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Stability of DeSitter Space Coupled to a Quantum Field

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Abstract

We consider the Einstein gravitational field equations coupled to the expectation value of the stress tensor in a real scalar field theory with arbitrary mass and coupling to curvature. We consider small oscillations of the metric and linearize the response of the quantum wave-functional and the stress tensor expectation value. In a small time approximation we find unstable oscillations for a minimally coupled field and stable oscillations in the case of conformal coupling.

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I. Introduction

One can often gain insight into the properties of interacting quantum field theories by maintaining a semiclassical approximation in which one or several fields are classical while the remaining fields are treated quantum mechanically. For example, such an approximation works well in obtaining the low energy theorems of QED in which the electron is quantized in a classical background electromagnetic field[1]. The back reaction may be approximated by allowing the electromagnetic field to couple to the expectation value of the electromagnetic current.

An analogous class of problems arises in general relativity. We may wish to treat the gravitational field classically but consider its response to quantum matter fields. Thus, we consider the Einstein field equation with the expectation value of the stress tensor appearing on "the right hand side." This expectation value is taken in a quantum state of the matter fields. The matter field state (wave functional) evolves in the background geometry determined by the Einstein equation. Thus, we seek the solutions to the coupled pair of equations:

$$
\begin{aligned}
G_{\mu\nu} - \Lambda g_{\mu\nu} &= 8\pi G_N \langle\Psi| T_{\mu\nu} |\Psi\rangle \\
H_g |\Psi\rangle &= i \frac{\partial}{\partial x^0} |\Psi\rangle
\end{aligned}
\tag{1}
$$

where the quantum field theory is described by a functional Schroedinger equation involving a Hamiltonian, H_g, which depends upon the metric g.

Much work has been done on the problem of quantum field theory in curved spacetime[2]. In particular, the stress-tensor has been evaluated successfully in various background geometries, which involves regularization of the usual short-distance infinities and renormalization. For example, $\langle T_{\mu\nu} \rangle$ has been evaluated by Bunch, Davies and others in deSitter space[3] in a particular deSitter invariant vacuum state (which we refer to subsequently as the "Bunch-Davies" vacuum). There exists a self-consistent solution of eqs.(1) in which the rhs of Einstein's equation is this expectation value of the stress-tensor. In the conformal limit, $\mu=0$ and $\xi=1/6$, the value of the Hubble constant determined by this solution is $H=6M_{Pl}\sqrt{10\pi/N}$ where N is the number of noninteracting real scalar fields. Thus, for a single scalar field we have unrealistically large curvature and the neglect of quantum gravity becomes an inconsistency. In the large N limit the semiclassical approximation becomes valid.

The question naturally arises whether the self-consistent deSitter space solution is stable against small fluctuations. We shall analyse this question in the present paper. A proper stability analysis in relativity requires that one study arbitrary metric and matter perturbations. However, it is quite difficult to calculate the renormalized stress-energy in the general case. We shall consider presently Robertson-Walker metrics with scale factor a(t). In deSitter space $H=a^{-1}da(t)/dt$ is a constant H_0. The gravitational fluctuations studied here will be small variations in H. We will then obtain the linearized response of the quantum wave-functional to the metric perturbation through the functional Schroedinger equation. This is used to compute the first order change of the expectation value of the stress tensor, $\langle T_{\mu\nu} \rangle$. Substituting $\langle T_{\mu\nu} \rangle$ into Einstein's equations gives a linear differential equation for the small geometry fluctuation which turns out to be a second order oscillator equation. The values of the parameters in this equation depend upon the mass and conformal coupling parameter of the quantum fields. We will see that the system has stable fluctuations for a conformally coupled field (though it may be unstable to more general metric fluctuations). On the other hand, the system is unstable to perturbations in H for a minimally coupled ($\xi = 0$) field, an unambiguous statement of deSitter space instability.

The dynamical back reaction problem has been studied for conformally coupled massless fields in Robertson-Walker spacetime[4-12]. Other formal approaches to quantum instability of deSitter space have been given[6,7,8]. Extensive numerical work has been done on the semiclassical equation for non-conformal fields, focusing primarily upon the quantum effects in the very early universe at the quantum gravity phase transition[9-12].

