_5, p_2)\cdot H(p_5+p_6, p_3) + \big[F(p_5, p_3) + s_{15}\big]\cdot H(p_5+p_6, p_2) \Big\} ,$$

$$D^A(4) = \frac{4}{s_{15}\, s_{34}\, t_{125}} \Big\{ F(p_2, p_3)\cdot H(p_5+p_6, p_5) - F(p_5, p_3)\cdot H(p_5+p_6, p_2) + \big[F(p_5, p_2) + s_{15}\big]\cdot H(p_5+p_6, p_3) \Big\} ,$$

$$D^A(5) = \frac{4}{s_{15}\, s_{24}\, s_{36}} \Big\{ F(p_5, p_2)\cdot H(p_5+p_6,\ p_6)$$
$$- \Big[F(p_5, p_6) + \frac{s_{15}}{2} \Big]\cdot H(p_5+p_6, p_2)$$
$$- \Big[F(p_6, p_2) + \frac{s_{24}}{2} \Big]\cdot H(p_5+p_6, p_5) \Big\} \quad ,$$

$$D^A(6) = \frac{4}{s_{16}\, s_{24}\, s_{35}} \Big\{ F(p_6, p_2)\cdot H(p_5+p_6,\ p_5)$$
$$- \Big[F(p_6, p_5) + \frac{s_{16}}{2} \Big]\cdot H(p_5+p_6,\ p_2)$$
$$- \Big[F(p_5, p_2) + \frac{s_{24}}{2} \Big]\cdot H(p_5+p_6, p_6) \Big\} \quad ,$$

$$D^A(7) = \frac{4}{s_{24}\, s_{36}\, t_{136}} \Big\{ F(p_6, p_2)\cdot H(p_5+p_6,\ p_3) - F(p_3, p_2)\cdot H(p_5+p_6, p_6)$$
$$- \Big[F(p_6, p_3) - \frac{s_{36}}{2} - \frac{s_{13}}{2} + \frac{s_{16}}{2} \Big]\cdot H(p_5+p_6, p_2) \Big\} \quad ,$$

$$D^A(8) = \frac{4}{s_{24}\, s_{35}\, t_{135}} \Big\{ F(p_5, p_2)\cdot H(p_5+p_6, p_3) - F(p_3, p_2)\cdot H(p_5+p_6,\ p_5)$$
$$- \Big[F(p_5, p_3) - \frac{s_{35}}{2} - \frac{s_{13}}{2} + \frac{s_{15}}{2} \Big]\cdot H(p_5+p_6, p_2) \Big\} \quad ,$$

$$D^A(9) = \frac{4}{s_{15}\, s_{36}\, t_{125}} \Big\{ F(p_5,p_6)\cdot H(p_5+p_6,p_3) - F(p_5,p_3)\cdot H(p_5+p_6,p_6)$$
$$- \Big[F(p_3,p_6) + \frac{s_{46}}{2} - \frac{s_{34}}{2} - \frac{s_{36}}{2} \Big]\cdot H(p_5+p_6,p_5) \Big\} ,$$

$$D^A(10) = \frac{4}{s_{16}\, s_{35}\, t_{126}} \Big\{ F(p_6,p_5)\cdot H(p_5+p_6,p_3) - F(p_6,p_3)\cdot H(p_5+p_6,p_5)$$
$$- \Big[F(p_3,p_5) + \frac{s_{45}}{2} - \frac{s_{34}}{2} - \frac{s_{35}}{2} \Big]\cdot H(p_5+p_6,p_6) \Big\} ,$$

$$D^A(11) = \frac{4}{s_{16}\, s_{25}\, t_{136}} \Big\{ F(p_6,p_5)\cdot H(p_5+p_6,p_2) - F(p_6,p_2)\cdot H(p_5+p_6,p_5)$$
$$- \Big[F(p_2,p_5) + \frac{s_{45}}{2} - \frac{s_{24}}{2} - \frac{s_{25}}{2} \Big]\cdot H(p_5+p_6,p_6) \Big\} ,$$

$$D^A(12) = \frac{2}{s_{24}\, s_{36}\, t_{124}} \Big[s_{35} + s_{36} - s_{56} \Big]\cdot H(p_5+p_6,p_2) ,$$

$$D^A(13) = \frac{2}{s_{24}\, s_{35}\, t_{124}} \Big[s_{35} + s_{36} - s_{56} \Big]\cdot H(p_5+p_6,p_2) ,$$

