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Semi-Classical Instability of Compactification

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ABSTRACT

We show that several schemes for compactification of the extra dimensions in Kaluza-Klein theories are unstable to a quantum gravitational process of barrier penetration: the universe can tunnel from a state with static extra dimensions to a de Sitter expansion of all dimensions. We estimate the tunneling rate and find that the present state of the universe is probably long-lived (in good agreement with observation).

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Soon after developing the general theory of relativity, Einstein realized that his field equations implied the Universe was dynamic, contrary to the then-current belief in an eternal, unchanging world. In order to make his equations conform to his view of the Universe, he inserted a cosmological term to obtain a static solution. Eddington, Lemaitre, and Tolman[1] subsequently showed, however, that even this static model is unstable to expansion or collapse. The advent of modern Kaluza-Klein theories (and higher dimensional theories in general) marks a return to what one might call partially static cosmologies. In these theories, one looks for solutions with three dimensions expanding in Friedmann fashion, while D extra spatial dimensions are static and curled up into a compact manifold of unobservably small size, typically of order l_{Pl}/e, where $l_{Pl} = (G\hbar c^{-3})^{1/2}$ is the Planck length and e is a gauge coupling constant. The radius of the compact dimensions is stabilized against small perturbations by balancing a positive 'bare' cosmological constant against the stress-energy of classical or quantum fields. We show below, however, that in several proposed schemes the compactification is semi-classically unstable: the compactified state is a 'false vacuum' which can decay by quantum tunneling through a potential barrier. In a separate paper[2], we will discuss the possibility of classically rolling over the barrier at finite temperature. Although the effective four-dimensional cosmological constant vanishes in the "static" compactified state, once tunneling occurs the Λ term drives an exponential (de Sitter) expansion of all $3 + D$ spatial dimensions.

We calculate the tunnel action in the semi-classical (lowest order in $\hbar$) approximation; the decay rate has the standard WKB form $\Gamma \sim \exp(-\alpha c/\Lambda\hbar K^2)$, where $K^2 = 16\pi G = 16\pi\hbar c/m_{Pl}^2$, Λ is the 'bare' cosmological constant appearing in the Lagrangian, and α is a dimensionless constant. (Henceforth, we use units in which $\hbar = c = 1$.) This tunneling action has the same form as that calculated for several instabilities in four-dimensional quantum gravity[3]; however, the decay rate in Kaluza-Klein theories, for which typically $\Lambda \sim e^2 m_{Pl}^2$, is much larger than in the four-dimensional case, for which the effective cosmological constant is restricted to $\Lambda/m_{Pl}^2 \lesssim 10^{-120}$.

In the Kaluza-Klein ansatz, the spacetime manifold has the form of a product space, which we take to be $R^1 \times Q^3 \times S^D$, where Q^3 stands for flat R^3, S^3, or the 3-hyperboloid for $k = 0,1,-1$. The ground state metric is $g_{MN} = \mathrm{diag}(-1,\, a^2(t)\,\tilde{g}_{mn},\, b^2(t)\,\tilde{g}_{\mu\nu}\,)$, where a(t) and b(t) are the scale factors for Q^3 and S^D, and $\tilde{g}_{mn}$ and $\tilde{g}_{\mu\nu}$ are metrics on the maximally symmetric unit 3-space and D-sphere. Upper case indices M,N run over all values, lower case latin indices $m,n = 1,2,3$ and lower case greek indices $\mu,\nu = 5,6,\ldots,4+D$. The Einstein equations can be written

$$R_{MN} = -8\pi\overline{G}\left[\,T_{MN} - \frac{g_{MN}}{D+2}\,T\,\right] - \frac{g_{MN}\Lambda}{D+2} \tag{1}$$

where T_{MN} is the stress energy of classical and/or quantum fields, T is its trace, and $\overline{G}$ is a gravitational constant in 4+D dimensions. In the compactification schemes under consideration, the non-vanishing components of the stress tensor can be written as

$$T_{00} = \rho\ ;\ \ T_{mn} = g_{mn}p_3\ ;\ \ T_{\mu\nu} = g_{\mu\nu}p_D \tag{2}$$

with trace $T = -\rho + 3p_3 + Dp_D$, where ρ, p_3, and p_D are polynomial functions of b(t) (see below). The dynamical equations for the evolution of the scale factors are

