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DEPRIT'S ALGORITHM, GREEN'S FUNCTIONS, AND MULTIPOLE PERTURBATION THEORY*

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Abstract Within the context of accelerator multipole theory, we describe a method for solving Deprit's equations using a periodic Green's function. This procedure, implemented in a MACSYMA program, has been applied to a Hamiltonian with a normal sextupole term, and the "averaged" or "renormalized" Hamiltonian has been calculated to fourth order in sextupole strength. Anticommutativity of the bracket algebra explains the observed slow growth in the number of induced resonances with order.

DEPRIT'S LIE TRIANGLE ALGORITHM

In 1969 Deprit[1] used the Lie transform to develop a procedure for bringing Hamiltonians perturbatively into normal form, or at least as close to it as possible. Compared to the more familiar "generating function" techniques, Deprit's algorithm possesses the singular advantage of being completely explicit. It therefore can be implemented simply in a symbolic algebra code for analytically expressing the transformation to any desired order in the perturbation parameter. The algorithm has been studied and used extensively in celestial dynamics and plasma physics.[2] We shall present a possible application to accelerator theory.

Let $\underline{x} = (x_1, x_2)$ denote transverse particle coordinates in an accelerator, the sense being defined by saying that (i) (1,2,3) form a right handed system, (ii) 2 is "up", and (iii) 3 lies along the beam current. The canonically conjugate momenta are $\underline{x}' = d\underline{x}/ds$, where s = arc length along the reference orbit. We change immediately from rectangular to polar coordinates: $(\underline{x},\underline{x}') \longrightarrow (\underline{\delta}^{\star},\underline{I}^{\star})$.

$$x_k = (\ 2I_k^{\star}\ \beta_k(\theta)\ )^{1/2} \sin[\ \tilde{\psi}_k(\theta) + \delta_k^{\star}\ ]\ , \quad k = 1,2 \qquad (1)$$

$$\tilde{\psi}_k(\theta) = \psi_k(\theta) - \nu_k\theta$$

Here, θ is the angular coordinate on the design orbit ($ds = R d\theta$), β_k and ψ_k are the linear (decoupled) lattice functions, and ν_k the horizontal and vertical tunes; the asterisks on the polar variables serves to distinguish them from yet another set of coordinates to be introduced shortly. The Floquet transformation that gives rise to the definitions of β_k and ψ_k also transforms the linear part of the Hamiltonian into that of a harmonic oscillator.

To set up the perturbation theory, let us assume that the full Hamiltonian depends on a "small parameter", ε, and admits a power series expansion about $\varepsilon = 0$.

$$H = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} H_n \quad , \quad H_0(\underline{\delta}^\star, \underline{I}^\star; \theta) = \underline{\nu} \cdot \underline{I}^\star \tag{2}$$

Our objective is to transform the coordinate functions, $(\underline{\delta}^\star, \underline{I}^\star) \longrightarrow (\underline{\delta}, \underline{I})$, so that the $(\underline{\delta}, \underline{I})$ dynamics are generated by a simpler Hamiltonian.

$$K = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} K_{0n}$$

The transformation is constructed as a Lie transform with generator S. As part of the perturbation theory, S also is expanded in a power series.

$$S = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} S_{n+1}$$

(The shifted index is conventional.) The relationships between S_n, H_n and K_{0n} comprise Deprit's equations.

$$DS_n = H_n + \Sigma_n^{(H)} - \Sigma_n^{(K)} - K_{0n} \quad ,$$

$$\Sigma_n^{(H)} = \sum_{m=1}^{n-1} \binom{n-1}{m-1} \{ H_{n-m} , S_m \} \quad ,$$

$$\Sigma_n^{(K)} = \sum_{m=1}^{n-1} \binom{n-1}{m-1} K_{m\, n-m} \quad ,$$

$$K_{m\ n-m} = \sum_{j=1}^{m} \binom{m-1}{j-1} \{ S_j , K_{m-j\ n-m} \} ,$$

$$Df = \partial f/\partial\theta + \{f,H_0\} .$$

The partial differential equations to be solved for the unknown functions S_n and K_{0n} have the same form at all orders.

$$DS_n = rhs_n$$

Each rhs_n is assembled by taking Poisson brackets of functions developed at earlier orders and subtracting K_{0n}. A minimal K_{0n} will contain only those terms in $H_n + \Sigma_n^{(H)} - \Sigma_n^{(K)}$ not in the range of the operator D, assuring the existence of S_n. As we shall see in the next section, these turn out to be shear and resonance terms.

