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Integrability, Duality, Monodromy, and the Structure of Bethe's Ansatz

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1.1 INTRODUCTION

The technology of integrable quantum systems has developed rapidly over the past several years.[1] The confluence of old and new ideas has produced some beautiful mathematical physics and provided a great deal of insight into the nature of these systems. In spite of all these developments, one central problem remains unsolved: that of calculating correlation functions for general Bethe's ansatz models.[2-8] In principle, the problem is straightforward -- all information relating to correlations is contained in the Bethe wave functions. We only need to calculate the expectation value of a product of field (or spin) operators for an N-body wave function in a box of length L and let $N \to \infty$ with N/L fixed. The problem is that such a calculation involves a number of terms of order $(N!)^2$, and the simplifications which presumably occur depend on intricate cancellations among these terms which are not well understood. In the 51 years since Bethe wrote down his ansatz, no one has succeeded in getting a complete answer in this way.

In these lectures I will discuss the problem of calculating correlation functions for integrable models, and some of the possibilities that have been raised by recent developments. Since I don't know how to solve the general problem, I will focus on ideas and directions, and on special limits in which the problem becomes more tractable. In particular, I want to explore the connection between two very important recent developments. One is the advent of the quantum inverse scattering method,[1] to which you have already been introduced by the lectures of Faddeev at this school. The other closely related but largely independent development is the treatment of correlation functions for certain special integrable models by Sato, Miwa, and Jimbo (SMJ).[7] The ideas which underlie these two approaches appear to be deeply related, but in a way that is not completely understood. The quantum inverse method, like its precursor, Bethe's ansatz, is quite generally applicable to integrable quantum systems. The SMJ approach, on the other hand, has only been formulated for certain special "free fermion" models like the 2-D Ising model, X-Y spin chain, and the impenetrable Bose gas. But within this limited domain it provides a powerful and complete method for calculating correlation functions. The formal and conceptual similarities between the two methods raises the hope that an understanding of the relationship between them will lead to a complete solution of the problem of calculating correlation functions for integrable systems.

One of the main purposes of these lectures will be to present a pedagogical discussion of the SMJ method in a context where its connection with Bethe's ansatz and the quantum inverse formalism will be more apparent. Before getting into the details, let me emphasize some of the essential points. The SMJ approach relies heavily on the idea of duality. This idea goes back to the Kramers-Wannier[9] observation that the high and low temperature phases of the 2-D Ising model are mathematically equivalent in the sense that the partition function in the two phases is related by a simple transformation $\tanh \beta \leftrightarrow e^{-2\beta}$, (where $\beta = 1/kT$). Kadanoff and Ceva[10] later completed the statement of equivalence by introducing "disorder" variables which were dual to the spin variables. The familiar fermion operators of the Ising model[11-13] could be regarded as the product of an order variable and a disorder variable. Thus, the disorder variable at site n is essentially the Jordan-Wigner tail $\prod_{j>n}(i\sigma_j^z)$ which is attached to a Pauli matrix to turn it into a fermion operator. In terms of these operators, the essence of duality can be expressed by the commutation relations between order and disorder, or equivalently, spin and fermion operators. Specifically, a spin at site i and a fermion at site j commute if $i<j$ and anticommute if $i\geqslant j$. This algebraic relation forms the basis of SMJ's treatment of correlations. The duality algebra and associated Jordan-Wigner transformation is in some sense a special case of Bethe's ansatz for those models in which the two-body S-matrix (Bethe's ansatz phase shift) is -1. The model I will discuss in detail is the nonlinear Schrödinger (NLS) model. The infinite coupling ($c=\infty$) case of this model[3-6] has a structure very similiar to the Ising model and the correlation functions for this case can be calculated by the SMJ appraoch. In addition there are some encouraging signs that the ideas behind the SMJ treatment have a natural

generalization to the finite c case via the quantum inverse method. The fermion variable at $c=\infty$ generalizes to the reflection coefficient operator $R(x)$, the Jordan-Wigner transformation generalizes to the Gel'fand-Levitan transformation,[14-16] and the duality algebra generalizes to the commutator $[R(x),\phi(y)]$ where $\phi(y)$ is the NLS field operator (which is analogous to a spin operator in the Ising model).

In order to present these ideas in a unified setting, it is convenient to use a graphical representation of Bethe's ansatz which was originally derived from a study of Feynman graphs.[17] More recently it has developed that this approach is also closely related to both the Gel'fand-Levitan method and to the SMJ formalism. I will therefore begin with an introduction to this graphical formalism.

1.2 BETHE'S ANSATZ GRAPHS

We'll consider the nonlinear Schrödinger model defined by the Hamiltonian

$$H = \int\{\partial_x\phi^*\partial_x\phi + c\phi^*\phi^*\phi\phi\}\, dx \equiv H_o + V \tag{1}$$

where $\phi(x)$ is a canonical nonrelativistic boson field

$$[\phi(x),\phi^*(y)] = \delta(x-y) \tag{2}$$

To treat the finite density system we will eventually want to add a chemical potential term, i.e. $H \rightarrow H - \mu N$ where $N = \int\phi^*\phi dx$

is the particle number operator, but for now we will consider the zero density case $\mu = 0$. The particle number N is conserved, and we can therefore diagonalize H separately in each N-body sector. The N-body wave functions of Bethe's ansatz are constructed by dividing the N dimensional space of particle coordinates $x_1,\ldots,x_n$, into N! sectors corresponding to specific orderings of the particles, e.g. $x_{P_N}<x_{P_{N-1}}<\ldots<x_{P_1}$ where $(P_1..P_N)$ is a permutation of $(1,..,N)$. In each sector the wave-function is proportional to an N-body plane wave, but the coefficients which appear in the different orderings involve products of two-body phase shifts. For example, the exact two-body wave function may be written

$$|k_1,k_2> = \int dx_1 dx_2\, e^{i(k_1x_1+k_2x_2)}\,\{\theta(x_2<x_1)+S(k_{21})\theta(x_1>x_2)\}$$

$$\times\ \phi^*(x_1)\ \phi^*(x_2)|0> \qquad (3)$$

where $k_{21} \equiv k_2 - k_1$, and S is the two-body S-matrix

$$S(k) = \frac{k - ic}{k + ic} \qquad (4)$$

More generally, if we adopt the convention that the ordering $x_N<\ldots<x_1$ (i.e. P=I=identity) has a coefficient of unity, then the coefficient of any other P is obtained by starting with I and constructing P by interchanging adjacent elements. There is a factor $S(k_{ij})$ for each interchange of adjacent elements i and j. For example, in the 3-body case, the ordering $x_2<x_1<x_3$ would have a factor of $S(k_{32})S(k_{31})$.

The graphical representation of Bethe's ansatz is best described by separating connected and disconnected parts of the wave functions. Thus, for the two-body case we would rewrite the integrand in (3) as

$$\theta(x_2<x_1) + S_{21}\,\theta(x_1<x_2) = 1 + \tau_{21}\,\theta(x_1<x_2) \tag{5}$$

where $S_{ij} \equiv S(k_{ij})$ and

$$\tau_{ij} = S_{ij} - 1 . \tag{6}$$

The first and second terms on the right hand side of (5) can be represented by a disconnected and a connected graph respectively, as shown in Fig. 1. These graphs consist of straight "particle lines" which represent a Bethe's ansatz particle with a given pseudomomentum, and wiggly "phonon" lines which represent an interaction between two modes. From the two-body case, we see that the phonon line represents both a factor τ_{21} and a coordinate space step function $\theta(x_1<x_2)$. In momentum space (i.e. Fourier transforming the wave function over the x_i's), the step function becomes a phonon propagator,

$$\frac{-i}{q-i\epsilon} \tag{7}$$

where q is the momentum carried by the phonon. The generalization of these graphical rules to the N-body wave function is straightforward. Choose a particular ordering of the k_i's along the bottom of the graph and draw all graphs with this ordering and with all phonon lines pointing from left to right as in Fig. 2. (The wave functions obtained by choosing different orderings of pseudomomenta differ only by overall factors of two-body phase shifts.) Any pair of particle lines can be connected by either zero or one phonon line, corresponding to the absence or presence of an interaction between the two modes. Thus, each phonon represents a propagator $(-i)/(q-i\varepsilon)$ and a dynamical factor τ_{ij}. The full N-body wave function (connected plus disconnected pieces) consists of $2^{N(N-1)/2}$ graphs. The graphs for the three-body wave function are shown in Fig. 2. It is not difficult to show that this graphical ansatz reproduces the N-body Bethe wave functions.[17] An important advantage of the graphical formulation is that the wave function is naturally written in a cluster-decomposed form in which the connectedness of a particular term is manifest. This is often of great help in practical calculations (e.g. of matrix elements in Bethe's ansatz states).