Regularization procedures necessarily spoil conformal invariance and there generally occurs an anomalous trace contribution to $\langle T_{\mu\nu} \rangle$. In a "conformal vacuum" the entire stress-tensor is determined by the trace anomaly. The trace anomaly may be written in a general form involving geometrical tensors and two free parameters which depend upon the matter content of the theory. Starobinsky[4] found that deSitter space was unstable to perturbations in H for a particular choice of the signs of these parameters and for zero background cosmological constant. In his analysis the rhs of the Einstein equation is taken to be the anomaly. Myrvold[5] allowed a general cosmological constant and found stability or instability for different choices of the signs of various parameters. One might think that these analyses are general because of the conformal symmetry restricting the expectation value of the stress tensor to be given by the trace anomaly. However, conformal symmetry is broken by the anomaly. The geometry will be driven away from a conformal metric and the vacuum away from a conformal one. Thus, the expectation value of the stress-tensor is expected to differ from the anomaly at subsequent times. In the present analysis we will consider more general couplings and arbitrary masses of the fields.

We take the metric signature to be (+,-,-,-), $R^{\alpha}{}_{\beta\gamma\delta} = -\Gamma^{\alpha}{}_{\beta\gamma,\delta} + \ldots$, $R^{\mu}{}_{\nu} = R^{\alpha\mu}{}_{\alpha\nu}$. d refers always to the spatial dimensionality.

II. Semiclassical Einstein Equations

In our analysis gravity is treated as a classical field theory and the stress tensor is given by real scalar quantum fields. We assume a massive scalar field theory coupled to the scalar curvature and described by the action:

$$S = \int d^{d+1}x\, \mathcal{L} = \frac{1}{2}\int d^{d+1}x\, |g|^{1/2}\left\{\nabla_{\mu}\varphi\nabla^{\mu}\varphi - \mu^2\varphi^2 + \xi R\varphi^2\right\} \tag{2}$$

We shall work in d+1 spacetime dimensions for the sake of dimensional regularization and shall perform DeWitt-Schwinger subtractions to obtain renormalized quantities (this is equivalent to Pauli-Villars with some small subtleties[13]). Consider a general Robertson-Walker metric of the form:

$$ds^2 = dt^2 - a(t)^2 \delta_{ij}\, dx^i dx^j \tag{3}$$

The gravitational stress tensor (i.e. the rhs of Einstein's equations) is obtained as usual by variation of the action wrt the metric:

$$T^{(g)}_{\mu\nu} = \frac{-2}{|g|^{1/2}} \frac{\delta S}{\delta g^{\mu\nu}} = \begin{cases} \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{2}\left(\nabla_\rho \varphi \nabla^\rho \varphi - \mu^2 \varphi^2\right) g_{\mu\nu} \\ + \xi G_{\mu\nu} \varphi^2 + \xi \left(\varphi^2{}_{;\mu;\nu} - g_{\mu\nu} \varphi^2{}_{;\rho;}{}^{\rho}\right) \end{cases} \tag{4}$$

Alternatively, the canonical stress tensor may be constructed directly from the Lagrangian as:

$$T^{(c)}_{\mu\nu} = \nabla_\mu \varphi \nabla_\nu \varphi - g_{\mu\nu} \mathcal{L} \tag{5}$$

These tensors are not equal in the presence of coupling to the scalar curvature, and the canonical object is not conserved. Nonetheless, the quantum mechanical Hamiltonian which evolves the wave-functional in coordinate time must be constructed out of the canonical stress tensor. In Heisenberg picture this guarantees consistency between the operator equations of motion and the Heisenberg equations of motion for field operators.[13]

Using the canonical stress tensor we construct a global Hamiltonian on the spacelike surfaces with unit normal η^μ, defined in the coordinate system[3]:

$$H_g = \int d\Sigma^\mu T^{(c)}_{\mu\nu}\eta^\nu = \frac{1}{2}(a(t))^d \int d^d x \left\{ \dot{\varphi}^2 + a(t)^{-2}\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + (\mu^2 - \xi R)\varphi^2 \right\} \tag{6}$$

and the canonical momentum becomes:

$$\pi(x) = |g|^{-1/2}\eta^\mu \frac{\delta S}{\delta \nabla^\mu \varphi} = \partial_0 \varphi \tag{7}$$