$$D^A(14) = \frac{2}{s_{15}\, s_{36}\, t_{145}} \Big[s_{23} - s_{26} - s_{36} \Big]\cdot H(p_5+p_6,p_5) ,$$

$$D^A(15) = \frac{2}{s_{16}\, s_{25}\, t_{146}} \Big[s_{23} - s_{35} - s_{25} \Big]\cdot H(p_5+p_6,p_6) ,$$

$$D^{A}(16) = \frac{4}{s_{25}\, s_{36}\, t_{125}} \Big\{ \Big[F(p_5, p_6) + \frac{t_{125}}{4} \Big] \cdot H(p_5+p_6, p_2)$$

$$- \Big[F(p_2, p_6) + \frac{t_{125}}{4} \Big] \cdot H(p_5+p_6, p_5)$$

$$- \Big[F(p_5, p_2) + \frac{s_{15}}{2} - \frac{s_{12}}{2} - \frac{s_{25}}{2} \Big] \cdot H(p_5+p_6, p_6) \Big\} ,$$

$$D^{A}(17) = \frac{4}{s_{25}\, s_{36}\, t_{136}} \Big\{ \Big[F(p_6, p_5) + \frac{t_{136}}{4} \Big] \cdot H(p_5+p_6, p_3)$$

$$- \Big[F(p_3, p_5) + \frac{t_{136}}{4} \Big] \cdot H(p_5+p_6, p_6)$$

$$- \Big[F(p_6, p_3) + \frac{s_{16}}{2} - \frac{s_{13}}{2} - \frac{s_{36}}{2} \Big] \cdot H(p_5+p_6, p_5) \Big\} ,$$

$$D^{A}(18) = \frac{1}{s_{25}\, s_{36}\, s_{14}} \Big\{ \big[s_{23} - s_{26} + s_{35} - s_{56} \big] \cdot H(p_5+p_6, p_5-p_2)$$

$$+ \big[s_{23} - s_{35} + s_{26} - s_{56} \big] \cdot H(p_5+p_6, p_3-p_6)$$

$$+ \big[s_{23} - s_{26} - s_{35} + s_{56} \big] \cdot H(p_5+p_6, p_2+p_5) \Big\} ,$$

$$D^{A}(19) = \frac{2}{s_{14}\, s_{36}\, t_{124}} \big[s_{56} - s_{35} - s_{36} \big] \cdot H(p_5+p_6, p_2) ,$$

$$D^{A}(20) = \frac{2}{s_{14}\, s_{35}\, t_{124}} \big[s_{56} - s_{35} - s_{36} \big] \cdot H(p_5+p_6, p_2) ,$$

$$D^A(21) = \frac{2}{s_{14}\, s_{36}\, t_{145}} \left[s_{26} + s_{36} - s_{23} \right] \cdot H(p_5 + p_6 \,,\, p_5) \quad ,$$

$$D^A(22) = \frac{2}{s_{14}\, s_{25}\, t_{146}} \left[s_{25} + s_{35} - s_{23} \right] \cdot H(p_5 + p_6 \,,\, p_6) \quad ,$$

$$D^A(23) = \frac{1}{s_{14}\, s_{36}} \; H(p_5 + p_6 \,,\, p_3 - p_6) \quad ,$$

$$D^A(24) = \frac{1}{s_{14}\, s_{25}} \; H(p_5 + p_6 \,,\, p_2 - p_5) \quad . \qquad \textbf{(12)}$$

The diagrams D^S are listed below:

$$D^s(1) = \frac{1}{s_{14}\, s_{36}\, t_{145}} \left[s_{45} - s_{15} \right] \cdot \left[s_{23} - s_{26} - s_{36} \right] ,$$

$$D^s(2) = \frac{1}{s_{36}\, t_{145}} \left[s_{23} - s_{26} - s_{36} \right] ,$$

$$D^s(3) = \frac{1}{s_{14}\, s_{25}\, t_{146}} \left[s_{46} - s_{16} \right] \cdot \left[s_{23} - s_{35} - s_{25} \right] ,$$

$$D^s(4) = \frac{1}{s_{25}\, t_{146}} \left[s_{23} - s_{35} - s_{25} \right] ,$$

$$D^s(5) = \frac{1}{s_{14}\, s_{36}\, t_{124}} \left[s_{12} - s_{24} \right] \cdot \left[s_{56} - s_{35} \right] ,$$