Starting from the graphical representation of Bethe's ansatz, all of the known results for this model (e.g. thermodymanics, spectral integral equation, norms of Bethe wave functions) can be derived and given a simple graphical interpretation. These results have been discussed elsewhere[17] and I will not review them here. Instead I will just give the graphical rules for computing inner products of Bethe's ansatz states $\langle p_1..p_N|k_1..k_N\rangle$ and matrix elements of the field operator $\langle p_1...p_N|\phi(x)|k_o...k_N\rangle$. For the inner product $\langle p_1...p_N|k_1...k_N\rangle$ one draws the same set of graphs as for the N-body wave function, but now: (1) There are pseudomomentum variables $k_1...k_N$ at the bottom of the graph and $p_1...p_N$ at the top of the graph, and (2) A phonon connecting line i with line j represents a phonon propagator $(-i)/(q-i\varepsilon)$ along with a dynamical factor

$$\tau(p_{ij}) + \tau(k_{ji}) + \tau(p_{ij})\tau(k_{ij}) = S(p_{ij})S(k_{ji}) -1 \qquad (8)$$

representing the fact that the two particles could have interacted in either the initial state, in the final state, or in both. The full inner product is obtained by summing over all permutations of the external pseudomomenta, with appropriate factors of $S(p_{ij})$ and $S(k_{ij})$ accompanying the different permutations. From these graphical rules, it is easy to show that when the k_i's are not equal to the p_i's, the inner product vanishes (i.e., the states are orthogonal). In addition, by letting $i\varepsilon$ be small but nonzero, one can compute the norm of the Bethe wave functions $\langle k_1..k_N|k_1..k_N\rangle$ obtaining a graphical result which is equivalent to the determinental expression discussed by Gaudin and Korepin.[18-19]

The graphical rules for a matrix element of the field operator $\langle p_1..p_N|\phi(x)|k_o..k_N\rangle$ are also quite simple. For each graph, we draw N+1 lines numbered 0 through N, with pseudomomentum labels $k_0,\ldots,k_N$ on the bottom and $p_1,\ldots,p_N$ on the top. Line zero represents the particle which is annihilated by the field operator and hence has no pseudomomentum label in the final state. As before, we draw all possible graphs with phonons connecting pairs of particle lines. A phonon connecting line i with line j incurs a factor of

$$S(p_{ij})S(k_{ji})-1 \tag{9}$$

if $i<j$ and $i\neq 0$, while a phonon connecting line zero with line j gives a factor of

$$S(k_{j0})-1 = \tau(k_{j0}) \tag{10}$$

(In each case, there is also the usual phonon propagator $(-i)/(q-i\varepsilon)$). These rules are just a hybrid version of the previous rules for inner product graphs and wave function graphs. As usual the full matrix element is obtained by summing over permutations of pseudomomenta with S-matrix factors. The graphs for matrix elements of the field operator are closely related to the quantum Gel'fand-Levitan transform, as we shall see shortly.

1.3 QUANTUM INVERSE SCATTERING AND THE GEL'FAND-LEVITAN FORMALISM

There have been several extensive reviews of the quantum inverse scattering method,[1] and I will not attempt to give a thorough introduction here. I will simply review the basic ideas and quote the results which are relevant to the study of correlation functions. To solve the nonlinear Schrödinger model by the inverse scattering method, we study the linear eigenvalue problem associated with the spatial component of a "Lax pair"

$$\partial_\mu \Psi = i Q_\mu \Psi \qquad \mu=0,1 \tag{11}$$

Here $Q_\mu(x,k)$ are 2×2 matrices which depend on the fields $\phi(x)$ and $\phi^*(x)$ and on an eigenvalue parameter k. In particular, the spatial component Q_1 is given by

$$Q_1 = \begin{pmatrix} k/2 & \sqrt{c}\,\phi \\ -\sqrt{c}\phi^* & -k/2 \end{pmatrix} \tag{12}$$

We regard the $\mu=1$ component of Eq. (11) as a problem of time independent scattering theory and construct the scattering data in terms of the field $\phi(x)$, the latter playing the role of the scattering potential. In the quantum theory, the scattering data become normal ordered functionals of the operators ϕ and ϕ^*. The formulation of the quantum inverse method that we will use is carried out in the zero density vacuum where $\phi(x)\to o$ (weakly) as $x\to\infty$. (Of course we are really interested in the correlations at finite density, and so we must eventually find a way to formulate our operator expressions in the finite density Hilbert space. More about this later.)

With the zero-density restriction, one can construct Jost solutions for the linear problem

$$\partial_1 \Psi = i{:}Q_1 \Psi{:} \tag{13}$$

For example, we may define a Jost solution ψ by its asymptotic behavior at $x\to-\infty$

$$\psi(x,k) \underset{x\to-\infty}{\sim} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ikx/2} \tag{14}$$

The scattering coefficients $a(k)$ and $b(k)$ are defined by the behavior of ψ as $x\to+\infty$:

$$\psi(x,k) \underset{x\to+\infty}{\sim} \begin{pmatrix} a(k)\ e^{ikx/2} \\ b(k)\ e^{-ikx/2} \end{pmatrix} \tag{15}$$

Solving Eq. (13) by a Born series, we obtain series expansions for the Jost solution and for the scattering coefficients in terms of ϕ and ϕ^*. In particular, $a(k)$ and $b(k)$ can be written

$$a(k) = 1+c \int dx_1 dy_1 \theta(x_1<y_1) e^{ik(x_1-y_1)} \phi^*(x_1)\phi(y_1)$$
$$\times c^2 \int dx_1 dy_1 dx_2 dy_2 \theta(x_1<y_1<x_2<y_2) e^{ik(x_1-y_1+x_2-y_2)}$$
$$\times\phi^*(x_1)\phi^*(x_2)\phi(y_1)\phi(y_2) \tag{16}$$

$$\frac{i}{\sqrt{c}}\, b(k) = \int dx_1 e^{ikx_1}\phi^*(x_1) + c\int dx_1 dy_1 dx_2 \theta(x_1<y_1<x_2) e^{ik(x_1-y_1+x_2)}$$
$$\times\phi^*(x_1)\phi^*(x_2)\phi(y_1)$$
$$+ \ldots \tag{17}$$

In the classical inverse method, the importance of the scattering data lies in the fact that, if the field $\phi(x)$ evolves in time according to the nonlinear Schrödinger equation, the time evolution of the functionals $a(k)$ and $b(k)$ is very simple. The important observation which gave birth to the quantum inverse method was that the scattering data, regarded as quantum operators, have simple algebraic properties and are closely related to Bethe's ansatz. Specifically,

$$[H,a(k)] = 0 \tag{18a}$$
$$[H,b(k)] = k^2 b(k) \tag{18b}$$

$$[a(k),a(k')] = [a(k),b(k')] = [b(k),b(k')] = 0 \tag{18c}$$

$$a(k)b(k') = \left(1 - \frac{ic}{k-k'-i\varepsilon}\right) b(k')a(k) \tag{18d}$$

(I should emphasize here that we are working in an infinite volume. The commutation relations in a finite box can also be constructed, but they contain extra terms associated with periodic boundary conditions.) From these relations, we see that the states obtained by applying b(k)'s to the vacuum,

$$|k_1 \ldots k_N\rangle = b(k_1)\ldots b(k_N)|0\rangle \tag{19}$$

are eigenstates of the Hamiltonian, and also of the one-parameter set of operators a(k). It can be shown that these states are precisely those constructed by Bethe's ansatz. The operator a(k) may be regarded as the generating function for an infinite number of conserved quantities. To discuss the inverse problem and Gel'fand-Levitan formalism, it is convenient to define the quantized reflection coefficient

$$R^*(k) = \frac{i}{\sqrt{c}}\, b(k)\ a^{-1}(k). \tag{20}$$

which also has simple algebraic properties,

$$[H, R^*(k)] = k^2 R^*(k) \tag{21a}$$

$$R(k)R(k') = S(k'-k)R(k')R(k) \tag{21b}$$

$$R(k)R^*(k') = S(k-k')R^*(k')R(k) + 2\pi\delta(k-k') \tag{21c}$$

where S is the usual two-body phase shift (4). The R* operators create normalized Bethe eigenstates.

Like Fourier transformation, the inverse scattering transformation consists of both a direct and an inverse transform. The direct transform, Eqs. (16)-(17), give the scattering coefficients a(k) and b(k) in terms of the field operators $\phi(x)$ and $\phi^*(x)$. In order to study correlation functions, we must construct the inverse transform which gives the field operators in terms of the scattering data. This is accomplished by the method of Gel'fand and Levitan.[20-21] The basic idea of this method is to use the analytic properties of the Jost solutions to construct a function which is analytic in the k-plane with a discontinuity across the real axis which is proportional to the reflection coefficient R(k). A dispersion relation for this function then provides an integral equation for the Jost solution $\chi(x,k)$ defined by the asymptotic behavior

$$\chi(x,k) \underset{x\to+\infty}{\sim} \begin{pmatrix}0\\1\end{pmatrix} e^{-ikx/2} \tag{22}$$

Specifically χ_1 and χ_2^* satisfy the coupled integral equations

$$\chi_2^*(x,k)e^{-ikx/2} = 1 + \frac{\sqrt{c}}{2\pi}\int dk' \frac{R^*(k')\chi_1(x,k')e^{-ik'x/2}}{k'-k-i\varepsilon} \tag{23a}$$

$$\chi_1(x,k)e^{ikx/2} = \frac{\sqrt{c}}{2\pi}\int dk' \frac{\chi_2^*(x,k')R(k')e^{ik'x/2}}{k'-k+i\varepsilon} \tag{23b}$$

In both the classical and quantum theory, the Gel'fand-Levitan equation follows from analyticity arguments. There is one interesting feature of the quantum derivation which is worth mentioning here. In that case, the construction of an analytic operator function in the complex k-plane depends not only on the analyticity of the Jost solutions but also on the fact that the commutator $[R^*(k),\phi^*(x)]$ is analytic in the lower half k-plane. As we will see in the next lecture, this commutator reduces, in the limit $c\to\infty$, to the duality commutator between fermion and boson fields, while the Gel'fand-Levitan transform itself becomes a Jordan-Wigner transformation.