To satisfy the equal time commutation relations:

$$[\varphi(x,t), \pi(x',t)] = i|g|^{-1/2}\delta^d(\vec{x}-\vec{x}') \tag{8}$$

substitute:

$$\pi(x) \longrightarrow -i|g|^{-1/2}\frac{\delta^{(d)}}{\delta\varphi(x)} \quad ; \quad \frac{\delta^{(d)}\varphi(x)}{\delta\varphi(y)} \equiv \delta^{d}(x-y) \tag{9}$$

Using the Hamiltonian and the operator representation for the canonical momentum we may write down the Schroedinger equation for the field theory:(13,14)

$$H_g\Psi(\varphi,t) = \frac{1}{2}\int a(t)^d d^d x \left\{-a(t)^{-2d}\frac{\delta^2}{\delta\varphi(x)^2} + a(t)^{-2}\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + (\mu^2-\xi R)\varphi^2\right\}\Psi(\varphi,t) = i\frac{\partial}{\partial t}\Psi(\varphi,t) \tag{10}$$

$\varphi(x)$ is an instantaneous field configuration. The time dependence is carried by $\Psi(\varphi,t)$, which is the amplitude to find the field configuration φ at time t. We expand φ in momentum space:

$$\varphi(\vec{x}) = \int\frac{d^d k}{(2\pi)^d}\,\alpha_k\, e^{i\vec{k}\cdot\vec{x}}$$

$$\Psi(\varphi,t) = \mathcal{N}\exp\left\{-\int\frac{d^d k}{(2\pi)^d}A(k,t)|\alpha_k|^2 - i\Omega(t)\right\} \tag{11}$$

and where we've chosen a gaussian ansatz for Ψ. Thus:

$$\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\,a(t)^d\left\{-a(t)^{-2d}\frac{\delta^2}{\delta\alpha_k\delta\bar{\alpha}_k}+\left(a(t)^{-2}\vec{k}^2+\mu^2-\xi R\right)|\alpha_k|^2\right\}\Psi=i\frac{\partial}{\partial t}\Psi \tag{12}$$

and we obtain the equation of motion for A(k,t), or equivalently for Γ(k,t):

$$i\dot{\Gamma}(k,t)=\Gamma(k,t)^2+\frac{d^2}{4}H^2+\frac{d}{2}\dot{H}-\vec{k}^2a(t)^{-2}-\mu^2+\xi R \tag{13}$$

where:

$$\begin{aligned}A(k,t)&=a(t)^d\left[\Gamma(k,t)+\frac{id}{2}H(t)\right]\\ H(t)&=\dot{a}(t)/a(t)\end{aligned}\tag{14}$$

Equation (13) is readily solved in deSitter space when $H=H_0$ is constant. For the choice of the Bunch-Davies vacuum[13]:

$$\Gamma(\kappa,t) \equiv \Gamma_0 = -i\frac{d}{dt}\left[\ln\left\{H_\nu^{(2)}(\kappa H_0^{-1}e^{-H_0 t})\right\}\right] \qquad (15)$$

where $\nu^2 = \frac{d^2}{4} - \frac{(\mu^2 - \xi R_0)}{H_0^2}$.

(b) Small Oscillations About the Bunch-Davies Vacuum

Consider a small fluctuation from the classical deSitter space metric as:

$$a(t) = 1 + I_0(t) \; ; \quad I_0(t) \equiv H_0 \int_{t_0}^{t} \delta(t')\,dt' \qquad (16)$$

To find $\delta(t)$ we will solve the following linear combination of the Einstein equations:

$$G_{00} + d^{-1}\sum_{i=1}^{d} g^{ii}\, G_{ii} = 8\pi \langle(\rho - P)\rangle / M_{Pl}^2 \qquad (17)$$

where:

$$\rho = \langle T_{00}^{(g)} \rangle \;; \qquad P = -d^{-1} \sum_{i=1}^{d} \langle T_{ii}^{(g)} \rangle \tag{18}$$

The expectation values are taken in the state Ψ_0. The Einstein-Schroedinger equations determine the self-consistent solution for the metric perturbation, $\delta(t)$ with $\delta\langle \rho - P \rangle = \langle \rho - P \rangle - \langle \rho - P \rangle_0$ as a source.

In the remainder of this section we calculate $\delta\langle \rho - P \rangle_{ren}$.