$$3\frac{\ddot{a}}{a} + D\frac{\ddot{b}}{b} = \frac{1}{D+2}\ \left[\,\Lambda - 8\pi\overline{G}\ ((D+1)\rho + 3p_3 + Dp_D)\,\right] \tag{3}$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + D\frac{\dot{a}\dot{b}}{ab} + \frac{2k}{a^2} = \frac{1}{D+2}\left[\Lambda - 8\pi\overline{G}(-\rho + Dp_D + (1-D)p_3)\right] \tag{4}$$

$$\frac{\ddot{b}}{b} + (D-1)\frac{\dot{b}^2}{b^2} + 3\frac{\dot{b}\dot{a}}{ba} + \frac{D-1}{b^2} = \frac{1}{D+2}\left[\Lambda - 8\pi\overline{G}(-\rho + 3p_3 - 2p_D)\right] \quad (5)$$

A number of compactification schemes have been considered in the literature, and we consider two representative examples. In six-dimensional Einstein-Maxwell theory, the classical ground state of the U(1) gauge field is assumed to be a magnetic monopole configuration on the compact 2-sphere[4]. In this case, the terms in the energy momentum tensor are ρ, p_3, $p_D \sim 1/e^2b^4$, where e is the U(1) coupling constant. In the second model[5], the energy-momentum tensor arises from one-loop quantum fluctuations in massless matter fields (due to the non-trivial topology of the spacetime), in analogy to the Casimir effect in quantum electrodynamics. In this case dimensional analysis gives $\rho = A/b^{4+D}$, $p_3 = B/b^{4+D}$, $p_D = C/b^{4+D}$, where A,B and C are model-dependent (dimensionless) constants. Note that this result for T_{MN} was obtained assuming $\dot{a} = \dot{b} = 0$, $a >> b$. We will discuss possible effects of correction terms below.

Now focus on the Casimir model. For $k = 0$, Eqs.3-5 have a static solution $a = a_o$, $b = b_o$ if $A = -B$, while conservation of energy-momentum requires $C = 4A/D$. This gives

$$b_o^{D+2} = \frac{8\pi\overline{G}A(4+D)}{D(D-1)} \quad (6)$$

$$\Lambda = \frac{D(D-1)(D+2)}{b_o^2(D+4)} \quad (7)$$

The only other solution to Eqs.3-5 with b=const. is $a \sim e^{Ht}$; to obtain power law behavior for a(t), one must consider additional stress-energy terms due, e.g., to radiation[2]. The solution (6-7) is stable against small perturbations: $\delta b(t) = b(t) - b_o$ has no exponentially growing modes. Consider, however, the case of large b(t), i.e., ρ, p_3, $p_D << \Lambda/\overline{G}$. In the limit $b(t) \rightarrow \infty$ (and

$k = 0$ or $a \to \infty$), the solutions to Eqs.3-5 become $a,b \sim e^{Ht}$ and $a,b \sim e^{-Ht}$, with $H^2 = \Lambda/(D+2)(D+3)$. Using initial conditions $b(0) >> b_o$, $\dot{b}(0) = 0$, the approximate solution (exact in the limit $t >> 1/H$) is $b(t) \sim b(0)\cosh Ht$; i.e, if the radius is ever large (and initially static), it subsequently increases without bound instead of relaxing to the equilibrium value b_o. This clearly represents an instability. Since it occurs whenever the compactification terms ρ, p_3, $p_D \to 0$ as $b \to \infty$, this analysis holds for the Einstein-Maxwell as well as the Casimir model.

We can make this more concrete by regarding the radius of the extra dimensions as a scalar field in a potential in four dimensions. Schematically, we start with a piece of the gravitational action, $S_k = -\int d^{4+D}x\sqrt{-g}(R_k/16\pi\overline{G})$, where R_k is that part of the Ricci scalar R containing time derivatives of b,

$$R_k = -D\left[2\frac{\ddot{b}}{b} + (D-1)\left(\frac{\dot{b}}{b}\right)^2 + 6\frac{\dot{a}\dot{b}}{ab}\right]. \tag{8}$$