Solving Eqs. (23) by iteration, we obtain a series expansion for the Jost solution in terms of the eigenmode operators $R^*(k)$ and $R(k)$. The local field operator is recovered from the asymptotic form of the Jost solution,

$$\chi_1(x,k)e^{ikx/2} \underset{k\to\infty}{\sim} -\frac{\sqrt{c}}{k}\,\phi(x) + O(1/k^2) \tag{24}$$

The series expansion for $\phi(x)$ may be written explicitly as

$$\phi(x) = \sum_{n=o}^{\infty} \phi^{(m)}(x) \tag{25a}$$

where

$$\phi^{(n)}(x) = (-c)^n \int \left(\prod_1^n \frac{dp_i}{2\pi}\right)\left(\prod_0^n \frac{dk_i}{2\pi}\right) \frac{e^{i(\sum_0^n k_i - \sum_1^n p_i)x}}{\prod_{i=1}^n [(p_i-k_{i-1}-i\varepsilon)(p_i-k_i-i\varepsilon)]}$$

$$\times R^*(p_1)\ldots R^*(p_n)R(k_n)\ldots R(k_o) \qquad (25b)$$

Note that this series is normal ordered with respect to the zero density vacuum (i.e. R^*'s to the left and R's to the right).

The integrand in (25b) is in the form which comes directly from iteration of the Gel'fand-Levitan integral equation. There are many other equivalent ways of writing these integrals. Let us write a generic operator of this form as

$$\int \left(\prod_1^n \frac{dp_i}{2\pi}\right)\left(\prod_0^n \frac{dk_i}{2\pi}\right) f(p_i;k_i)\ R^*(p_1)\ldots R^*(p_n)R(k_n)\ldots R(k_o), \qquad (26)$$

where $f(p_i;k_i)$ is some c-number function. Then, by permuting integration variables and using the commutation relations (21b), we can make the replacement

$$f(p;k) \longrightarrow S_P(p_i)\ S_Q(k_i) f(p_{P_i};k_{Q_i}) \qquad (27)$$

where P is any permutation of the p_i's and Q is any permutation of the k_i's, and S_P and S_Q are the products of two-body S-matrices associated with those permutations. By utilizing this freedom to interchange integration variables, we may construct a variety of equivalent expressions for an operator of the form (26). The only part of the function $f(p_i,k_i)$ that is relevant is the "R-symmetrized" function $f^{(S)}(p;k)$ which is obtained by summing over all permutations P and Q with appropriate S-matrix factors. Now let us consider the matrix element of the field operator $\langle p_1 \ldots p_n|\phi(x)|k_o \ldots k_n\rangle$ which we have already discussed in terms of Bethe's ansatz graphs. If all the p's are different from all the k's, then it is easily seen that only the nth term in the GL series (25a) contributes to this matrix element. Thus, the R-symmetrized integrand appearing in $\phi^{(n)}(x)$ is just the matrix element $\langle p_1 \ldots p_n|\phi(x)|k_o \ldots k_n\rangle$, provided that the p's and k's are all different. For general p and k, these two functions can differ from each other only in the signs of $i\varepsilon$'s. The Gel'fand-Levitan integrands can thus be obtained from the Bethe's ansatz matrix elements of $\phi(x)$ by first calculating these matrix elements ignoring $i\varepsilon$'s and then inserting the correct $i\varepsilon$ prescription by letting all p's have a negative imaginary part, $p_i \to p_i - i\varepsilon$. In this way the graphical formalism for the matrix elements of $\phi(x)$ can also be used to represent the Gel'fand-Levitan series. Such a direct graphical construction of the Gel'fand-Levitan integrands leads to expressions which differ from those in (25b) but are equivalent after R-symmetrization. The graphical expression is in some ways more convenient, particularly for treating the $c \to \infty$ limit. Since the graphs consist entirely of phonon propagators $(-i)/(q-i\varepsilon)$ and factors of $\tau(k) = S(k)-1$

(where $k = k_{ij}$ or p_{ij}), the limit $c\to\infty$ is thus easily taken by setting $\tau(k)\to -2$. For the case $c=\infty$, the graphical formalism provides a nice intuitive picture of the SMJ method for calculating correlation functions. For most of the remaining lectures I will be discussing the case $c=\infty$ and the application of the SMJ procedure. However, it should be emphasized that the graphical formalism applies as well to the finite c case and is directly related to Bethe's ansatz and to the Gel'fand-Levitan transformation. I remain hopeful that as our understanding of integrable systems progresses, some generalization of the SMJ arguments to Bethe's ansatz systems will be found.

2.1 GEL'FAND-LEVITAN EQUATION AS A GENERALIZED JORDAN-WIGNER TRANSFORMATION

In the previous lecture we saw that the nonlinear Schrödinger model could be solved at the quantum level either by an explicit Bethe ansatz or by the quantum inverse method, which provides an algebraic form of Bethe's ansatz. By studying the direct scattering transform associated with the linear Zakharov-Shabat eigenvalue problem, we constructed the scattering data operators $a(k)$ and $b(k)$. These operators are functionals of the NLS field operators $\phi(x)$ and $\phi^*(x)$. The operator $b(k)$ produces eigenstates of H by repeated application on the vacuum, while the operator $a(k)$ is diagonal on these states. Thus, the reflection coefficient operator $R(k)=b(k)a^{-1}(k)$ also creates eigenstates of H. The commutation relations (21c) show that in fact $R(k)$ creates the properly normalized eigenstates. The inverse problem is the problem of reconstructing the field operators $\phi(x)$ and $\phi^*(x)$ in terms of the operators $R(k)$ and $R^*(k)$. This construction is accomplished by the quantum Gel'fand-Levitan procedure and leads to the series expression (25). The basic goal of the Gel'fand-Levitan procedure is to calculate correlation functions for the field operators by expressing them in terms of the R operators. Until now this goal has not been realized for the general finite c case due to the combinatorial problems which arise in taking expectation values of the

Gel'fand-Levitan series (25). However, if we take the limit $c\to\infty$, the problem simplifies considerably. This is somewhat surprising since the series (25) appears to be an expansion in powers of c. However, there is an implicit c dependence in the R operators due to their commutation relations, and after symmetrizing the integrands it is found that the $c\to\infty$ limit can be taken term by term in the series. As I mentioned before, the $c=\infty$ limit follows most directly from the graphical representation of the Gel'fand-Levitan integrands, but it is interesting to see how the result follows from the expressions in (25b). I will illustrate the point by considering the first two terms in the series. The first term requires no symmetrization and is just the Fourier transform of the reflection coefficient

$$\phi^{(0)}(a) = \int \frac{dk_0}{2\pi}\, e^{ik_0 a}\, R(k_0) \equiv r(a) \tag{28}$$

For the second term we symmetrize over the variables k_0 and k_1 and use the commutation relation (21b) to make the replacement

$$\frac{(-c)}{(p_1-k_0)(p_1-k_1)} \longrightarrow \frac{(-c)}{(p_1-k_0)(p_1-k_1)} \times \frac{1}{2}\,[1+S_{10}] \tag{29}$$

The right hand expression now has a finite limit as $c\to\infty$, and in this limit we can write

$$\phi^{(1)}(a) = -2\int\frac{dk_0}{2\pi}\,\frac{dk_1}{2\pi}\,\frac{dp_1}{2\pi}\; e^{i(k_0+k_1-p_1)a}\;\frac{ik_{10}}{(p_1-k_0)(p_1-k_1)}$$

$$\times R^+(p_1)R(k_1)R(k_0) \qquad (30)$$

Next we observe that for $c=\infty$, the R's become canonical fermion operators with local anticommutation relations in both momentum and position space,

$$\{R(k),\ R^+(k')\} = 2\pi\delta(k-k') \qquad (31a)$$

$$\{R(k),\ R(k')\} = 0 \qquad (31b)$$

and

$$\{r(x),\ r^+(x')\} = \delta(x-x') \qquad (32a)$$

$$\{r(x),\ r(x')\} = 0 \qquad (32b)$$

So, finally, we can write

$$\phi^{(1)}(a) = (-2)\int\frac{dk_0}{2\pi}\,\frac{dk_1}{2\pi}\,\frac{dp_1}{2\pi}\;\frac{(-i)\,e^{i(k_0+k_1-p_1)a}}{p_1-k_1-i\varepsilon}\;R^+(p_1)R(k_1)R(k_0)$$

$$= -2\left[\int_x^\infty dz\ r^+(z)\ r(z)\right]\ r(a) \qquad (33)$$

By carrying out a similar procedure of symmetrization on each term in the Gel'fand-Levitan series, it has been found that the $c\to\infty$ limit can be taken term by term, with the remarkably simple result

$$\phi^{(N)}(a) = \frac{(-2)^N}{N!} : \left[\int_a^\infty dz r^+(z) r(z)\right]^N r(x): \tag{34}$$