$$D^s(6) = \frac{1}{s_{14}\, t_{124}} \left[s_{12} - s_{24} \right] ,$$

$$D^s(7) = \frac{1}{s_{36}\, t_{124}} \left[s_{56} - s_{35} \right] ,$$

$$D^s(8) = \frac{1}{t_{124}} ,$$

$$D^s(9) = \frac{1}{s_{14}\, s_{35}\, t_{124}} \left[s_{12} - s_{24} \right] \cdot \left[s_{56} - s_{36} \right] ,$$

$$D^s(10) = \frac{1}{s_{35}\, t_{124}} \left[s_{56} - s_{36} \right] ,$$

$$D^s(11) = \frac{1}{s_{25}\, s_{36}\, t_{125}} \left[s_{12} - s_{15} - s_{25} \right] \cdot \left[s_{46} - s_{34} + s_{36} \right] ,$$

$$D^s(12) = \frac{1}{s_{36}\, t_{125}} \left[s_{46} - s_{34} + s_{36} \right] ,$$

$$D^s(13) = \frac{1}{t_{125}} \quad ,$$

$$D^s(14) = \frac{1}{s_{35}\, t_{135}} \left[s_{13} - s_{15} - s_{35} \right] \quad ,$$

$$D^s(15) = \frac{1}{s_{25}\, s_{36}\, t_{136}} \left[s_{13} - s_{16} - s_{36} \right] \cdot \left[s_{45} - s_{24} + s_{25} \right] \quad ,$$

$$D^s(16) = \frac{1}{s_{36}\, t_{136}} \left[s_{13} - s_{16} - s_{36} \right] \quad ,$$

$$D^s(17) = \frac{1}{s_{35}\, t_{126}} \left[s_{45} - s_{34} + s_{35} \right] \quad ,$$

$$D^s(18) = \frac{1}{t_{126}} \quad ,$$

$$D^s(19) = \frac{1}{2\, s_{25}\, s_{36}} \left[s_{23} - s_{26} - s_{35} + s_{56} \right] \quad ,$$

$$D^s(20) = \frac{1}{2\, s_{14}\, s_{36}} \left[s_{13} - s_{16} - s_{34} + s_{46} \right] \quad ,$$

$$D^s(21) = \frac{1}{2\, s_{14}\, s_{25}} \left[s_{12} - s_{24} - s_{15} + s_{45} \right] \quad ,$$

$$\begin{aligned} D^s(22) = \frac{1}{2\, s_{14}\, s_{25}\, s_{36}} \Big\{ & \left[s_{12} - s_{24} - s_{15} + s_{45} \right] \cdot \left[s_{16} - s_{13} + s_{46} - s_{34} \right] \\ & + \left[s_{23} - s_{26} - s_{35} + s_{56} \right] \cdot \left[s_{24} - s_{12} + s_{45} - s_{15} \right] \\ & + \left[s_{13} - s_{16} - s_{34} + s_{46} \right] \cdot \left[s_{35} - s_{23} + s_{56} - s_{26} \right] \Big\} . \end{aligned} \tag{13}$$

The diagrams D^G are listed below:

$$D^G(1) = \frac{2}{s_{36}\, t_{124}} \left[s_{35} - s_{56} \right] \cdot \left[s_{12} - s_{24} + s_{14} \right] ,$$

$$D^G(2) = \frac{2}{s_{25}\, t_{134}} \left[s_{26} - s_{56} \right] \cdot \left[s_{13} - s_{34} + s_{14} \right] ,$$

$$D^G(3) = \frac{2}{s_{36}\, t_{145}} \left[s_{26} - s_{23} \right] \cdot \left[s_{45} - s_{15} + s_{14} \right] ,$$

$$D^G(4) = \frac{2}{s_{25}\, t_{146}} \left[s_{35} - s_{23} \right] \cdot \left[s_{46} - s_{16} + s_{14} \right] ,$$

$$D^G(5) = \frac{2}{s_{36}} \left(p_1 - p_4 \right) \left(p_3 - p_6 \right) ,$$

$$D^G(6) = \frac{2}{s_{25}} \left(p_1 - p_4 \right) \left(p_2 - p_5 \right) ,$$

$$\begin{aligned} D^G(7) = \frac{4}{s_{25}\, s_{36}} \Big\{ & \left[(p_2 - p_5)(p_3 - p_6) \right] \cdot \left[(p_1 - p_4)(p_3 + p_6) \right] \\ & - \left[(p_2 - p_5)(p_3 + p_6) \right] \cdot \left[(p_1 - p_4)(p_3 - p_6) \right] \\ & + \left[(p_2 + p_5)(p_3 - p_6) \right] \cdot \left[(p_1 - p_4)(p_2 - p_5) \right] , \end{aligned}$$