An integration by parts and over the internal D dimensions gives $S_k = -D(D-1)m_{Pl}^2\int d^4x\sqrt{-g_4}(b/b_o)^{D-2}(\dot{b}/b_o)^2/16\pi$, where $m_{Pl}^{-2} = \overline{G}/V_o^D$ is the 4-dimensional Newton constant (V_o^D is the volume of the compact D-sphere with radius b_o), and g_4 is the determinant of the 4-dimensional part of the metric. We define the new variable $\phi(b) \equiv m_{Pl}(b/b_o)^{D/2}((D-1)/2\pi D)^{1/2}$ which has the usual action for a homogeneous scalar field in four dimensions. With this change of variable, Eq.5 becomes

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{\dot{\phi}^2}{\phi} = -\frac{dV}{d\phi} \tag{9}$$

where, using Eqs.6,7 for the Casimir model,

$$V(\phi) = \frac{(D-1)\Lambda m_{Pl}^2}{8\pi(D+2)} \left\{ -\left(\frac{\phi}{\phi_o}\right)^2 + \left(\frac{\phi}{\phi_o}\right)^{-8/D} + \left(\left(\frac{\phi}{\phi_o}\right)^{2(D-2)/D} - 1\right)\left(\frac{D+4}{D-2}\right) \right\} \tag{10}$$

where $\phi_o \equiv \phi(b_o)$ is the static solution. The integration constant in Eq.10 has been chosen so that $V(\phi_o) = 0$. Note that at $\phi = \phi_o$, $dV/d\phi = 0$ and $d^2V/d\phi^2 > 0$, so this point is a local minimum of the potential[6]. At large ϕ, the potential is dominated by the negative quadratic term, and it is unbounded from below.

It is natural to ask for the lifetime of the compactified state $b = b_o$ against quantum tunneling. Except for the extra term $\dot{\phi}^2/\phi$, Eq.9 is the equation of motion for a homogeneous, minimally coupled scalar field in a 4-dimensional Friedmann universe; if we extend our original ansatz for the metric so that $b = b(\bar{x},t)$ (where $\bar{x}$ lives in ordinary 3-space), Eq.9 would be replaced by the equation of motion for an ordinary 4-d scalar field $\phi(\bar{x},t)$ in a potential $V(\phi)$. Treating the 4-dimensional gravitational degrees of freedom ($a(t)$) as a classical background, this problem is similar to the false vacuum decay discussed by Coleman and De Luccia[7]. For an explicit calculation, we shall work with the Casimir model in 11 dimensions (D=7). In this case, $V(\phi)$ has a local minimum at $\phi_o \simeq 0.37 m_{Pl}$, a local maximum at $\phi_m \simeq 1.96\phi_o$ and a point degenerate with the local minimum at $\phi_T \simeq 2.58\phi_o$. (The form of V is only weakly dependent on D.)

The tunneling solutions will be bubble-like configurations with thick walls[8]: at the instant of tunneling, ϕ is a few ϕ_T at the bubble center and falls gradually to $\phi \approx \phi_o$ at infinity. In terms of the 4-geometry, the bubble interior is approximately de Sitter, while the exterior is asymptotically flat. (Note this is the opposite of the usual case in which the effective cosmological constant *decreases* in the decay.) To estimate the tunnel action, we approximate Eq.10 (for D=7) with a polynomial

$$V(\bar{\phi}) \approx 0.093\Lambda\bar{\phi}^2 - 0.159\Lambda\bar{\phi}^3/m_{Pl} \tag{11}$$

where the shifted field $\bar{\phi} \equiv \phi - \phi_o$ is used to place the metastable minimum at the origin. ($V(\bar{\phi})$ matches the true potential (10) at ϕ_o, ϕ_m, and ϕ_T and is generally accurate to better than 30% for $\bar{\phi}$ less than a few $\bar{\phi}_T$, which is the range of $\bar{\phi}$ which dominates the tunnel action.) Eq.11 has the form $V(\bar{\phi}) = M^2\bar{\phi}^2/2 - \delta\bar{\phi}^3/3$ which has been studied by Linde[9]. In the flat space approximation, he finds the tunnel action $S \simeq 205M^2/\delta^2 \approx 165m_{Pl}^2/\Lambda$. The decay rate per unit 4-volume is $\Gamma/V_4 \simeq m^4e^{-S}$, where the determinant m is a characteristic mass scale in the problem; here $m \sim m_{Pl}$. In a matter-dominated universe, the probability for a given point to remain in the compactified phase becomes small after a time $\tau \simeq (9\pi\Gamma/165V_4)^{-1/4} \simeq (1/m_{Pl})\exp(41m_{Pl}^2/\Lambda)$; this is longer than the age of the universe if $\Lambda \lesssim 0.3m_{Pl}^2$. From Eqs.6,7, this corresponds to the requirements $A \gtrsim 17$, $b_o \gtrsim 11\ l_{Pl}$. The results for the Einstein-Maxwell theory are similar.