where : : means normal ordering, i.e., all r^+'s to the left and all r's to the right. Thus, the Gel'fand-Levitan transformation exponentiates into a Jordan-Wigner transformation for the case $c=\infty$,

$$\phi(a) = : \exp\left[-2\int_a^\infty r^+(z) r(z)\right] r(a): \tag{35}$$

This expression has a form which is familiar from other free fermion models (e.g., the two dimensional Ising model) which involve a boson-to-fermion transformation. It is this result and the associated duality algebra between the boson $\phi(a)$ and the fermion $r(x)$ which allows us to establish a link between the quantum inverse method and the SMJ treatment of correlation functions. The duality relations are somewhat more transparent if we imagine putting the r operators on a spatial lattice and replacing the integral from a to ∞ by a discrete sum. Then the operators $r^+(z)r(z)$ commute on different lattice sites and we may normal order the exponential on each site separately, using

$$: e^{-2r^+(z)r(z)} : = 1-2r^+(z)r(z) \tag{36}$$

Then noting that

$$r^{+}(z)\ [1-2r^{+}(z)r(z)] = -\ [1-2r^{+}(z)r(z)]\ r^{+}(z) \tag{37}$$

we obtain the duality relations

$$\phi(a)r^{+}(x) = \varepsilon(x-a)\ r^{+}(x)\phi(a) \tag{38a}$$

$$\phi(a)r(x) = \varepsilon(x-a)\ r(x)\phi(a) \tag{38b}$$

2.2 TRANSITION TO THE FINITE DENSITY VACUUM

So far we have dealt only with the zero density system where $\phi(x)\to 0$ as $x\to\pm\infty$. The expressions we have obtained by the quantum inverse method are thus normal ordered with respect to the bare (zero density) vacuum state $|0\rangle$. In order to address the problem of correlations in the finite density system we must find a way to relax the asymptotic condition on $\phi(x)$. There are several possibilities. We might quantize the system in a box of length L and only take the limit $L\to\infty$ at the end of the calculation. This is the traditional approach to deriving spectral results from Bethe's ansatz, but in the context of the quantum inverse method, the R operators lose their simple properties in a finite box. Alternatively, one might quantize the classical inverse problem with boundary condition $\phi(x)\to\text{const.}\neq 0$. The classical problem has been treated by Zakharov and Shabat.[22] However, for $c=\infty$ Jimbo, Miwa, Mori, and Sato[6] have introduced a more direct procedure which is based on the duality relations (38). Let me first outline their method and then go on to describe it in more detail. The essential point is that the boson field $\phi(a)$ is completely determined (up to an overall normalization factor) by the duality relations. Thus if we introduce a new set of fermion creation and annihilation operators which are defined with respect to the physical vacuum, we need only to "solve" the duality relations to obtain an expression for $\phi(a)$ in terms of these physical fermion operators. In the remainder of this lecture I will show how this procedure is carried out. Then in the last lecture I will show how the resulting expression for the boson field is used to compute correlation functions.

To discuss the SMJ calculation, it is convenient to consider only the real part of the NLS field

$$\sigma(x) \equiv \frac{1}{2}\,[\phi(x)+\phi^*(x)] \tag{39}$$

It is easy to see that the correlation $\langle\sigma^*(x)\sigma(y)\rangle$ is the same as the correlation $\langle\phi^*(x)\phi(y)\rangle$ (for $x\neq y$) since $\langle\phi(x)\phi(y)\rangle = \langle\phi^*(x)\phi^*(y)\rangle = 0$ by particle number conservation. However, we now wish to introduce a small violation of particle number conservation so that we may consider the case $\langle\sigma(x)\rangle\to\text{const.}\neq 0$ as $x\to\infty$, analogous to the classical vacuum at finite density. This may be done by introducing a small term in the Hamiltonian of the form $\lambda(\phi^*\phi^*+\phi\phi)$ which introduces a small mass gap at the Fermi surface and allows us to consider $\langle\sigma(x)\rangle\neq 0$. (Jimbo, Miwa, Mori, and Sato (JMMS) accomplish the same thing by considering the $c=\infty$ NLS model as the scaling limit of the X-Y Heisenberg spin chain where $\sigma(x)$ is the scaling limit of a Pauli spin operator σ_x.) In the limit $\lambda\to 0$ we recover the symmetric vacuum $\langle\sigma(x)\rangle = 0$ and the correct correlation functions. It will also be convenient to introduce the real and imaginary parts of the r operator,

$$p(x) = r^+(x) + r(x) \tag{40a}$$

$$q(x) = r^+(x) - r(x) \tag{40b}$$

The duality algebra now reads

$$\sigma(a)p(x) = \varepsilon(x-a)p(x)\sigma(a) \tag{41a}$$

$$\sigma(a)q(x) = \varepsilon(x-a)q(x)\sigma(a) \tag{41b}$$

Since we are taking $\langle\sigma(a)\rangle\neq 0$ we could imagine inverting $\sigma(a)$ and writing these relations as

$$\sigma(a)p(x)\sigma(a)^{-1} = \varepsilon(x-a)p(x) \tag{42a}$$

$$\sigma(a)q(x)\sigma(a)^{-1} = \varepsilon(x-a)q(x) \tag{42b}$$

Thus the duality relations can be regarded as the statement that $\sigma(a)$ induces a linear (orthogonal) transformation on the fermion basis formed by $p(x)$ and $q(x)$. A fundamental theorem of Clifford groups says that such an operator can be expressed (up to a normalization factor) as an exponential

$$\sigma(a) = \langle\sigma(a)\rangle : e^{\rho(a)/2}: \tag{43}$$

where $\rho(a)$ is a quadratic form in the fermion operators. We will see how this theorem is satisfied by explicitly constructing the operator $\rho(a)$. First we introduce physical fermion creation and annihilation operators by defining

$$\psi(p) = \frac{1}{2}\left\{\sqrt{a(p)}\ P(p) - \frac{1}{\sqrt{a(p)}}\ Q(p)\right\} \tag{44}$$

where $P(p)$ and $Q(p)$ are the Fourier transforms of $p(x)$ and $q(x)$, and $a(p)$ is a step function at the Fermi surface

$$a(p) = \varepsilon(k_F - |p|) \tag{45}$$

To see that $\psi(p)$ and $\psi^+(p)$ are the physical operators, note that for $|p|>k_F$,

$$\psi(p) = \frac{i}{2}\ [P(p)+Q(p)] = iR(p) \tag{46}$$

while for $|p|<k_F$,

$$\psi(p) = \frac{1}{2}\ [P(p)-Q(p)] = R^+(-p) \tag{47}$$

Thus, $\psi(p)$ annihilates particles above the Fermi surface and annihilates holes below the Fermi surface. Similarly $\psi^+(p)$

creates particles and holes above and below, respectively.

The Clifford group argument tells us that, once we select a fermion basis, the expression for $\sigma(a)$ in that basis is uniquely determined (up to a normalization) by the duality relation. More specifically, it must be an exponential of a quadratic form, Eq. (43). Thus, if we can find an expression for $\rho(a)$ which is quadratic in the operators $\psi(p)$ and $\psi^{+}(p)$ and satisfies the duality relations (41), we will have an expression for $\sigma(a)$ which is normal ordered with respect to the physical vacuum. This operator expression can then be used to calculate correlation functions. In the graphical language introduced in the first lecture, $\sigma(a)$ has a graphical expansion of the form shown in Fig. 3, where each phonon vertex contains not only a "scattering" piece (corresponding to $\psi^{+}\psi$ terms in the operator $\rho(a)$) but also a pair creation ($\psi^{+}\psi^{+}$) and a pair annihilation ($\psi\psi$) piece. Rather than deriving the result, let me first motivate the general form of the phonon vertex, state the result, and then show that duality is in fact satisfied by this result. The Clifford group argument then assures us that it is unique.

The key trick used by JMMS to construct $\rho(a)$ is a Weiner-Hopf factorization of the step function $\varepsilon(k_F-|p|)$:

$$\varepsilon(k_F-|p|) = \left[\frac{(p+k_F-i\eta)(p-k_F-i\eta)}{(p+k_F+i\eta)(p-k_F+i\eta)}\right]^{1/2} \equiv \frac{b(p)}{b(-p)} \tag{48}$$

where

$$b(p) = [(p+k_F+i\eta)(p-k_F+i\eta)]^{-1/2} \tag{49}$$

and we will eventually take the limit $i\eta \to 0$. Note that b(p) and b(-p) are analytic in the upper and lower half-plane respectively. Thus, we regard the ε step function as the limiting case of an analytic function with cuts slightly above and below the real axis running from $-k_F$ to k_F, as in Fig. 4. We also will need to introduce the function

$$\omega(p) = \frac{1}{b(p)b(-p)} \tag{50}$$

This is a real function which, in the limit $\eta\to 0$ becomes the excitation energy $\omega(p)\to p^2-k_F^2$. Thus the introduction of the parameter η is equivalent to introducing a small mass gap at the Fermi surface.