To calculate the response of the wave-functional to the perturbation of the metric let:

$$\Gamma(K,t) = \Gamma_0 + \Delta\Gamma(K,t) \tag{19}$$

The linearized equation for $\Delta\Gamma$ follows upon substituting the perturbed expression for H(t):

$$i \frac{d}{dt} \Delta\Gamma(K,t) - 2\,\Delta\Gamma(K,t)\,\Gamma_0 = S_K(t) \tag{20}$$

where:

$$S_\kappa(t) = \frac{1}{2} d^2 H_0^2 \delta(t) + \frac{d}{2} H_0 \dot{\delta}(t) + 2\vec{k}^2 e^{-2H_0 t}\, I_0(t) + \xi \Delta R \tag{21}$$

and ΔR is given in eq.(A-9). The solution may be written:

$$\frac{\Delta\Gamma}{(Re\,\Gamma_0)^2} = \frac{-i}{H_0^2}\,\frac{\pi^2}{4}\left[H_\nu^{(1)}(\kappa H_0^{-1} e^{-H_0 t})\right]^2 \int_{t_0}^{t} S_\kappa(\lambda)\left[H_\nu^{(2)}(\kappa H_0 e^{-H_0\lambda})\right]^2 d\lambda \tag{22}$$

Next we take the expectation value of $\rho-P$ in the state defined by $\Gamma_0+\Delta\Gamma$. This requires performing a gaussian functional integral. The details are given in ref.(13) and the formal result is:

$$\langle\rho-P\rangle = \int \mathcal{D}\varphi\, \Psi^*(\varphi)(\rho-P)_{op}\Psi(\varphi)$$

$$= \frac{a(t)^{-d}}{2}\int \frac{d^d k}{(2\pi)^d}\left\{\left(\frac{d-1}{d}\right)\frac{\vec{k}^2}{a(t)^2} + \mu^2 - \xi(R-R_{00})\left(\frac{d-1}{d}\right)\right\}\frac{1}{Re\,\Gamma(\kappa,t)} - \xi\Box\langle\varphi^2\rangle \tag{23}$$

Everything in the integrand is known in terms of the metric perturbation $\delta(t)$, using eq.(16,20,22).

The linearized variations of $\langle\Psi|(\rho-P)|\Psi\rangle$ come from the explicit variations of the operators ρ and P and from the linearized response of the wave functional. The typical integrals required for $\delta\langle\rho-P\rangle$ are of the form:

$$\mathrm{Re}\left[-i\int\frac{d^dk}{(2\pi)^d}\,k^p\left[H_\nu^{(1)}(kH_0^{-1}e^{-H_0 t})\,H_\nu^{(2)}(kH_0^{-1}e^{-H_0\lambda})\right]^2\right. \tag{24}$$

These integrals are made difficult by the appearance of two different time arguments, t and λ. As the perturbation vanishes, $\lambda \to t$, and the above integral becomes zero (since the product of Hankel functions becomes real). It is therefore possible to obtain a systematic expansion in small λ-t. The product of 4 Hankel functions reduces to 2, and the integrals become of a standard Weber-Schafheitlin type. The formal details of this expansion are given in the appendix. Essentially this implies that our results are valid if the characteristic small oscillation frequency, ω , is large compared to H_o.

We compute the momentum space integrals which are straightforward (see ref.(13)) and have dimensional poles at d=3 spatial dimensions. We renormalize this regularized expression by expanding in a large mass limit and subtracting the nonvanishing terms of this expansion (DeWitt-Schwinger renormalization as used by Bunch and Davies in ref.(3); no point splitting ambiguities arise when regularization is dimensional). One obtains:

$$\delta\langle\rho-P\rangle_{ren} \equiv \left(\langle\rho-P\rangle - \langle\rho-P\rangle_0\right)_{renormalized}$$
$$= \frac{H_0^4}{16\pi^2}\Big\{\left(-\ln(m^2)+\psi_+ +\psi_-\right)\left[m^4 I_0\Delta - 2\xi f\Delta + m^2(\xi f - 2I_0\Delta^2)\right]$$
$$+ m^2 I_0\Delta(2\Delta+\tfrac{1}{3}) + \frac{I_0\Delta}{15} - 2I_0\Delta^3 + \tfrac{1}{3}\xi f + 2\Delta\xi f\Big\} \quad (25)$$

where:

$$\Delta = 1-6\xi\,; \quad \nu_0 = \left(\tfrac{9}{4}-12\xi - m^2\right)^{1/2}; \quad m^2 = \mu^2/H_0^2\,;$$
$$f = 12I_0(1+4\xi) + 4\delta(t)(1+9\xi) + 2\dot\delta(t)(1+6\xi)/H_0 \quad (26)$$
$$\psi_\pm = \psi(\tfrac{1}{2}\pm\nu_0),$$

where ψ is the digamma function.

