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Two dimensional Liouville, sine-Gordon, and non-linear σ-models

K. Tanaka

Fermi National Accelerator Laboratory

Batavia, Illinois 60510

and

Department of Physics

The Ohio State University

Columbus, Ohio 43210

Abstract

The Liouville, sine-Gordon and non-linear σ-model in 2 dimensions are obtained from a common formalism. The topological charge of the Liouville model is found to be zero, if the definiton of the topological charge of the sine-Gordon model is utilized.

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The two-dimensional classical models are of interest because they usually have solutions in closed form and are amenable to studies on their relationships, symmetry properties, and mathematical methods.[1-5]

Consider the SL(2C) operator g of the following form considered by Bohr, Hou, and Saito[1]

$$g=i\begin{pmatrix}-\alpha & u\\ v & \alpha-1\end{pmatrix}(1+uv)^{-1/2} \tag{1}$$

For technical convenience, we choose $\alpha=-1$, and $u=v^*$, so that we have an SU(2) group element with two parameters,

$$g=i\begin{pmatrix}1 & u\\ v & -1\end{pmatrix}(1+uv)^{-1/2}\ , \tag{2}$$

where the determinant of g is unity, and u and v are complex variables. We use light-cone coordinates in two-dimensional Euclidean space $x_\pm=\frac{1}{2}(x_1\pm ix_2)$, and take the field strength to have zero curvature

$$F\pm = \partial_+A_- - \partial_-A_+ + [A_+,A_-] = 0 \tag{3}$$

so that

$$A_\pm = g^{-1}\partial_\pm g. \tag{4}$$

From (2) and (4)

$$A_\pm = \frac{u\partial_\pm v - v\partial_\pm u}{2(1+uv)}\sigma_3 + \frac{\partial_\pm u}{(1+uv)}\sigma_+ - \frac{\partial_\pm v}{1+uv}\sigma_- \tag{5}$$

where

$$A_+^\dagger = -A_-,$$

$$\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \sigma+=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\ \sigma-=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

The common Lagrangian for the classical models is

$$L = -\mathrm{Tr}(A_+A_-)$$

$$= -\frac{1}{2}\frac{(u^2\partial_+v\partial_-v+v^2\partial_+u\partial_-u)}{(1+uv)^2} + \frac{(2+uv)(\partial_+u\partial_-v+\partial_+v\partial_-u)}{2(1+uv)^2} \tag{6}$$

It was shown in BHS that when one restricts the model to

$$\partial_- u = \partial_+ v = 0,$$

and defines

$$e^{\varphi_+} = \frac{\partial_+u}{1+uv} \qquad e^{\varphi_-} = -\frac{\partial_-v}{1+uv}$$

$$\varphi_\pm = \frac{1}{2}\rho \pm 2i\beta \tag{7}$$

the Euler-Lagrange (E-L) equations for ρ and β are [1]

$$\partial_-\partial_+\rho + 4e^{\rho} = 0 \tag{8}$$

$$\partial_-\partial_+\beta = 0. \tag{9}$$

A solution of the Liouville equation (8) is

$$\rho = -2\ln\cosh\delta \tag{10}$$

where

$$\delta = (1+i)x_+ + (1-i)x_- = x - vt. \tag{11}$$

This model that contains two independent field variables ρ and β is regarded as the extended Liouville model. When instanton like solutions of the corresponding σ-model are utilized in the extended Liouville model, the variable β appears as a winding number and the topological charge $q = \frac{1}{4\pi}\int d\cos\theta d\psi$ of the Liouville model yields a fractional charge.[1]

We obtain a corresponding sine-Gordon equation from the Lagrangian (6) and use the definition of the topological charge of the sine-Gordon equation to examine the topological charge of the Liouville model.

One can obtain the E-L equations for u and v of (6) and then carry out a transformation of variables to θ and ψ, but we take the simplier approach of carrying out the transformation first and then obtain the E-L equations for θ and ψ , with no restrictions on u and v.

Let us change the u,v variables to θ,ψ in L of (6) by the transformation

$$u = \tan\theta e^{-i\psi}$$

$$v = \tan\theta e^{i\psi} \tag{12}$$

and obtain

$$L = 2\,(\partial_+\theta\partial_-\theta+\sin^2\theta\;\partial_+\psi\partial_-\psi). \tag{13}$$

The E-L equations for θ and ψ are from (13)

$$\partial_-\partial_+\theta - \sin\theta\cos\theta\;\partial_-\psi\partial_+\psi = 0 \tag{14}$$

$$\partial_-\partial_+\psi+ \cot\theta(\partial_+\theta\partial_-\psi+\partial_-\theta\partial_+\psi) = 0 \tag{15}$$

Equations (14) and (15) may be regarded as E-L equations for an extended sine-Gordon model, and will be used to obtain the E-L equations of the non-linear σ-model.