In order to solve the duality relation in terms of the physical fermion basis we must express P's and Q's in terms of ψ's and ψ^+'s, i.e.

$$P(p) = \frac{1}{\sqrt{a(p)}} \left[\psi^+(-p)+\psi(p)\right] \tag{51a}$$

$$Q(p) = \sqrt{a(p)} \left[\psi^+(-p)-\psi(p)\right] \tag{51b}$$

The structure of the phonon vertex (i.e., of the operator $\rho(a)$) is motivated by the observation that

$$\frac{\sqrt{\omega(p)}}{\sqrt{a(p)}} = \frac{1}{b(p)} \tag{52}$$

is analytic in the upper half-plane, while

$$\frac{1}{\sqrt{a(p)}} \times \frac{1}{\sqrt{\omega(p)}} = b(-p) \tag{53}$$

is analytic in the lower half-plane. With these observations, we state the result of JMMS

$$\rho(a) = \int \frac{dp}{2\pi} \frac{dp'}{2\pi} \frac{(-i)e^{i(p+p')a}}{p+p-i\varepsilon}$$

$$\left\{ \left(\frac{\sqrt{\omega}}{\sqrt{\omega'}} - \frac{\sqrt{\omega'}}{\sqrt{\omega}}\right) [\psi^+(-p)\psi^+(-p')-\psi(p)\psi(p')] \right.$$

$$\left. - \left(\frac{\sqrt{\omega}}{\sqrt{\omega'}} + \frac{\sqrt{\omega'}}{\sqrt{\omega}}\right) [\psi^+(-p)\psi(p')+\psi^+(-p')\psi(p)] \right. \tag{54}$$

In terms of graphs, the scattering vertex, Fig. 5a, is given by

$$- \left(\frac{\sqrt{\omega(p)}}{\sqrt{\omega(p')}} + \frac{\sqrt{\omega(p')}}{\sqrt{\omega(p)}} \right) \tag{55}$$

while the creation and annihilation vertices, Fig. 5b, are given by

$$\left(\frac{\sqrt{\omega(p)}}{\sqrt{\omega(p')}} - \frac{\sqrt{\omega(p')}}{\sqrt{\omega(p)}} \right) \tag{56}$$

It is amusing to see how duality is satisfied by this result by first considering some few body matrix elements of the operator duality relation. In fact, the essential point can be seen by taking the vacuum to one-particle matrix element. We want to compare the matrix elements

$$\langle p' | \sigma(a) P(p) | \Omega \rangle \tag{57a}$$

and

$$\langle p' | P(p) \sigma(a) | \Omega \rangle \tag{57b}$$

where Ω is the physical vacuum, and $|p'\rangle$ is a physical one particle state (i.e., a mode or a hole depending on whether $|p'| \gtrless k_F$). The graphical expressions for these matrix elements are shown in Figs. 6a and b. Recall that $\sigma(a)$ is of the form

$$\sigma(a) = \langle\sigma\rangle : 1 + \frac{1}{2}\rho(a) + \ldots : \tag{58}$$

Thus, for the matrix element of $\sigma(a)P(p)$, the operator $P(p)$ acting on the vacuum creates a particle, and the operator $\sigma(a)$ either does nothing (the 1 in (58) and the first graph in Fig. 6a) or it scatters the particle (the $\psi^+\psi$ term in $\rho(a)$ and the second graph in Fig. 6a). For the matrix element of $P(p)\sigma(a)$, the operator $\sigma(a)$ either does nothing, or it creates a pair. In the latter case, one of these particles is eaten by the annihilation part of $P(p)$ while the other becomes the particle in the final state. To go back to x-space and verify the duality relation

$$\sigma(a)p(x) = \varepsilon(x-a)p(x)\sigma(a) \tag{59}$$

we multiply by e^{ipx} and integrate over p. First consider the case $x>a$. The first graph in Figs. 6a and b is obviously the same, so we must only verify that the second graph gives the same contribution. In each of these graphs there are two terms. One is proportional to $\sqrt{\omega(p')}/\sqrt{\omega(p)}$ and has the same sign in the two graphs (see (55) and (56)). The other is proportional to $\sqrt{\omega(p)}/\sqrt{\omega(p')}$ and has opposite signs for the two graphs. But by virtue of Eq. (52), this term is analytic in the upper half p-plane and thus gives no contribution for $x>a$. This verifies duality for $x>a$. For $x<a$ we must close contours

in the lower half p-plane. By virtue of Eq. (53), the term in each of the one-phonon graphs proportional to $\sqrt{\omega(p')}/\sqrt{\omega(p)}$ is analytic in the lower half p-plane except for the pole represented by the phonon propagator $(-i)/(p'-p-i\varepsilon)$. The contribution from this pole exactly cancels off the noninteracting graph in each matrix element. The remaining term proportional to $\sqrt{\omega(p)}/\sqrt{\omega(p')}$ has a relative minus sign for the two orderings, giving (59) for $x<a$. This completes the demonstration of duality for the vacuum to one-particle matrix elements. The corresponding demonstration for general n-particle matrix elements is no more complicated, with the duality relation again following from the properties of the phonon vertex which we have just discussed. In general, for $x<a$ there is a cancellation between the $(n-1)$-phonon graph and the pole residues of the n-phonon graph. Because there are n such residues, the cancellation dictates a factor of $1/n!$ for the n-phonon graph, and hence an exponentiation of $\rho(a)$.

Thus, for $c=\infty$, we have an expression for $\sigma(x)$ which is normal ordered with respect to the physical, finite density ground state. In the next lecture we will use this expression to calculate correlation functions.

3.1 WAVE FUNCTIONS AND THEIR MONODROMY PROPERTIES

The main result of the last lecture was an expression for the real part of the NLS field $\sigma(a)$ which is normal ordered with respect to the finite density ground state,

$$\sigma(a) = \langle\sigma\rangle : \exp \frac{1}{2} \rho(a) : \tag{60}$$

where $\rho(a)$ is given by (54). The graphical representation of $\sigma(a)$ is shown in Fig. 3. Using this expression, one may construct graphical series expansions for any n-point function $\langle\sigma(a_1)\ldots\sigma(a_n)\rangle$ by expanding each exponential and using Wick's theorem. For example, a typical graph for the 4-point function is shown in Fig. 7. Representing the sources $\sigma(a_i)$ by horizontal dotted lines and imagining the time evolution of the graph as proceeding from bottom to top, it is easy to distinguish scattering vertices from pair creation and annihilation vertices. For example in Fig. 7, $\sigma(a_4)$ creates a pair of particles, one is scattered by $\sigma(a_3)$ and the other by $\sigma(a_2)$, and they are both annihilated by $\sigma(a_1)$. The rules for evaluating a graph follow directly from the expression for $\sigma(a)$ and include: (1) A propagator $(-i)/(q-i\varepsilon)$ for each phonon line, (2) A vertex factor (55) or (56) for each internal phonon vertex, (3) A factor e^{iqa_i} for the attachment of a phonon with momentum q to a source line a_i, and (4) A momentum integration for each closed loop which remains after the dotted source lines are removed (i.e., for closed loops of internal fermion lines).

An important consequence of the graphical expansion for correlation functions is that the logarithm of the correlation function is just the sum of "connected" graphs where connectedness is defined after removal of external source lines, e.g., Fig. 8a is disconnected while Fig. 8b is connected. When viewed in this way, the graphs for a correlation function have a connectedness structure which we would usually associate with a partition function (i.e., the full set of graphs is just the exponential of the connected graphs). This is not entirely unexpected -- in the 2-D Ising model, the correlation function $\langle\sigma(x)\sigma(y)\rangle$ can be regarded as a partition function for a lattice with a line defect between x and y.[10] In the present case the exponential form of the correlation functions follows directly from the application of Wick's theorem to exponential operators of the form (43).

From here on we will confine our attention to the two-point function

$$G(x) = \langle\sigma(x)\sigma(0)\rangle \ . \tag{61}$$

The graphical series for log G(x) is shown in Fig. 9. This series may be loosely described as a fermion bouncing back and forth (albeit in both space and time) between fixed sources at 0 and x. In order to calculate log G(x) it is convenient to also consider "wave function" graphs which represent an external "probe" fermion impinging on the sources, bouncing back and forth a certain number of times, and emerging. (The conceptual similarity between this construction and that of Jost solutions in the inverse scattering method is amusing.) These wave function graphs can be divided into four distinct sets according to whether there is an even or odd number of bounces (phonons) and whether the first bounce is at 0 or at x. In this way we define four functions $V_i(p,p')$ $i = 0,1,2,3$. The graphical series for these four functions is shown in Fig. 10. In addition to their implicit dependence on x, these functions also depend on the initial and final momentum of the probe fermion, i.e., p and p'. Actually, in order to calculate the two-point function, it suffices to consider functions of a single momentum variable $v_i(p)$, $i = 0,\ldots,3$, which are obtained from the V_i's by taking the limit $p'\to\infty$. This can be represented graphically by contracting the last phonon to a point, as in Fig. 11.