$$
\begin{aligned}
D^G(8) = \frac{4}{s_{25}\, s_{36}\, t_{125}} \Big\{ & \big[(p_1+p_2-p_5)(p_4+p_3-p_6)\big] \cdot H(p_2, p_3) \\
& - \big[(p_1+p_2-p_5)(p_4-p_3+p_6)\big] \cdot H(p_2, p_6) \\
& - \big[(p_1-p_2+p_5)(p_4+p_3-p_6)\big] \cdot H(p_5, p_3) \\
& + \big[(p_1-p_2+p_5)(p_4-p_3+p_6)\big] \cdot H(p_5, p_6) \\
& - \big[p_1(p_2-p_5)\big] \cdot H(p_3-p_6, p_3+p_6) \\
& - \big[p_4(p_3-p_6)\big] \cdot H(p_2+p_5, p_2-p_5) \\
& + 2\, s_{14} \cdot \big[p_1(p_2-p_5)\big] \cdot \big[p_4(p_3-p_6)\big] \Big\} ,
\end{aligned}
$$

$$
\begin{aligned}
D^G(9) = \frac{2}{s_{25}\, s_{36}} \Big\{ & s_{14} \cdot \big[(p_2-p_5)(p_3-p_6)\big] \\
& - H(p_3-p_6, p_2-p_5) + H(p_2-p_5, p_3-p_6) \Big\} ,
\end{aligned}
$$

$$
D^G(10) = \frac{2\, s_{14}}{t_{125}} ,
$$

$$
D^G(11) = \frac{2}{s_{25}\, t_{125}} \Big\{ s_{14}\, (s_{12}-s_{15}) - H(p_2+p_5, p_2-p_5) \Big\} ,
$$

$$
D^G(12) = \frac{2}{s_{36}\, t_{125}} \Big\{ s_{14}\, (s_{46}-s_{34}) + H(p_3-p_6, p_3+p_6) \Big\} ,
$$

$$D^G(13) = \frac{4}{s_{12}\, s_{36}\, t_{125}} \Big\{ \big[(p_1+p_2-p_5)(p_4-p_3+p_6)\big] \cdot H(p_2, p_6)$$
$$- \big[(p_1+p_2-p_5)(p_4+p_3-p_6)\big] \cdot H(p_2, p_3)$$
$$+ \big[p_4(p_3-p_6)\big] \cdot H(p_2, p_2-p_5) \Big\} ,$$

$$D^G(14) = \frac{4}{s_{25}\, s_{34}\, t_{125}} \Big\{ \big[(p_1-p_2+p_5)(p_4+p_3-p_6)\big] \cdot H(p_5, p_3)$$
$$- \big[(p_1+p_2-p_5)(p_4+p_3-p_6)\big] \cdot H(p_2, p_3)$$
$$+ \big[p_1(p_2-p_5)\big]\, H(p_3-p_6, p_3) \Big\} ,$$

$$D^G(15) = \frac{4}{s_{15}\, s_{36}\, t_{125}} \Big\{ \big[(p_1-p_2+p_5)(p_4-p_3+p_6)\big] \cdot H(p_5, p_6)$$
$$- \big[(p_1-p_2+p_5)(p_4+p_3-p_6)\big] \cdot H(p_5, p_3)$$
$$- \big[p_4(p_3-p_6)\big]\, H(p_5, p_2-p_5) \Big\} ,$$

$$D^G(16) = \frac{4}{s_{25}\, s_{46}\, t_{125}} \Big\{ \big[(p_1-p_2+p_5)(p_4-p_3+p_6)\big] \cdot H(p_5, p_6)$$
$$- \big[(p_1+p_2-p_5)(p_4-p_3+p_6)\big] \cdot H(p_2, p_6)$$
$$- \big[p_1(p_2-p_5)\big]\, H(p_3-p_6, p_6) \Big\} ,$$

$$D^G(17) = \frac{4}{s_{12}\, s_{36}\, t_{124}} \left[s_{56} - s_{35} \right] H(p_2, p_2) \quad ,$$