III. Solutions of the Einstein Equations

To linear order in $\delta(t)$, the Einstein equation (17) becomes:

$$\dot\delta(t) + 6H_0\delta(t) = 4\pi H_0^{-1} M_{Pl}^{-2}\,\delta\langle\rho-P\rangle_{ren} \quad (27)$$

where now the rhs is known in terms of $\delta(t)$ from eq.(24).

One may specify initial conditions by giving $\delta(t_0)$ and then eq.(27) determines $\dot\delta(t_0)$. Then the Einstein constraint equation determines:

$$\delta\rho(t_0) = \frac{3H_0^2\,\delta(t_0)}{4\pi G_N} \tag{28}$$

and one of the dynamical equations fixes δp. Physically, one thinks of generating pressure fluctuations, which induce density fluctuations. The Einstein equations then determine the metric perturbations.

Note that the initial conditions, eq.(28) and the Einstein equation (17) insures that all the Einstein equations are satisfied (see the discussion of Appendix B).

Substituting eq.(25) into eq.(27) and once differentiating yields a harmonic oscillator equation for $\delta(t)$:

$$(1+\alpha)\,\ddot{\delta}(t) + 6H_0(1+\beta)\,\dot{\delta}(t) + \omega^2\,\delta(t) = 0 \tag{29}$$

α, β and ω^2 are determined from $\delta\langle\rho - P\rangle_{ren}$

Among the various solutions of eq.(27) is the case of pure curvature fluctuations when the contribution of the matter fields is much smaller than the perturbation from the gravitational field. That is, if $H_0 \ll M_{Pl}$ and $\omega^2 \ll 1/T^2$ (where T is the characteristic time scale for variations in δ) then $\ddot{\delta} + 6H_0\dot{\delta} \approx 0$. These are decaying exponentials.

It is easy to verify that this solution, and the ones which follow below, are not gauge modes. The d=3 variation in the scalar curvature is $\Delta R = -6H_0^2(4\,\delta(t) + \dot{\delta}(t)/H_0)$. Since $\Delta R \neq 0$, the metric perturbation is physical.

Now consider presently the solutions for various values of the coupling parameter ξ.

(i) Minimal Coupling, $\xi=0$

When $\xi=0$ we have by eq.(25,26) that $\alpha = \beta = 0$ and:

$$\omega^2 = \frac{-H_o^4}{4\pi M_{Pl}^2}\left\{\mathcal{M}^2\left(\ln \mathcal{M}^2 - \Psi_+ - \Psi_-\right)\left(\mathcal{M}^2 - 2\right) + \frac{7}{3}\mathcal{M}^2 - \frac{29}{15}\right\} \quad (30)$$

One can check that in both high and low mass limits that ω^2 is negative:

$$\omega^2 \longrightarrow \frac{-H_o^4}{\pi M_{Pl}^2}\begin{cases} \frac{7}{24}\mathcal{M}^{-2} & \text{when } \mathcal{M} \equiv \frac{\mu}{H_o} \to \infty \\ \frac{61}{60} & \text{when } \mathcal{M} \equiv \frac{\mu}{H_o} \to 0 \end{cases} \quad (31)$$

This implies that the small oscillations are unstable (exponentials; (in deriving the high mass limit we use eq.(A.11)). Consistency with the short time approximation used in deriving eq.(25) demands that $M_{Pl} \ll \sqrt{N}\, H_o$, where N is the number of noninteracting real scalar fields.