Next we note that, for the two-point function, all graphs contain only pair creation and annihilation vertices (i.e., there are no scattering vertices). The creation and annihilation vertices have the important property that they vanish at zero momentum transfer (c.f. eq. (56)) and thus cancel the pole of the attached phonon propagator. This means that the sign of the $i\varepsilon$ in the propagator is irrelevant and may be chosen for convenience. For reasons which will become clear shortly, we will reverse the sign of the $i\varepsilon$'s in all the phonon propagators attached to source line x. Graphically this means reversing the direction of the arrows on the upper phonon lines, as in Fig. 12. This leads to a convenient simplification. To illustrate, let's consider a typical graph, Fig. 13. In addition to the phonon propagators, this graph contains a product of vertex factors of the form

$$\left(\frac{\sqrt{\omega}}{\sqrt{\omega_1}} - \frac{\sqrt{\omega_1}}{\sqrt{\omega}}\right)\left(\frac{\sqrt{\omega_1}}{\sqrt{\omega_2}} - \frac{\sqrt{\omega_2}}{\sqrt{\omega_1}}\right)\left(\frac{\sqrt{\omega_2}}{\sqrt{\omega_3}} - \frac{\sqrt{\omega_3}}{\sqrt{\omega_2}}\right)\left(\frac{\sqrt{\omega_3}}{\sqrt{\omega_4}} - \frac{\sqrt{\omega_4}}{\sqrt{\omega_3}}\right) \frac{1}{\sqrt{\omega_4}}$$

$$= \frac{1}{\sqrt{\omega}} \times \frac{\omega_1}{\omega_2}\,\frac{\omega_3}{\omega_4} \;+\; \text{other terms} \tag{62}$$

Here, the other terms are all characterized by the fact that one or more of the ω_i's are absent. It is easy to see that all these other terms vanish upon integration for $x>0$, because the p_i integration associated with the missing ω_i is of the form

$$\int dp_i \, \frac{e^{-ip_i x}}{(p_{i-1}+p_i-i\varepsilon)(p_i+p_{i+1}-i\varepsilon)} = 0 \tag{63}$$

(Here and elsewhere, we take $x>0$.) Thus, we get simplified series for the functions $v_i(p)$, shown in Fig. 14. After removing external factors of $\omega^{\pm 1/2}$, we can write these series as

$$v_0(p) = 1+(i\lambda)^2\int\frac{dp_1}{2\pi}\,\frac{dp_2}{2\pi}\,\frac{e^{-i(p_1+p_2)x}}{(p+p_1-i\varepsilon)(p_1+p_2-i\varepsilon)}\,\frac{\omega_1}{\omega_2} + \ldots \tag{64a}$$

$$v_1(p) = i\lambda\int\frac{dp_1}{2\pi}\,\frac{e^{-ip_1x}}{p+p_1-i\varepsilon}\,\frac{1}{\omega_1} + (i\lambda)^3\int\frac{dp_1}{2\pi}\,\frac{dp_2}{2\pi}\,\frac{dp_3}{2\pi}$$

$$\times\frac{e^{-i(\Sigma p_i)x}}{(p+p_1-i\varepsilon)(p_1+p_2-i\varepsilon)(p_2+p_3-i\varepsilon)}\quad\frac{1}{\omega_1}\,\frac{\omega_2}{\omega_3} + \ldots \tag{64b}$$

$$v_2(p) = 1 + (i\lambda)^2\int\frac{dp_1}{2\pi}\,\frac{dp_2}{2\pi}\,\frac{e^{-i(p_1+p_2)x}}{(p+p_1-i\varepsilon)(p_1+p_2-i\varepsilon)}\,\frac{\omega_2}{\omega_1} + \ldots \tag{64c}$$

$$v_3(p) = (i\lambda)\int\frac{dp_1}{2\pi}\,\frac{e^{-ip_1x}}{p+p_1-i\varepsilon}\,\omega_1 + (i\lambda)^3\int\frac{dp_1}{2\pi}\,\frac{dp_2}{2\pi}\,\frac{dp_3}{2\pi}$$

$$\times\frac{e^{-i(\Sigma p_i)x}}{(p+p_1-i\varepsilon)(p_1+p_2-i\varepsilon)(p_2+p_3-i\varepsilon)}\quad\omega_1\,\frac{\omega_3}{\omega_2} + \ldots \tag{64d}$$

Here (following JMMS) we have introduced a parameter λ which counts the number of phonons. Eventually, we'll set $\lambda = 1$.

The key to determining the correlation function is a monodromy property of a 2×2 matrix constructed from these wave functions. Define

$$Y(p) = \begin{pmatrix} v_2(-p) & v_3(p) \\ v_1(-p) & v_0(p) \end{pmatrix} \begin{pmatrix} e^{ipx}\,\omega(p) & 0 \\ 0 & 1 \end{pmatrix} \tag{65}$$

From the series (64) we see that the functions $v_i(p)$ are analytic in the lower half p-plane. We also define functions $\tilde{v}_i(p)$ which are analytic in the upper half-plane by making the replacement $(p-i\varepsilon)\to(p+i\varepsilon)$ in the first propagator of each term in the series. The discontinuity $v_i(p) - \tilde{v}_i(p)$ is obtained by replacing the first propagator $(-i)/(p+p_1-i\varepsilon)$ by $2\pi\delta(p+p_1)$. Graphically this amounts to a removal of the first phonon line. A fundamental property of these series is that if the first phonon is removed from the set of graphs for $v_0(p)$ it becomes essentially the set of graphs for $v_1(-p)$ and vice-versa, and similarly for $v_2(p)$ and $v_3(-p)$. This leads to the discontinuity conditions

$$v_0(p) - \tilde{v}_0(p) = -\lambda e^{ipx}\,\omega(p)\; v_1(-p) \tag{66a}$$

$$v_1(p) - \tilde{v}_1(p) = -\lambda e^{ipx}\,\frac{1}{\omega(p)}\,v_0(-p) \tag{66b}$$

$$v_2(p) - \tilde{v}_2(p) = -\lambda e^{ipx}\,\frac{1}{\omega(p)}\,v_3(-p) \tag{66c}$$

$$v_3(p) - \tilde{v}_3(p) = -\lambda e^{ipx}\,\omega(p)\; v_2(-p) \tag{66d}$$

Because of these conditions we can write

$$Y(p) = \begin{pmatrix} v_2(-p) & \tilde{v}_3(p) \\ v_1(-p) & \tilde{v}_0(p) \end{pmatrix} \begin{pmatrix} e^{ipx}\,\omega(p) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \tag{67}$$

The first factor in this expression is manifestly analytic in the upper half-plane, and so we have isolated the upper half-plane singularities of $Y(p)$ within the explicit function $\omega(p)$ appearing in the second factor. $\omega(p)$ has a square root branch cut both above the real axis from $c_1 = -k_F + i\eta$ to $c_2 = k_F + i\eta$ and below the real axis from $c_4 = -k_F - i\eta$ to $c_3 = k_F - i\eta$. Let us denote by γ_r a closed curve around the branch point c_r, as shown in Fig. 15. We will demonstrate that $Y(p)$ has the following fundamental property: when it is analytically continued around the closed curve γ_r, $r=1,\ldots 4$, it returns to the same function multiplied by a constant matrix,

$$Y(p) \xrightarrow[\gamma_r]{} Y(p)\, M_r \tag{68}$$

The constant matrix M_r is called the monodromy matrix. The monodromy matrices are somewhat analogous to the scattering data in the inverse scattering method. To demonstrate the monodromy property (68) around the branch points c_1 and c_2, we simply use (67) and observe that $\omega(p)$ acquires a minus sign from going around either branch point. This gives (68) with

$$M_1 = M_2 = \begin{pmatrix} -1 & 2\lambda \\ 0 & 1 \end{pmatrix} \tag{69}$$

Similarly we can isolate the lower half-plane singularities of Y(p) by writing

$$Y(p) = \begin{pmatrix} \tilde{v}_2(-p) & v_3(p) \\ \tilde{v}_1(-p) & v_0(p) \end{pmatrix} \begin{pmatrix} e^{ipx}\,\omega(p) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}. \tag{70}$$

This leads to (68) for r = 3 and 4, with

$$M_3 = M_4 = \begin{pmatrix} -1 & 0 \\ -2\lambda & 1 \end{pmatrix} \tag{71}$$

We also note the asymptotic behavior of Y(p) which can be read off directly from the definition (65) and the series (64),

$$Y(p) \underset{p\to\infty}{\sim} \begin{pmatrix} p^2 e^{ipx} & 0 \\ 0 & 1 \end{pmatrix} [1+0(1/p)] \tag{72}$$

The monodromy property of Y(p) leads directly to the result that Y(p) satisfies a set of first order linear differential equations. Near each of the branch points c_r, $r = 1,\ldots,4$, we may write

$$Y(p) = \hat{Y}(p)\ (p-c_r)^{L_r} \tag{73}$$

where $\hat{Y}(p)$ is regular and nonvanishing at $p = c_r$, and L_r is a constant matrix related to the monodromy matrix by

$$M_r = e^{2i\pi L_r} \tag{74}$$

Now consider the function $(\partial Y/\partial p)\ Y^{-1}$. It is easy to see from (73) that this function has a simple pole at c_r with residue

$$A_r = \hat{Y}(c_r)\ L_r\ \hat{Y}^{-1}(c_r) \tag{75}$$

Combining this observation with the asymptotic behavior implied by (72), the function $(\partial Y/\partial p)\ Y^{-1}$ can be constructed explicitly, leading to the linear differential equation

$$\frac{\partial Y}{\partial p} = \left\{ \sum_{r=1}^{4} \frac{A_r}{p-c_r} + \begin{pmatrix} ix & 0 \\ 0 & 0 \end{pmatrix} \right\} Y \tag{76}$$