$$D^G(18) = \frac{4}{s_{45}\, s_{36}\, t_{145}} \left[s_{23} - s_{26} \right] H(p_5, p_5) \quad ,$$

$$D^G(19) = \frac{4}{s_{13}\, s_{25}\, t_{134}} \left[s_{56} - s_{26} \right] H(p_3, p_3) \quad ,$$

$$D^G(20) = \frac{4}{s_{25}\, s_{46}\, t_{146}} \left[s_{23} - s_{35} \right] H(p_6, p_6) \quad ,$$

$$D^G(21) = \frac{2}{s_{12}\, s_{36}} H(p_2, p_3 - p_6) \quad ,$$

$$D^G(22) = \frac{-2}{s_{36}\, s_{45}} H(p_3 - p_6, p_5) \quad ,$$

$$D^G(23) = \frac{2}{s_{34}\, t_{125}} H(p_3 - p_6, p_3) \quad ,$$

$$D^G(24) = \frac{-2}{s_{46}\, t_{125}} H(p_3 - p_6, p_6) \quad ,$$

$$D^G(25) = \frac{2}{s_{12}\, t_{125}} H(p_2, p_2 - p_5) \quad ,$$

$$D^G(26) = \frac{-2}{s_{15}\, t_{125}} H(p_5, p_2 - p_5) \quad ,$$

$$D^G(27) = \frac{2}{s_{13}\, s_{25}} H(p_3, p_2 - p_5) \quad ,$$

$$D^G(28) = \frac{-2}{s_{25}\, s_{46}} H(p_2 - p_5, p_6) \quad ,$$

$$D^G(29) = \frac{8}{s_{12}\, s_{36}\, s_{45}} \left[(p_3 - p_6)(p_4 + p_5) \right] \cdot H(p_2, p_5) \quad ,$$

$$D^G(30) = \frac{4}{s_{12}\, s_{46}\, t_{125}} \left[(p_1 + p_2 - p_5)(p_4 - p_3 + p_6) \right] \cdot H(p_2, p_6) \quad ,$$

$$D^G(31) = \frac{4}{s_{12}\, s_{34}\, t_{125}} \left[(p_1 + p_2 - p_5)(p_4 + p_3 - p_6) \right] \cdot H(p_2, p_3) \quad ,$$

$$D^G(32) = \frac{4}{s_{15}\, s_{34}\, t_{125}} \left[(p_1 - p_2 + p_5)(p_4 + p_3 - p_6) \right] \cdot H(p_5, p_3) \quad ,$$

$$D^G(33) = \frac{4}{s_{15}\, s_{46}\, t_{125}} \left[(p_1 - p_2 + p_5)(p_4 - p_3 + p_6) \right] \cdot H(p_5, p_6) \quad ,$$

$$D^G(34) = \frac{8}{s_{13}\, s_{25}\, s_{46}} \left[(p_2 - p_5)(p_4 + p_6) \right] \cdot H(p_3, p_6) \quad ,$$