(ii) Conformal Coupling

When $\xi=1/6$ one finds that $\alpha = \omega^2/(5H_o^2)$, $\beta = \omega^2/(12H_o^2)$, and:

$$\omega^2 = \frac{-H_o^4}{\pi M_{Pl}^2} \left\{ \frac{5}{6} \right\} \left[-\mathcal{M}^2 \left(\ln \mathcal{M}^2 - \psi_+ - \psi_- \right) + \frac{1}{3} \right] \tag{32}$$

In this case ω^2 is positive in both high and low mass limits and the theory is stable against fluctuations in the metric scale factor:

$$\omega^2 \rightarrow \frac{H_o^4}{\pi M_{Pl}^2} \begin{cases} \frac{1}{18} \mathcal{M}^{-2} & \text{when } \mathcal{M} \rightarrow \infty \\ \frac{5}{9} & \text{when } \mathcal{M} \equiv \frac{\mu}{H_o} \rightarrow 0 \end{cases} \tag{33}$$

There are several different parameter regimes within the conformally coupled case that are consistent with the short time approximation, though all are exponentially decaying (overdamped) perturbations. If $M_{Pl} \gg H_o$ then the matter terms are unimportant and one has pure curvature fluctuations. If $M_{Pl} \ll H_o$ then the curvature terms are unimportant and one has pure matter fluctuations with $\delta\rho \sim \delta P$. If $M_{Pl} \gtrsim H_o$ then $\delta(t) = \delta(t_o)\exp(-6H_o\sigma t)$ with $\sigma = (1+\omega^2/(12H_o^2))(1+\omega^2/(5H_o^2))^{-1}$ which is order unity.

(iii) General Coupling

The stability or instability of small oscillations of the metric apparently depends sensitively on the coupling of the field to curvature. We have not yet carried out a complete study of the behavior of ω^2 for arbitrary ξ and μ^2, however in general ω^2 has an infrared singularity.

The low mass limit of ω^2 for arbitrary ξ is:

$$\omega^2 \to \frac{H_0^4}{4\pi M_{pl}^2}\Big[(-\ln \mathcal{M})\,24\xi\Delta(1+4\xi) + 2(1+4\xi)(1-\nu_0)\left(\tfrac{3}{2}+\nu_0\right)\left(\tfrac{1}{2}+\nu_0\right)$$
$$-\frac{\Delta}{15} + 2\Delta^3 + 12\xi(1+4\xi)\left(2\Delta+\tfrac{1}{3}\right)\Big] \qquad (34)$$

The coefficient of the infrared singular logarithm vanishes only when $\xi=0$ or $\xi=1/6$.

IV. Discussion

We have studied the stability of the self-consistent deSitter space solution where matter consists of a quantized real scalar field (or the large N generalization thereof).

When the field is minimally coupled to curvature the small fluctuations become unstable on a time scale of $T \sim M_{Pl}/H_0^2$, valid for $H_0\sqrt{N} \gg M_{Pl}$. The conformally coupled case gives stable oscillations of $\delta(t)$ when $M_{Pl} < H_0\sqrt{N}$, and decaying exponentials when $M_{Pl} \gg H_0\sqrt{N}$.

We remark that the fact that the result does not depend strictly upon the combination $\mu^2 - \xi R$ indicates that the sensitivity to the conformal coupling involves the the renormalization effects, and is probably tied to the conformal anomaly in some way. Further work is required to elucidate this result.

A more fundamental issue is the validity of the semiclassical equations (1). Horowitz[15] has found that eq.(1) leads to the conclusion that Minkowski space with a conformal coupling to curvature (hence a "new-improved" stress-tensor) is unstable. Since Minkowski space is phenomenologically stable we have that (i) either eq.(1) is invalid, (ii) there are no conformally coupled scalar fields (iii) or, perhaps the most optimistic possibility, there exist physical boundary conditions which should be imposed to rule out unphysical runaway solutions of the semiclassical equations. If scalar fields exist (they need not be elementary since our analysis applies in limits in which effective scalar Lagrangians can be valid) and are not conformally coupled, we would conclude that deSitter space is unstable. This may provide a natural mechanism for the decay of an inflationary phase in the early Universe[16].

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Appendix A

The integral in eq.(24) can be evaluated in d-spatial dimensions in a short time approximation. It has the usual dimensional pole at d=3. We renormalize by subtracting the matrix element in a DeWitt-Schwinger scheme.