GREEN'S FUNCTIONS

We now solve Deprit's equations by using a Green's function. Under the assumption that H_0 is as in Eq.(2), D simplifies to

$$D = \partial/\partial\theta + \underline{\nu}\cdot\partial/\partial\underline{\delta} ,$$

whose 2π-periodic eigenfunctions are the complex exponentials.

$$D[f(\underline{I})\ e^{i(n\theta + \underline{m}\cdot\underline{\delta})}] = i(n + \underline{m}\cdot\underline{\nu})\ f(\underline{I})\ e^{i(n\theta + \underline{m}\cdot\underline{\delta})} .$$

Taking our cue from this, we shall expand functions of $\underline{\delta}$ and $\underline{I}$ in the following basis.

$$\{ \phi_{\underline{\mathbf{em}}} \mid \underline{e} = (e_1,e_2) , \underline{m} = (m_1,m_2) ; e_1,e_2,m_1,m_2 : \text{integers} \}$$

$$\phi_{\underline{\mathbf{em}}}(\underline{\delta}, \underline{I}) = I_1^{e_1/2}\ I_2^{e_2/2}\ e^{i\underline{m}\cdot\underline{\delta}} , \qquad \underline{m}\cdot\underline{\delta} = m_1\delta_1 + m_2\delta_2$$

By employing these the bracket operation becomes a matter of bookkeeping, not differentiation; the algebra is defined by specifying it on the basis.

$$\{ \phi_{\underline{\underline{em}}} , \phi_{\underline{e}'\underline{m}'} \} = \frac{i}{2} (e_1' m_1 - e_1 m_1') \phi_{(e_1+e_1'-2,\ e_2+e_2'),\ \underline{\underline{m}}+\underline{\underline{m}}'}$$

$$+ \frac{i}{2} (e_2' m_2 - e_2 m_2') \phi_{(e_1+e_1',\ e_2+e_2'-2),\ \underline{\underline{m}}+\underline{\underline{m}}'}$$

Because the operator D is linear and does not mix the $\phi_{\mathbf{em}}$, we can solve Deprit's equations one component at a time.

$$D[S_{n\ \underline{\underline{em}}}(\theta)\ \phi_{\underline{\underline{em}}}] = rhs_{n\ \underline{\underline{em}}}(\theta)\ \phi_{\underline{\underline{em}}}$$

$$S_{n\ \underline{\underline{em}}}(\theta) = \int_0^{2\pi} d\theta'\ G_{\underline{\underline{m}}}(\theta - \theta')\ rhs_{n\ \underline{\underline{em}}}(\theta')$$

For $\underline{\mathbf{m}}\cdot\underline{\nu} \neq$ integer --- that is, off resonance --- the Green's function $G_{\mathbf{m}}$ must satisfy the differential equation,

$$D[G_{\underline{\underline{m}}}(\theta - \theta')\ \phi_{\underline{\underline{em}}}] = \sum_{n=-\infty}^{\infty} \delta(\theta - \theta' - 2\pi n)\ \phi_{\underline{\underline{em}}}$$

$$= \delta_{per}(\theta - \theta')\ \phi_{\underline{\underline{em}}} \quad .$$

The solution is found easily.

$$G_{\underline{\underline{m}}}(\tau) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \frac{e^{in\tau}}{n + \underline{\underline{m}}\cdot\underline{\nu}} = \frac{e^{-i\underline{\underline{m}}\cdot\underline{\nu}\,(\ \mathrm{mod}(\tau) - \pi\ )}}{2i\ \sin(\ \pi\ \underline{\underline{m}}\cdot\underline{\nu}\ )}$$

where $\mathrm{mod}(\tau) = \tau\ (\mathrm{mod}\ 2\pi) \in [0,2\pi)$.

Near a resonance, the situation is only a little more complicated. Suppose that for some integer n_o we have $n_o + \underline{\mathbf{m}}\cdot\underline{\nu} = 0$. Then, because D annihilates the function $e^{in_o\theta}\ \phi_{\mathbf{em}}$, we must have

$$\int_0^{2\pi} d\theta\ rhs_{n\ \underline{\underline{em}}}(\theta)\ e^{-in_o\theta} = 0 \ , \quad \text{for all } \underline{e}.$$

As we mentioned before, this is accomplished by filtering any offending terms into the definition of K_{0n}. The Green's function can be modified as well by subtracting the contribution coming from the resonance term under consideration.

$$G_{\underline{\underline{m}}}(\tau) = \frac{e^{-i\underline{\underline{m}}\cdot\underline{\nu}(\ \mathrm{mod}(\tau) - \pi\ )}}{2i\ \sin(\ \pi\ \underline{\underline{m}}\cdot\underline{\nu}\ )} - \frac{1}{2\pi i}\ \frac{e^{in_o\tau}}{n_o + \underline{\underline{m}}\cdot\underline{\nu}} \tag{3}$$

This assures a finite result as the resonance is approached.