The fact that the monodromy matrices for Y are independent of x implies that Y also satisfies a differential equation in x. By noting that, for $p \approx c_r$,

$$\frac{\partial Y}{\partial x}\ Y^{-1} = \frac{\partial \hat{Y}}{\partial x}\ (p-c_r)^{L_r}\ (p-c_r)^{-L_r}\ \hat{Y}^{-1} = \frac{\partial \hat{Y}}{\partial x}\ \hat{Y}^{-1} \tag{77}$$

we see that $\partial Y/\partial x\ \ Y^{-1}$ is an entire function of p, namely a polynomial. Let us write the asymptotic expansion of $v_1(p)$ and

$v_3(p)$ as

$$v_i(p) \underset{p\to\infty}{\sim} \frac{1}{p}\, v_i^{(0)} + 0\left(\frac{1}{p^2}\right) \qquad i = 1,3 \tag{78}$$

where $v_i^{(0)}$ are functions of x only. Then the function $\partial Y/\partial x$ Y^{-1} may be explicitly constructed, giving

$$\frac{\partial Y}{\partial x} = \begin{pmatrix} ip & -iv_3^{(0)} \\ iv_1^{(0)} & 0 \end{pmatrix} Y \tag{79}$$

Note that this procedure may be used to construct a differential equation for Y with respect to any parameter which appears in Y but not in its monodromy matrices (e.g., for our case we could construct equations for $\partial Y/\partial c_r$). The totality of such equations may be written

$$dY = \Omega Y \tag{80}$$

where Ω is a differential form describing variations with respect to each parameter which leaves the monodromy invariant. Poincare's lemma, d^2(anything) = 0 (i.e., the equality of mixed second partial derivatives), leads to consistency conditions on Ω itself

$$d\Omega = \Omega \wedge \Omega \tag{81}$$

This is quite analogous to the "zero curvature" condition which arises as the consistency condition for a Lax pair in the inverse scattering method. In that case the coefficient matrices in the Lax pair depend on the local field variables, and the consistency condition becomes the nonlinear equation of motion. In the present case it turns out that the coefficient matrices in Ω can be related to the two-point correlation function. The consistency condition (81) reduces to an ordinary nonlinear (Painlevé) differential equation for the correlation function.

The full set of linear equations satisfied by Y is given by (80) with

$$\Omega = \sum_{r=1}^{4} A_r \, d\log(p-c_r) + \begin{pmatrix} ix & 0 \\ 0 & 0 \end{pmatrix} dp + \begin{pmatrix} ip & -iv_3{}^{(0)} \\ iv_1{}^{(0)} & 0 \end{pmatrix} dx \, . \tag{82}$$

For the purpose of determining the two-point function

$$G(x) = \langle \sigma(x)\sigma(0) \rangle, \tag{83}$$

we only need to consider variations with respect to p and x. Moreover, we can now finally let our mass gap parameter η go to zero, allowing the branch points to coalesce on the real axis. The monodromy matrices around γ_1 and γ_4 (c.f. Fig. 15) at $-k_F$ can be combined to give

$$M_- = M_4M_1 = \begin{pmatrix} 1 & -2\lambda \\ 2\lambda & 1-4\lambda^2 \end{pmatrix} \tag{84}$$

while the combined monodromy at $+k_F$ is

$$M_+ = M_2M_3 = \begin{pmatrix} 1-4\lambda^2 & 2\lambda \\ -2\lambda & 1 \end{pmatrix} \tag{85}$$

Note that both M_+ and M_- have eigenvalues κ_1 and κ_2 where

$$\kappa_1 = 1 - 2\lambda^2 + 2i\lambda\sqrt{1-\lambda^2} \tag{86a}$$

$$\kappa_2 = 1 - 2\lambda^2 - 2i\lambda\sqrt{1-\lambda^2} \tag{86b}$$

With these observations, we are led to the linear system

$$dY = \Omega Y \tag{87a}$$

with

$$\Omega = \left[\frac{A_+}{p-k_F} + \frac{A_-}{p+k_F} + \begin{pmatrix} ix & 0 \\ 0 & 0 \end{pmatrix}\right] dp + \begin{pmatrix} ip & -iv_3^{(0)} \\ iv_3^{(0)} & 0 \end{pmatrix} dx \tag{87b}$$

3.2 PAINLEVÉ EQUATION FOR THE TWO-POINT FUNCTION

The analysis of the linear system (87) which leads to the nonlinear equation for the two-point function has been discussed in detail by JMMS (Ref. 6, Section 6 and Appendix 4). I will just review the essential points and then state the result. The crucial observation is that the logarithmic derivative of the correlation function may be obtained from the large p limit of the wave function Y(p), specifically, from one of the two diagonal components v_0 or v_2. To see this, first imagine taking the derivative $\partial/\partial x$ of the series for log G(x) shown in Fig. 9. The graphs in Fig. 9 have a factor $e^{-iq_i x}$ for each phonon attached to the upper dotted line, so $\partial/\partial x$ simply introduces a factor $-i\sum_1^n q_i$ in the nth graph. Each q_i in this sum can be used to cancel the associated pole, represented graphically by contracting the phonon to a point. The n contracted graphs obtained are all equal by cyclic symmetry, so we obtain a factor n to cancel the 1/n in the series for log G(x). In this way, we obtain the series for $\partial/\partial x$ [log G(x)] shown in Fig. 16. Next we consider the large p behavior of the graphs for $v_0(p)$. The O(1/p) term is obtained by dropping the first graph (which is just unity) and contracting the last phonon on the left in all the rest. The two fermion lines which now terminate on the lower dotted line can be brought to the same point without affecting the value of the graph. Thus we again recover the graphical series shown in

Fig. 16. The result is that the correlation function is related to the large p behavior of the wave function by

$$v_0(p) \sim 1 + \frac{-i}{p}\frac{\partial}{\partial x} \log G(x) + 0\left(\frac{1}{p^2}\right) \tag{88}$$

Using this relation and the linear equation for $\partial Y/\partial p$ in (87), the logarithmic derivative of the correlation function can be expressed in terms of elements of the matrices A_+ and A_-. We also know that the matrices A_+ and A_- are related to the log of the monodromy matrix by a similarity transformation (c.f. Eq. (75)), and hence their eigenvalues are known from Eg. (86). For $\lambda = 1$, all four eigenvalues are equal to 1/2. With this information, the consistency conditions (81) can be reduced to a single nonlinear equation for

$$f(x) \equiv x\frac{\partial}{\partial x} \log G(x) \tag{89}$$

The equation,

$$(xf'')^2 = -4(xf^1-1-f)(xf^1+(f^1)^2-f) \tag{90}$$

is equivalent to a Painlevê equation of the fifth kind.

3.3 CONCLUSION

The results of Jimbo et al constitute a complete theory of correlation functions for the impenetrable ($c=\infty$) delta-function gas. At the present time, the theory for the general (finite c) Bethe's ansatz model is not as well developed. In these lectures I have tried to described the $c=\infty$ results in the context of the more general quantum inverse method and Gel'fand-Levitan transform, in the hope that the latter will eventually lead to a more complete theory. In order to clarify the nature of the problem for finite c, let me recapitulate the essential elements of the $c=\infty$ treatment with a view toward finite c. The JMMS treatment is based on the properties of free fermion operators and Jordan-Wigner transformations. For finite c these both have a natural generalization in the R operators and the Gel'fand-Levitan transform respectively. By using the Gel'fand-Levitan expressions for the nonlinear Schrödinger fields $\phi^*(x)$ and $\phi(y)$, one can generate series expansions for the finite c correlation functions. The problem of cutting off the infrared singularities represented by vanishing momentum denominators is more subtle than it is at $c=\infty$, but this problem has been solved, and it can be shown that the terms in the series expansion are well-defined to all orders. The Gel'fand-Levitan series for the two-point function reproduces the results of JMMS at $c=\infty$ and has also been used to study higher order terms in the $1/c$ expansion.[7,23] Although the

$O(1/c)$ term can be written in closed form in terms of Painlevé V functions, the analysis is rather cumbersome and the possibility of calculating all terms in the large c expansion and summing them up appears remote without some further insight into the problem. The essential complication which makes the finite c problem more difficult is that the R operators are not fermion operators, but are rather "Zamolodchikov operators", i.e., they commute by a two-body S-matrix. Thus, when we compute matrix elements of expressions involving R operators, the various Wick contractions contain different products of S-matrices. So far this complication has prevented an analysis of the finite c correlations along the lines of the monodromy arguments of Jimbo et al for the $c=\infty$ case. On the other hand, the similarity between the $c=\infty$ monodromy arguments and the quantum inverse method itself is quite suggestive. The series expansions for the wave functions $v_i(p)$, Eqs. (64) have the same form as the Gel'fand-Levitan series for the Jost solution of the quantum inverse problem, with the functions $\omega(p)$ and $\omega^{-1}(p)$ in the wave functions being replaced by the operators $R(p)$ and $R^+(p)$ in the Jost solutions. In fact, the linear equation (79) for $Y(p)$ is precisely analogous to the Zakharov-Shabat eigenvalue problem. Further pursuit of these intriguing connections may eventually lead to a general theory of correlation functions for integrable systems.

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FIGURE CAPTIONS

1. Two-body wave function.
2. Three-body wave function.
3. Graphical expansion of the Jordan-Wigner transformation.
4. Analytic structure of Eq. (48) in the complex p plane.
5. (a) A scattering vertex in the finite density ground state, Eq. (55).