$$D^G(35) = \frac{2}{t_{124}} \left[s_{24} - s_{12} - s_{14} \right] \quad ,$$

$$D^G(36) = \frac{2}{t_{134}} \left[s_{34} - s_{13} - s_{14} \right] \quad ,$$

$$D^G(37) = \frac{2}{t_{145}} \left[s_{15} - s_{45} - s_{14} \right] \quad ,$$

$$D^G(38) = \frac{2}{t_{146}} \left[s_{16} - s_{46} - s_{14} \right] \quad ,$$

$$D^G(39) = \frac{4}{s_{12}\, t_{124}} H(p_2, p_2) ,$$

$$D^G(40) = \frac{4}{s_{45}\, t_{145}} H(p_5, p_5) ,$$

$$D^G(41) = \frac{4}{s_{13}\, t_{134}} H(p_3, p_3) ,$$

$$D^G(42) = \frac{4}{s_{46}\, t_{146}} H(p_6, p_6) ,$$

$$D^G(43) = \frac{-4}{s_{12}\, s_{45}} H(p_2, p_5) ,$$

$$D^G(44) = \frac{-4}{s_{12}\, s_{46}} H(p_2, p_6) ,$$

$$D^G(45) = \frac{4}{s_{12}\, s_{34}} H(p_2, p_3) ,$$

$$D^G(46) = \frac{-4}{s_{15}\, s_{34}} H(p_5, p_3) ,$$

$$D^G(47) = \frac{4}{s_{15}\, s_{46}} H(p_5, p_6) ,$$

$$D^G(48) = \frac{-4}{s_{13}\, s_{46}} H(p_3, p_6) . \qquad \textbf{(14)}$$

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams D are calculated by using eqs. (11,12,13,14). The result is substituted to eq. (8) to obtain the vectors $\mathscr{D}_0$ and $\mathscr{D}_2$. After generating the vectors $\mathscr{D}_{0_\pi}$ and $\mathscr{D}_{2_\tau}$ by the appropriate permutations of momenta, eq. (6) is used to obtain the functions B_0 and B_2. Finally, the total cross section is calculated by using eq. (5). The FORTRAN5 program based on such a scheme generates two Monte-Carlo points in less than a second on the heterotic CDC CYBER 175/875.

The following testing procedures can be used in the numerical calculations based on the algorithm presented in this paper. First, the function $B_2(p_1,p_2,p_3,p_4,p_5,p_6)$ must be symmetric under arbitrary permutations of the momenta (p_1,p_3,p_4). Another, very important test examines the singular behaviour of the cross section in the kinematical limit of any two partons i and j moving collinearly, i.e. with s_{ij} going to zero. The double poles of the form $(s_{ij})^{-2}$ should be absent and further, in the leading $(s_{ij})^{-1}$ pole approximation, the answer should reduce to the appropriate two goes to three cross section [4], convoluted with the Altarelli-Parisi probability [5] for the decay of the final particles into partons i and j. For example, when the quark and antiquark momenta become parallel, the invariant quark - antiquark - four gluon matrix element squared must factorize into the five gluon matrix element squared and the Altarelli-Parisi probability for gluon branching into a quark-antiquark pair. It is worth mentioning that, similarly to the case of the lepton - antilepton - four parton matrix element squared [6], the factorization holds only after averaging over the azimuthal angle of the

branching process plane.

Our result has succesfully passed both these numerical checks.

While this manuscript was in preparation, we received a message from Z. Kunszt, via BITNET, that he has also completed a numerical routine for this cross section [7] using a different set of techniques. Together, we have made a numerical comparison of the two results and complete agreement was found.

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Table I. Matrices K(I,J) [I=1-16,J=1-16].

Matrix $K^{(0)}$

−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Matrix $K^{(2)}$

3	2	−1	−2	1	0	−1	−1	1	1	1	1	0	0	0	0
2	3	−2	−1	0	1	−1	−1	1	1	1	1	0	0	0	0
−1	−2	3	2	1	0	1	1	−1	−1	−1	−1	0	0	0	0
−2	−1	2	3	0	1	1	1	−1	−1	−1	−1	0	0	0	0
1	0	1	0	3	2	1	1	−1	−1	−1	−1	0	0	0	0
0	1	0	1	2	3	1	1	−1	−1	−1	−1	0	0	0	0
−1	−1	1	1	1	1	2	2	0	0	0	0	0	0	0	0
−1	−1	1	1	1	1	2	2	0	0	0	0	0	0	0	0
1	1	−1	−1	−1	−1	0	0	2	2	2	2	0	0	0	0
1	1	−1	−1	−1	−1	0	0	2	2	2	2	0	0	0	0
1	1	−1	−1	−1	−1	0	0	2	2	2	2	0	0	0	0
1	1	−1	−1	−1	−1	0	0	2	2	2	2	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table I continued

Matrix $K^{(4)}$

−3	−1	1	1	0	1	0	−1	−1	−1	−1	−1	0	0	1	−1
−1	−3	1	1	1	0	1	1	−1	−2	1	0	0	2	−1	−1
1	1	−3	−1	0	1	0	0	0	1	1	0	1	−1	0	0
1	1	−1	−3	1	0	0	−1	−1	1	0	0	−1	−1	0	2
0	1	0	1	−3	−1	−1	−1	2	1	0	−1	1	1	1	1
1	0	1	0	−1	−3	1	0	1	1	1	1	−1	1	−1	1
0	1	0	0	−1	1	−4	−2	1	2	−1	0	−1	−1	0	2
−1	1	0	−1	−1	0	−2	−4	2	1	0	−1	−1	−1	0	2
−1	−1	0	−1	2	1	1	2	−4	−2	0	2	0	0	−2	−2
−1	−2	1	1	1	1	2	1	−2	−4	2	0	2	2	−2	−2
−1	1	1	0	0	1	−1	0	0	2	−4	−2	−2	−2	2	2
−1	0	0	0	−1	1	0	−1	2	0	−2	−4	0	0	2	2
0	0	1	−1	1	−1	−1	−1	0	2	−2	0	−4	−4	2	2
0	2	−1	−1	1	1	−1	−1	0	2	−2	0	−4	−4	2	2
1	−1	0	0	1	−1	0	0	−2	−2	2	2	2	2	−4	−4
−1	−1	0	2	1	1	2	2	−2	−2	2	2	2	2	−4	−4