In deSitter space we have:

$$\langle \rho - P \rangle_0 = 2\langle \rho \rangle_0 = \left(\mu^2 - \xi(R - R_{00})\right) J_{(0)} + \left(\frac{d-1}{d}\right) H_0 J_{(2)} \tag{A-1}$$

where:

$$J_{(n)} = \frac{\pi H_0^{d-1}}{4} \int \frac{d^d K}{(2\pi)^d} K^n H_\nu^{(1)}(K)\, H_\nu^{(2)}(K) \tag{A-2}$$

The J_n can be readily evaluated[13,17].

To calculate the first order correction to $\langle \rho - P \rangle$, substitute the perturbed metric and wave-functional (eqs.(16) and (22)) into eq.(23) for $\langle \rho - P \rangle$. Then we have:

$$\delta\langle\rho-P\rangle = \langle\rho-P\rangle - 2\langle\rho\rangle_0$$
$$= \left[-\mu^2 I_0 d + \xi(d-1)(R-R_{00})I_0 - \xi\left(\frac{d-1}{d}\right)(\delta R-\delta R_{00})\right] J_{(0)}$$
$$- I_0 H_0^2\left(\frac{d-1}{d}\right)(d+2)J_{(2)} - \xi\,\Box\langle\varphi^2\rangle + F(\nu) \qquad \text{(A-3)}$$

where:

$$F(\nu) = \mathrm{Re}\left\{ \frac{i\pi^2}{8} H_0^{d-2}\int_0^t d\lambda \int \frac{d^d y}{(2\pi)^d}\, s_y(\lambda)\, \tau^d \right. \qquad \text{(A-4)}$$
$$\left. \cdot \left[\mu^2 + \left(\frac{d-1}{d}\right)H_0^2\tau^2 y^2 - \xi(R-R_{00})\left(\frac{d-1}{d}\right)\right]\left[H_\nu^{(1)}(\tau y)\, H_\nu^{(2)}(y)\right]^2 \right\}$$

Here $\tau = \exp(-H_0(t-\lambda))$ and $s_y(\lambda)$ is s_k evaluated at argument $k=H_0 y\exp(H_0\lambda)$ in eq.(21). $F(\nu)$ contains the product of four Hankel functions. We adopt the short time approximation, $H_0(t-t_0)\ll 1$. One can then utilize the Bessel function multiplication theorem (see e.g. Abramowitz and Stegun) to expand $H_\nu^{(1)}(\tau y)$ for $|\tau^2-1|\ll 1$ and reduce $F(\nu)$ to sums of the $J_{(n)}$. Using the Wronskian one finds for real ν:

$$\mathrm{Re}\left[-i\left(H_\nu^{(1)}(\tau y)\,H_\nu^{(2)}(y)\right)^2\right]$$
$$= \frac{2}{\pi}\tau^{2\nu}(\tau^2-1)H_\nu^{(1)}(y)H_\nu^{(2)}(y) + \mathcal{O}(\tau^2-1) \qquad \text{(A-5)}$$

Further, noting that:

$$\int_0^t d\lambda\, s_y(\lambda)\, \tau^{2\nu-d}(\tau^2-1) = -d H_0 I_0 (1-4\xi) + \mathcal{O}(\tau^2-1) \quad \text{(A-6)}$$

one finds:

$$F(\nu) = I_0 d(1-4\xi)\left[\left(\mu^2-\xi(R-R_{00})\left(\frac{d-1}{d}\right)\right)J_{(0)} + \left(\frac{d-1}{d}\right)H_0^2 J_{(2)}\right] \quad \text{(A-7)}$$

The computation of the gravitational stress-tensor also requires $\Box\langle\varphi^2\rangle$. This is straightforward, since:

$$\langle\varphi^2\rangle = (1-4d\xi I_0)\langle\varphi^2\rangle_0 = (1-4d\xi I_0)J_{(0)} \quad \text{(A-8)}$$

Also, in a Robertson-Walker metric with spatially flat hypersurfaces:

$$\begin{aligned} R &= -d\left[(d+1)H^2 + 2\dot{H}\right] \\ R - R_{00} &= -d\left[H^2 + \dot{H}\right] \end{aligned} \qquad \text{(A-9)}$$

in d+1 spacetime dimensions.

