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POLARIZATION PRECESSION IN QUADRUPOLES AND SEXTUPOLES*

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ABSTRACT

Lie algebraic techniques are applied to the precession of polarization in a magnetic field. The rotation matrix is represented as the exponential of an antisymmetric matrix. The equation of motion is derived as a series of commutators. The lowest order approximation gives the rotation in terms of the integral of the magnetic field along the particle trajectory. Higher-order terms yield rotations about the longitudinal axis. Explicit expressions are given for quadrupoles and sextupoles.

INTRODUCTION

Higher order terms in the equation of motion for the precession of the polarization of a moving particle in a magnetic field may be significant in beam lines and accelerators. The lowest order terms indicate that the polarization vector will be restored to its original direction upon passage through a magnetic system if the momentum vector is unaltered. However, a snake rotates the polarization vector by a large angle while leaving the momentum vector unchanged. In an accelerator, even if the higher-order terms are small, the repeated passage of particles through the system may cause large effects.

EQUATION OF MOTION

The equation of motion of the polarization vector P, of a particle of charge e, mass m, and ratio g of magnetic moment to classical moment is

$$\frac{d\underline{P}}{dt} = \frac{e}{m\gamma}\,\underline{P} \times \left[\frac{g}{2}\,\underline{B}_L + \left(1 + \frac{g-2}{2}\,\gamma\right)\underline{B}_T\right] \qquad (1)$$

Here the magnetic field is separated into longitudinal and transverse components, indicated by appropriate subscripts. This arises because the magnetic field is expressed in the laboratory system, while the polarization vector is expressed in the particle rest frame.

At high energies, the angle a trajectory makes with the reference trajectory will be small, so that the fields from bending magnets, quadrupoles, and sextupoles will be essentially transverse. The transverse field will also be favored by a factor of γ associated with the anomalous part of the magnetic moment. The magnetic field can be incorporated into an antisymmetric matrix F, whose form is given below.

For a particular trajectory, we may define a polarization transfer matrix M, so that the polarization is given in terms of its initial value by

$$\underline{P} = M\underline{P}_o \qquad (2)$$

An orthogonal transformation can be written as the exponential of an antisymmetric matrix, so that M takes the form

$$M = e^{H} \tag{3}$$

and F and H become

$$F = \begin{bmatrix} 0 & 0 & -B_y \\ 0 & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 0 & -\theta_y \\ 0 & 0 & \theta_x \\ \theta_y & -\theta_x & 0 \end{bmatrix} \tag{4}$$

Dividing equation (1) by βc to express the motion as a function of distance along the reference orbit, we derive

$$\frac{d}{dz} e^{H} = \frac{e}{p} \left(1 + \frac{g-2}{2} \gamma\right) F e^{H} \tag{5}$$

The matrix H is the dual of a rotation vector θ, just as F is the matrix dual of the magnetic field B. The matrix M has the form

$$M = \begin{bmatrix} n_x^2+n_y^2 C & n_x n_y (1-C) & -n_y S \\ n_x n_y (1-C) & n_x^2 C+n_y^2 & n_x S \\ n_y S & -n_x S & C \end{bmatrix} \tag{6}$$

$$\theta_x = n_x \theta_p \qquad \theta_y = n_y \theta_p \qquad \theta_p = \sqrt{\theta_x^2+\theta_y^2}$$

$$S = \sin\theta_p \qquad C = \cos\theta_p$$

SOLUTION OF THE EQUATION OF MOTION

Since H is a matrix, we must expand the left side of equation (5) as a series of commutators. Truncating the series after the first four terms yields

$$\frac{d}{dz} e^{H} = \frac{dH}{dz} e^{H} + \frac{1}{2}[H,\frac{dH}{dz}]e^{H} + \frac{1}{6}[H,[H,\frac{dH}{dz}]]e^{H} + \frac{1}{24}[H,[H,[H,\frac{dH}{dz}]]]e^{H} \tag{7}$$

All terms save the first may be transferred to the right and the equation solved by successive approximation. Including only the first two terms, we derive

$$H = \frac{e}{p}(1+\frac{g-2}{2}\gamma) \int_0^{\ell} F dz - \frac{1}{2} \frac{e^2}{p^2}(1+\frac{g-2}{2} \gamma)^2 \int_0^{\ell}\int_0^{z} [F(\zeta),F(z)]d\zeta dz \tag{8}$$

The second term involves the integral of a commutator over a triangular domain. The matrix obtained from the commutator is the dual of a vector formed from the cross product of the magnetic field vectors B at two different longitudinal locations. Such a term represents a rotation about the longitudinal axis, which, however, can be large, since it is produced by the transverse field and its coefficient contains the factor γ. It is because of this and other similar higher-order terms that snakes can rotate or reverse the polarization.

If the first-order approximation to the polarization transfer matrix is evaluated for each magnetic element and the results multiplied together, then the result will contain that portion of the commutator where the two longitudinal locations are in different elements. It will still not contain the contribution from a single element, for example, what might be called the "self-snaking" term for a quadrupole. It may be useful to examine the double and multiple integrals of commutators for a lattice as well as the lowest order field integral for a periodic structure to maximize the degree of polarization preservation.

For a quadrupole, we use a power series expansion of the orbit transfer matrix to obtain an approximate solution to equation (7). The result is

$$H = \frac{e}{p}\left(1+\frac{g-2}{2}\gamma\right) \int_0^{\ell} F dz - \frac{1}{12} \frac{e^2}{p^2}\left(1+\frac{g-2}{2}\gamma\right)^2 \frac{B_o^2}{a^2} \ell^3 (x_o' y_o - x_o y_o') E_z \qquad (9)$$

with
$$E_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (10)$$

The field integrals are given by

$$\int_0^{\ell} B_x dz = \frac{B_o}{a}\left[\frac{y_o}{k} \sinh(k\ell) + y_o' \frac{1}{k^2} (\cosh(k\ell)-1)\right] \qquad (11)$$

$$\int_0^{\ell} B_y dz = \frac{B_o}{a}\left[\frac{x_o}{k} \sin(k\ell) + x_o' \frac{1}{k^2} (1-\cos(k\ell))\right]$$

For a sextupole, we integrate along the first-order trajectory and derive

$$\int_0^{\ell} B_x dz = \frac{2B_o}{a^2}\left[x_o y_o \ell + \frac{1}{2}(x_o y_o' + x_o' y_o)\ell^2 + \frac{1}{3} x_o' y_o' \ell^3\right] \qquad (12)$$

$$\int_0^{\ell} B_y dz = \frac{B_o}{a^2}\left[(x_o^2 - y_o^2)\ell + (x_o x_o' - y_o y_o')\ell^2 + \frac{1}{3}(x_o'^2 - y_o'^2)\ell^3\right]$$