In this analysis, we have neglected the 'gravitational' contribution to the tunnel action. Since the bubble interior is not flat, this is not necessarily a good approximation, and in fact dimensional analysis suggests the gravitational corrections can be comparable to the flat space result. The behavior of the curved space solution is, however, qualitatively similar to the flat space approximation as long as [10] $M \gtrsim H$, where M is the mass parameter in the potential and H is the de Sitter Hubble constant for the bubble interior. From above, $M^2 \simeq 0.19\Lambda$, while at large ϕ, $H^2 = 0.01\Lambda$, so the above analysis should be accurate to an order of magnitude or so.

In the preceding analysis, we treated the 4-dimensional gravitational degrees of freedom classically and then argued that their contribution did not qualitatively affect the tunnel results. In principle, this treatment is inconsistent, because we have quantized only part of the gravitational field, the radius of the extra dimensions. In practice, however, the case of cosmological interest is decay from a state with a $>>$ b (presumably the present state of our universe), for which quantum corrections to the 4-geometry should be small. Alternatively, we could return to our original ansatz in which the metric is restricted to two quantum degrees of freedom, the radii a(t), b(t). As a result, we would now have a quantum mechanical rather than quantum field tunneling problem and could consider only homogeneous decays (i.e., no spatial dependence, so no finite bubble). This kind of truncation of the infinite gravitational degrees of freedom is called minisuperspace, and the resulting dynamics, governed by the Wheeler-DeWitt equation[11], would be that of a 4+D dimensional mixmaster model[12]. In the limit a $>>$ b, this case is similar to the 4-dimensional tunneling problem considered by Hartle and Hawking[3]. In the WKB approximation, the decay probability is $P \simeq \exp(-\alpha m_{Pl}^2/\Lambda)$, where α is of order unity; not surprisingly, the exponen~~t is in qual~~itative agreement with the field theoretic treatment.

We conclude with a few comments about our results and approximations. 1) In the Casimir model, the one-loop stress energy was calculated in the static limit $\dot{a} = \dot{b} = 0$. In general, the time dependence of the scale factors will generate additional quantum corrections to both the potential V [13] and the kinetic terms appearing in the action[14]. The corrections to V are suppressed by inverse powers of a, while the kinetic corrections can change the sign of the kinetic terms. The latter effect would presumably destabilize the compactification itself (changing the sign of the kinetic term corresponds to inverting the potential); we conclude that these corrections will not allow one to circumvent the instability of compactification. 2) It has previously been pointed out that compactification is *unstable* at high temperature[5,15], but it is amusing to find that it is also *metastable* at low temperature. 3) A semi-classical instability of the original five-dimensional Kaluza-Klein theory was discussed by Witten[16], but the case discussed in this paper is different; in particular, the present case does not involve a change in spacetime topology. In the five dimensional theory, it is not so surprising to find an instability because there is no dynamical mechanism for compactification; instability in higher dimensions and for a broad class of compactification schemes is perhaps more disturbing. 4) Rubakov[17] has shown that the 4-dimensional quantum gravitational tunneling from closed Robertson-Walker to de Sitter space of Hartle and Hawking[3] is accompanied by catastrophic particle creation. He speculates that the backreaction of this effect on the tunneling may lead to a decay rate which is only power law- rather than exponentially suppressed. This would render the Kaluza-Klein instability much more dangerous. 5) Finally, we believe that the considerations raised in this paper may be relevant to superstring theories. Although the status of the cosmological constant in these theories is at present unclear, recently Nepomechie, Wu, and Zee[18] have considered superstring compactification of the generalized monopole (Freund-Rubin) type. Also, in the 'low energy' limit, superstrings give rise to higher derivative terms in the gravitational action[19]. For higher derivative Lagrangians of the form $a_1R^2 + a_2R_{mn}R^{mn} + a_3R_{mnpq}R^{mnpq} + a_4R + a_5$, the instability discussed here can be avoided if[20] $a_3/a_2 = -5/6$ (for D=6 and assuming S^6 compactification), while the ghost-free string action[21] has $a_3/a_2 = -1/4$.

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