Matrix $K^{(6)}$

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	1	0	0	0	−1	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	2	0	0	0	1	0	0	0	−1	0
0	0	0	0	0	0	0	2	0	0	0	1	0	1	0	0
0	0	0	0	−1	0	0	0	2	0	0	0	−1	0	0	0
0	1	0	0	0	0	0	0	0	2	0	0	0	0	0	2
0	0	0	0	0	0	1	0	0	0	2	0	0	0	−2	0
0	0	0	0	0	0	0	1	0	0	0	2	0	1	0	0
0	0	0	0	0	0	0	0	−1	0	0	0	4	0	1	0
0	0	0	1	0	0	0	1	0	0	0	1	0	4	0	1
0	0	0	0	0	0	−1	0	0	0	−2	0	1	0	4	0
0	1	0	0	0	0	0	0	0	2	0	0	0	1	0	4

Table I continued

Matrix $K_p^{(0)}$

−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Matrix $K_p^{(2)}$

2	1	−2	−3	0	−1	−1	−1	1	1	1	1	0	0	0	0
1	2	−3	−2	−1	0	−1	−1	1	1	1	1	0	0	0	0
−2	−3	2	1	0	−1	1	1	−1	−1	−1	−1	0	0	0	0
−3	−2	1	2	−1	0	1	1	−1	−1	−1	−1	0	0	0	0
0	−1	0	−1	−2	−3	−1	−1	1	1	1	1	0	0	0	0
−1	0	−1	0	−3	−2	−1	−1	1	1	1	1	0	0	0	0
−1	−1	1	1	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	1	1	−1	−1	0	0	0	0	0	0	0	0	0	0
1	1	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0
1	1	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0
1	1	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0
1	1	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table I continued

Matrix $K_\rho^{(4)}$

−1	0	1	0	0	0	0	1	0	0	−1	1	−1	−1	2	0
0	−1	0	1	0	0	−1	1	0	−1	−1	0	−1	1	0	0
1	0	−1	0	0	0	−1	−1	0	−1	2	1	2	0	−1	−1
0	1	0	−1	0	0	−1	−2	1	1	1	1	0	0	−1	1
0	0	0	0	1	0	2	1	−1	−1	−1	−1	1	−1	1	−1
0	0	0	0	0	1	1	1	1	0	−1	−2	1	1	1	1
0	−1	−1	−1	2	1	2	0	−1	−1	1	1	−2	−2	0	0
1	1	−1	−2	1	1	0	0	−1	1	−1	1	−2	−2	2	2
0	0	0	1	−1	1	−1	−1	2	0	1	1	0	2	−1	−1
0	−1	−1	1	−1	0	−1	1	0	0	−1	1	0	2	−1	−1
−1	−1	2	1	−1	−1	1	−1	1	−1	0	0	−2	0	1	1
1	0	1	1	−1	−2	1	1	1	1	0	2	−2	0	1	1
−1	−1	2	0	1	1	−2	−2	0	0	−2	−2	−2	−2	2	2
−1	1	0	0	−1	1	−2	−2	2	2	0	0	−2	−2	2	2
2	0	−1	−1	1	1	0	2	−1	−1	1	1	2	2	−2	−2
0	0	−1	1	−1	1	0	2	−1	−1	1	1	2	2	−2	−2

Matrix $K_\rho^{(6)}$

0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	0	0	0	1	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	−1	0	0	0	1	0	0	0	0	0	−1	0
0	0	0	1	0	0	0	0	0	0	0	0	0	2	0	0
0	0	0	0	0	0	1	0	0	0	0	0	−1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	−1	0	0	0	0	0	0	0	0	0	0	0	−1	0
0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0
1	0	0	0	0	0	0	0	−1	0	0	0	2	0	0	0
0	0	0	0	0	0	0	2	0	0	0	1	0	2	0	0
0	0	1	0	0	0	−1	0	0	0	−1	0	0	0	2	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	2

Table II. Matrices $C^F(I,J)$ [I=1-16,J=1-22], $C^A(I,J)$ [I=1-16,J=1-24], $C^S(I,J)$ [I=1-16,J=1-22] and $C^G(I,J)$ [I=1-16,J=1-48]. Indices I and J specify row numbers and column numbers, respectively.