Finally, performing the momentum integration in (A-3) yields:

$$\delta\langle\rho - P\rangle = \frac{1}{2}\frac{H_0^{d+1}}{\pi^{d/2}} \frac{\Gamma(1-d)\Gamma(\frac{d}{2}-\nu_0)\Gamma(\frac{d}{2}+\nu_0)}{\Gamma(1-\frac{d}{2})\Gamma(\frac{1}{2}+\nu_0)\Gamma(\frac{1}{2}-\nu_0)} \cdot \left\{ \frac{2\mu^2}{H_0^2} I_0 \left(\frac{d-1-4\xi d}{d+1}\right) + \xi f \right\} \qquad \text{(A-10)}$$

where:

$$f = 2I_0(d-1)d(1+2\xi d - 2\xi) + 2\delta(d-1+2\xi d^2) + \frac{\dot{\delta}}{H_0}(d-1+4\xi d) \qquad \text{(A-11)}$$

valid for $H_0(t-t_0) \ll 1$ and ν_0 real. In actuality, eq.(A-10) is also valid by analytic continuation to ν_0 pure imaginary, as follows.

$F(\nu)$ is not an analytic function of ν. However, $F(\nu)$ does have the same functional form for ν either real or pure imaginary,

$$F(\nu) = \int d^d y \; P(y,\nu) \qquad (A-12)$$

where P(ν) is an analytic function of ν. Define N(ν) by (A-12). Then N(ν) is an analytic function of ν which agrees with F(ν) on the real and imaginary ν axes. We have just evaluated N(ν) for ν real. Therefore, we know N(ν) on the entire ν plane, and in particular we know F(ν) on the imaginary axis.

The expression $\delta\langle \rho - P \rangle$ has a pole when d=3. Let d=3- ϵ and expand in ϵ; define $M=m/H_0$, $\Delta=1-6\xi$ and $\Psi_\pm = (1/2 \pm \nu_0)$ where $\Psi = \Gamma'/\Gamma$ is the digamma function. Then:

$$\delta\langle \rho - P \rangle = \frac{H_0^4}{16\pi^2}\Big\{ \frac{1}{\epsilon}(\text{polynomial in } M^2) \qquad (A-13)$$

$$+ (\Psi_+ + \Psi_-)\big(M^4 I_0 \Delta - M^2(\lambda I_0 \Delta^2 - f\xi) - 2\Delta f \xi\big)\Big\}$$

f is evaluated in d=3 spatial dimensions.

To subtract this expression expand eq.(A13) for large M, keeping terms of order M^4 to order $M^{(0)}$, and define the renormalized quantity as:

$$\delta\langle p-P\rangle_{ren} = \delta\langle p-P\rangle - \delta\langle p-P\rangle_{heavy} \qquad (A\text{-}14)$$

where $\delta\langle p-P\rangle_{heavy}$ is the truncated large mass asymptotic expression. This requires the large M expansion of the digamma functions:

$$\psi_+ + \psi_- \to \ln M^2 - \frac{1}{M^2}\left(2\Delta+\frac{1}{3}\right) - \frac{1}{M^4}\left(\frac{1}{15}+\frac{2\Delta}{3}+2\Delta^2\right)$$
$$-\frac{1}{M^6}\,\frac{4}{3}\left(\frac{\Delta}{5}+\Delta^2+2\Delta^3+\frac{1}{40}\right)+\cdots \qquad (A\text{-}15)$$

(Here we display terms through order M^{-6} which are required in taking subsequent limits).

Substituting eq.(A14) into eq.(A12) to get $\delta\langle p-P\rangle_{heavy}$ and subtracting yields the renormalized expression of eq.(25,26).

Appendix B

We note that all the Einstein equations have been satisfied[18]. Recall how the classical Friedman models are solved. One satisfies the constraint equation and a dynamical equation:

$$G^0{}_0 \cong \langle T^0{}_0 \rangle \quad ; \quad G^i{}_j \cong \langle T^i{}_j \rangle \tag{B-1}$$

and the equation of state.

In the present work the equation of state is implicit since ρ and P are both defined by $\mathcal{L}$. Equation (17) is a linear combination of eq.(B-1), so one might worry that the constraints by themselves are not satisfied. However, $(G^{\mu\nu} - T^{\mu\nu})_{;\nu}=0$, and eq.(17) together imply that $G^{00} - T^{00}$ =constant. At $t=t_0$ the initial conditions of eq.(28) fix the constant to be zero.

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