Even in the absence of non-trivial resonances, one must perform a subtraction for $\underline{m} = \underline{0}$, $n_o = 0$. Applying Eq.(3) in this limit we get the result

$$G_{\underline{0}}(\tau) = \frac{1}{2} [1 - \frac{1}{\pi} \operatorname{mod}(\tau)]$$

SEXTUPOLE INDUCED TUNE SHIFT

Consider the normal sextupole Hamiltonian,

$$H = \underline{\nu}\cdot\underline{I} + \varepsilon \frac{1}{3} \frac{B_0 R}{|B\rho|} b_2(\theta)(x_1^3 - 3x_1 x_2^2),$$

where x_1 and x_2 are the functions given in Eq.(1). We write this in terms of our set of basis functions as follows.

$$H_1 = -i/6\sqrt{2} \sum_{\{\underline{e},\underline{m}\}} c_{\underline{em}} \, \mathrm{latt}_{\underline{em}}(\theta) \, \phi_{\underline{em}} ,$$

$$\mathrm{latt}_{\underline{em}}(\theta) = (B_0 R/B\rho) \, b_2(\theta) \, \beta_1(\theta)^{e_1/2} \, \beta_2(\theta)^{e_2/2} \, e^{i\underline{m}\cdot\underline{\tilde{\psi}}(\theta)} ,$$

$$c_{(30)(30)} = -1 , \quad c_{(30)(10)} = c_{(12)(12)} = c_{(12)(1\,-2)} = 3 ,$$
$$c_{(12)(10)} = -6 .$$

Deprit's algorithm has been implemented in a MACSYMA program and applied to this Hamiltonian. So far, calculations have been carried out to fourth order in ε and assuming that the working point (tunes) was far from induced resonances. The lowest order non-trivial term in the new Hamiltonian, $K_{02} = \langle\{H_1,S_1\}\rangle$, is sufficiently simple to display here. Begin by defining the following integrals.

$$f(\underline{e};\underline{e}';\underline{m}) = \int \frac{d\theta \, d\theta'}{(2\pi)^2} \, \mathrm{Re}\Big[\, \mathrm{latt}_{\underline{e}\,-\underline{m}}(\theta) \, 2\pi i \, G_{\underline{m}}(\theta - \theta') \, \mathrm{latt}_{\underline{e}'\,\underline{m}}(\theta') \, \Big]$$

$$= \frac{\pi}{\sin(\pi\, \underline{m}\cdot\underline{\nu})} \iint d\xi(\theta,\theta')\, b_2(\theta)\, b_2(\theta')$$
$$\beta_1(\theta)^{e_1/2}\, \beta_1(\theta')^{e_1'/2}\, \beta_2(\theta)^{e_2/2}\, \beta_2(\theta')^{e_2'/2}$$
$$\cos[\, |\underline{m}\cdot(\underline{\psi}(\theta) - \underline{\psi}(\theta'))| - \pi\, \underline{m}\cdot\underline{\nu}\,]$$

$$d\xi(\theta,\theta') = (B_0R/B\rho)^2\ \frac{d\theta}{2\pi}\ \frac{d\theta'}{2\pi}\ , \quad \theta,\theta' \in [0,2\pi)$$

The new Hamiltonian is written below to second order as a superposition of these integrals.

$$\begin{aligned} K(\underline{I}) = \underline{\nu}\cdot\underline{I} - \varepsilon^2 \{ & I_1^2 [\, \tfrac{1}{8} f(3,0;3,0;3,0) \\ & \qquad + \tfrac{3}{8} f(3,0;3,0;1,0)\,] \\ & + I_2^2 [\, \tfrac{1}{2} f(1,2;1,2;1,0) \\ & \qquad + \tfrac{1}{8} f(1,2;1,2;1,2) \\ & \qquad + \tfrac{1}{8} f(1,2;1,2;1,-2)\,] \\ & + I_1 I_2 [\, - f(3,0;1,2;1,0) \\ & \qquad + \tfrac{1}{2} f(1,2;1,2;1,2) \\ & \qquad - \tfrac{1}{2} f(1,2;1,2;1,-2)\,]\ \} \\ & + O(\varepsilon^4) \end{aligned}$$

The number of induced resonances grows more slowly with order than one would predict naively. For example, at the fourth order we may have expected resonances as high as $12\nu_1$ to appear; in fact, $6\nu_1$ is the highest. Anticommutativity of brackets explains this phenomenon. We tabulate below the actual resonances ($\underline{m}\cdot\underline{\nu}$ = integer, $m_1 \geq 0$) that do appear at each order of the calculation.

CONCLUSIONS

There is reason for optimism that Deprit's algorithm will be useful for doing perturbation theory on accelerator Hamiltonians. Because of its connections with resonance theory, it is not difficult to interpret its mathematical structure physically. The algorithm is most useful in the presence of at most one induced

TABLE I   Sextupole resonances induced at each order.

1		2		3		4		= order
m_1	m_2	m_1	m_2	m_1	m_2	m_1	m_2	
							±6	
			±4				±4	
			±2				±2	
				1	±4			
1	±2			1	±2			
1				1				
						2	±4	
		2	±2			2	±2	
		2				2		
				3	±2			
3				3				
						4	±2	
		4				4		
				5				
						6		

resonance, for we can construct constants of the motion in such cases. Nevertheless, we benefit from it even with multiple resonances, because the final Hamiltonian contains only slowly varying terms, all rapidly fluctuating terms having been filtered out by the transformation. The algorithm tailors equations of motion for use with standard (but symplectic[3]) numerical integrators. One then can hope to reap more information per computational step, simply because all that is known analytically about the system has been utilized before invoking numerical procedures.

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