Matrix C^F

0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1
1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	1	0	0	1	1	0	0	1	1
0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0
1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
1	1	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	1	1	1	1	1	0	0	1	1
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	1	1	1	1	0	0	0	0	0	0	0	0
1	1	2	2	1	1	1	1	0	0	0	0	0	0	0	0

Table II continued

Matrix C^A

0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0
0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	0	0	0	-1	0	1	0	0	0	0	0	0	0
-1	0	0	0	-1	0	1	0	0	0	0	0	0	0	0
1	-1	0	0	1	-1	-1	1	0	0	0	0	0	0	0
0	0	0	0	1	-1	-1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	-1	1	1	-1	-1	1	0	0	1	-1
-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
1	-1	0	0	-1	1	1	-1	0	0	0	0	0	0	0
1	-1	-2	2	1	-1	-1	1	0	0	0	0	0	0	0

Table II continued

Matrix C^S

0	0	0	0	0	1	0	1	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	−1	0	−1	1	1	0	0	0	0	0	0	−2	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	−1	0	−1	1	1	0	0	−4	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	1	−1	−1	0	0	0	0	0	0	2	−4	0	0	0	0
0	0	0	0	0	1	0	−1	−1	1	0	0	0	2	0	0	0	0	0	0	0	0
0	0	0	0	0	−1	0	1	1	−1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	−2	1	−2	1	1	−1	−2	0	0	0	0	0	0	1	−1	1	−1
0	0	0	0	−1	2	−1	2	−1	−1	0	0	0	0	−1	0	2	0	1	1	−1	1
0	0	0	0	−1	1	1	−1	0	0	1	0	0	0	0	0	0	0	−1	1	−1	1
0	0	0	0	1	−1	−1	1	0	0	0	0	0	0	1	−2	0	0	−1	−1	1	−1
0	0	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−2	0
1	−1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	−1	−1	−1	1
−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	−1	1	1	−1

Table II continued

Matrix C^G [columns J=1-24]

1	0	0	0	1	0	1	−1	−1	−1	−1	1	1	1	0	0	1	0	0	0	−1	0	1	0
−1	0	0	0	−1	0	−1	1	1	1	1	−1	−1	−1	0	0	−1	0	0	0	1	0	−1	0
0	0	1	0	1	0	−1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	−1	0	0
0	0	−1	0	−1	0	1	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	1	0	0
0	0	0	0	0	0	0	1	1	1	1	−1	−1	−1	0	0	0	0	0	0	−1	−1	−1	0
0	0	0	0	0	0	0	−1	−1	−1	−1	1	1	1	0	0	0	0	0	0	1	1	1	0
0	0	0	0	0	0	0	−1	−1	1	−1	−1	0	1	−1	0	0	0	0	0	0	0	−1	0
0	0	0	0	0	0	0	1	1	−1	1	1	0	−1	1	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	2	2	0	0	−1	0	−1	0	0	0	0	0	0	−1	−1
0	0	0	0	0	0	0	0	0	−2	−2	0	0	1	0	1	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	1	1	1	−1	1	0	0	1	1	0	0	0	0	0	0	0	−1
0	0	0	0	0	0	0	−1	−1	−1	1	−1	0	0	−1	−1	0	0	0	0	0	0	0	1
0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	−1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0
0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	−1	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0

Table II continued

Matrix C^G [columns J=25-48]

1	0	0	0	0	0	−1	0	0	0	1	0	0	0	1	0	0	0	0	0	−1	0	0	0
−1	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	−1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0
−1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
1	0	0	0	−1	0	−1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
0	−1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	1	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
−2	0	1	1	0	1	1	0	0	−1	0	0	0	0	0	0	0	0	0	1	1	0	0	−1
2	0	−1	−1	0	−1	−1	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	−1
0	−1	−1	−1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	1	1	1	0	0	0	0	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	−1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0
0	0	−1	−2	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	1
0	0	1	2	0	0	0	0	0	−1	0	0	0	1	0	0	0	1	0	0	0	0	0	1