A sine-Gordon like solution is given by taking

$$\psi = (1-i)x_+ +(1+i)x_-, \tag{16}$$

$$\theta = 4\tan^{-1}e^{\delta}, \tag{17}$$

where δ is given by Eq. (11).

When (16) is put into (14) it becomes

$$\partial_-\partial_+\theta-2\sin\theta\cos\theta = \partial_-\partial_+\theta- \sin 2\theta = 0$$

or

$$\partial_-\partial_+\theta-2\sin\theta = 0, \tag{18}$$

and when(16) and (17) is substituted in (15), it is identically zero.

It was pointed out by Dolan[3] that the curvature scalar R of the Liouville equation and the sine-Gordon equation are identical as from Eqs. (8) and (18)

$$R = -\partial_-\partial_+\theta/\sin\theta = \frac{1}{2}\partial_-\partial_+\rho e^{-\rho} = -2.$$

Indeed, a solution (10) of the Liouville equation and a solution (17) of the sine-Gordon equation are related by

$$-\partial-\partial+\rho = 4 + \sinh\delta\ \partial-\partial+\theta \tag{19}$$

We define the topological charge q of the sine-Gordon model by

$$q = \frac{1}{2\pi}[\theta(x+=\infty,x-)-\theta(x+=-\infty,x-)] = \frac{1}{2\pi}[\theta(x=\infty,\ t)-\theta(x=-\infty,\ t)] \tag{20}$$

and obtain from (11), (17) and (20)

$$q = \frac{1}{2\pi}(2\pi-0) = 1.$$

The Liouville model is known[6] to have a stable classical solution with minimum energy only at $\rho = -\infty$. Nevertheless, if we assume that the same definition for the topological charge is valid for the Liouville model (because of the close relation between the two models), we find

$$q = \frac{1}{2\pi}[\rho(x+=\infty,x-) - \rho(x+=-\infty,x-)] = 0, \tag{21}$$

with the aid of Eq. (10) as ρ is an even function of δ. The topological charge in BHS is given by

$$q = \frac{1}{4\pi}\int d\cos\theta\ d\psi = -\frac{1}{2}[f(\infty) - f(0)] \tag{22}$$

where

$$\cos\theta = f(|u|) \tag{23}$$

The topological charge for the Liouville model is from (12), (22) and (23), $q = -\frac{1}{2}(0-1) = \frac{1}{2}$, where $\cos\theta = f(|\tan\theta|) = (1+\tan^2\theta)^{-\frac{1}{2}}$.

For the O(3) non-linear σ-model, one chooses [1] the vector $\underset{\sim}{N} = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$ and $g = i\,\underset{\sim}{N}\cdot\underset{\sim}{\sigma}$ corresponding to Eq.(2), where σ are the Pauli matrices. It is convenient to consider the fields on a coset space SU(2)/U(1) defined on a two sphere S^2 i.e.

$$W = \frac{N_1 - iN_2}{1-N_3} = \cot\frac{\theta}{2}\, e^{-i\psi}, \tag{24}$$

$$Y = \frac{N_1 + iN_2}{1-N_3} = \cot\frac{\theta}{2}\, e^{i\psi},$$

We then invert (24) and obtain

$$\theta = \cos^{-1}\frac{WY-1}{WY+1}, \tag{25}$$

$$\psi = \frac{i}{2}\ln\left(\frac{W}{Y}\right),$$

and substitute (25) into the equations of motion of the coupled θ and ψ given in Eqs. (14) and (15) and obtain

$$\partial_-\partial_+ W - \frac{2Y\partial_- W\partial_+ W}{1+WY} = 0, \tag{26}$$

$$\partial_-\partial_+ Y - \frac{2W\partial_- Y\partial_+ W}{1+WY} = 0.$$

The instanton solutions of (26) are obtained when one imposes the conditions[1]

$$\partial_- W = 0, \qquad \partial_+ Y = 0.$$

It has been shown above that the BHS formalism can be extended to also incorporate the two-dimensional sine-Gordon equation.

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Reference

1. H. Bohr, B. Hou, and S. Saito, Nuovo Cimento 84 A, 237, (1984) This paper is refered to as BHS.

2. K. Pohlmeyer, Comm. Math. Phys. 46 207 (1976)

3. L. Dolan, Phys. Rev. 15, 2337 (1977)

4. A comprehensive review article is H. C. Thacker, Rev. of Mod. Phys. 53, 253 (1981)

5. L. Johansson, A. Kihlberg, and R. Marnelius, Phys. Rev. D29, 2798 (1984)

6. E. D' Hoker and R. Jackiw, Phys. Rev. Lett. 50, 1719 (1983)

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