   (b) A pair creation vertex, Eq. (56).
6. (a) Graphs for Eq. (57a).

   (b) Graphs for Eq. (57b).
7. A typical graph for the four-point function.
8. (a) A disconnected graph for the two-point function.

   (b) A connected graph for the two-point function.
9. Series expansion for log G(x).
10. Series expansions for $V_i(p,p')$, i = 0, 1, 2, and 3.
11. Reduction of a graph for $V_i(p,p')$ to a graph for $v_i(p)$ in the limit $p'\to\infty$.
12. Graphical result of changing the signs of the $i\varepsilon$'s in the upper phonon propagators.
13. A typical graph for $v_2(p)$.
14. Series expansions for $v_i(p)$, i = 0, 1, 2, and 3.
15. Contours for defining the monodromy matrices M_i, i = 1, 2, 3, 4.
16. Graphical series for $(\partial/\partial x)$ log G(x).

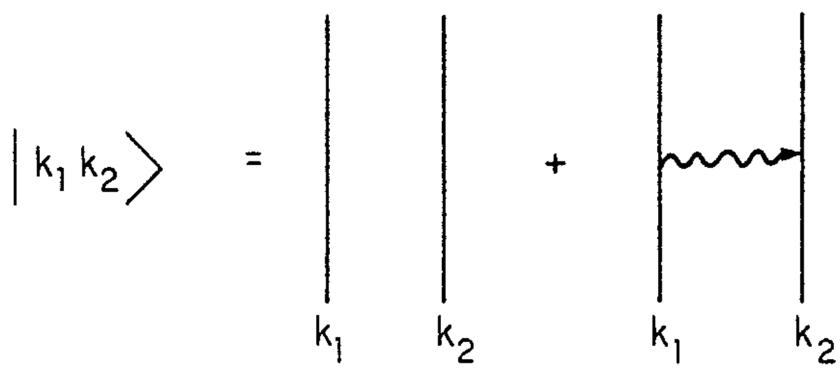


Fig. 1

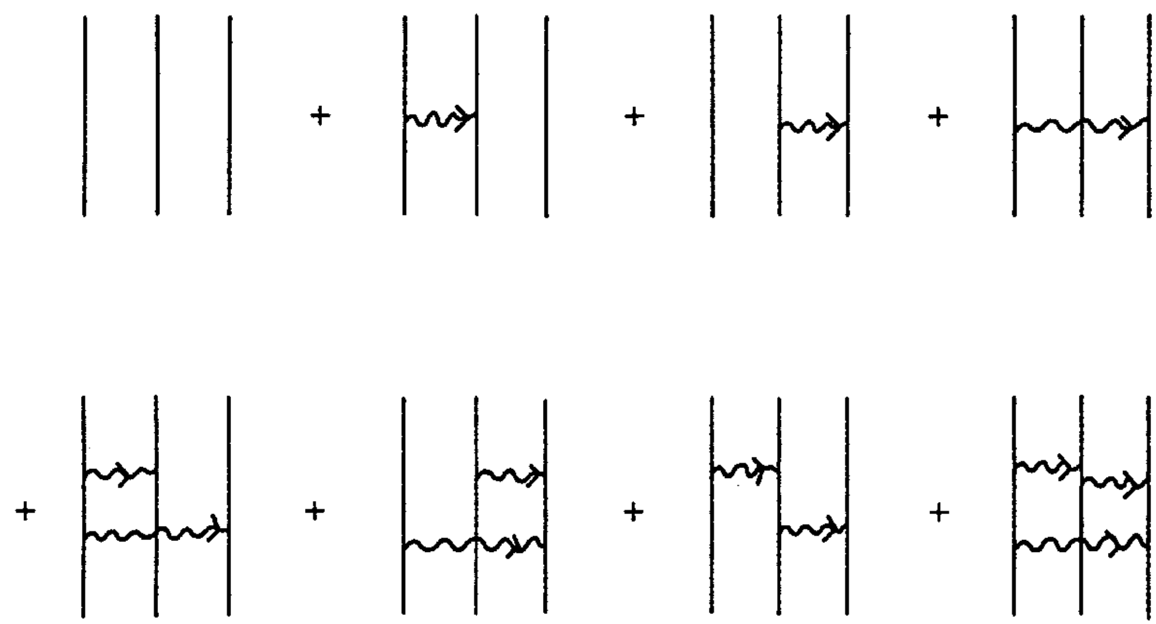
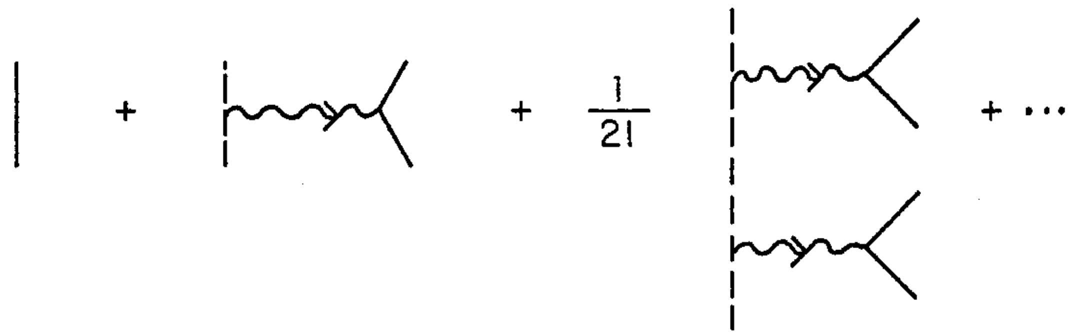


Fig. 2



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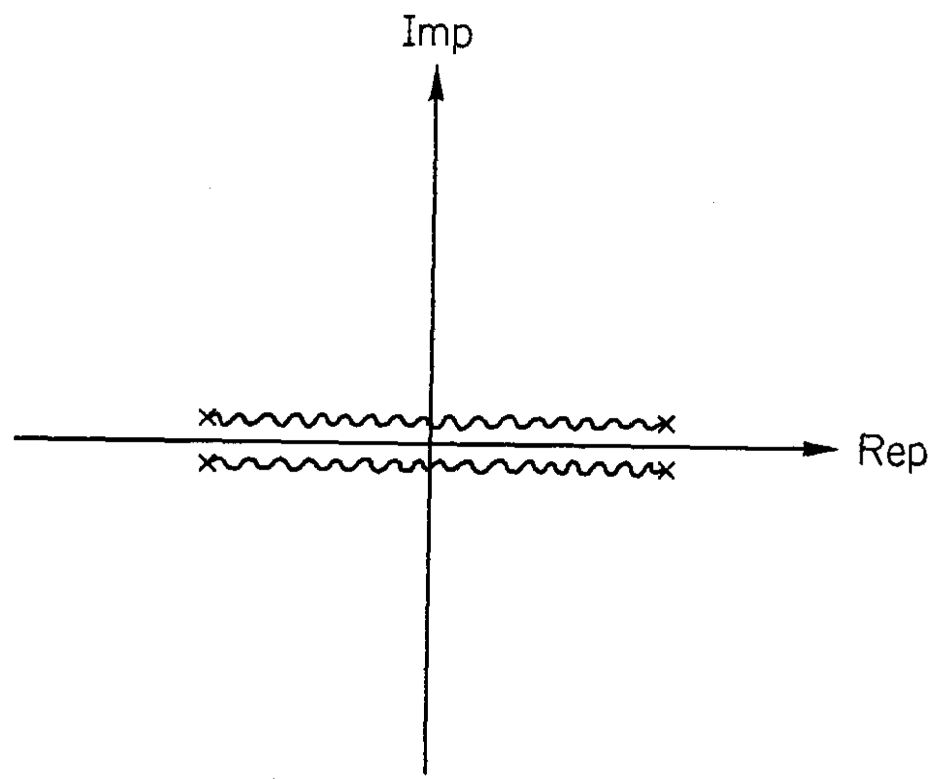
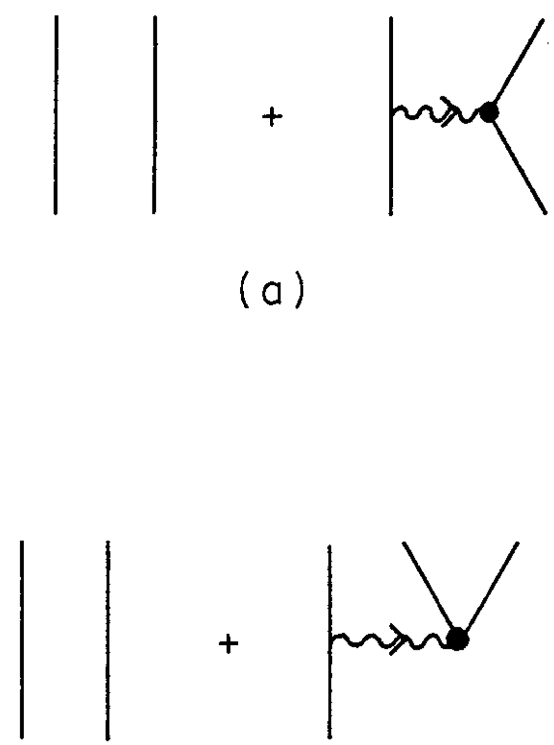


Fig. 4

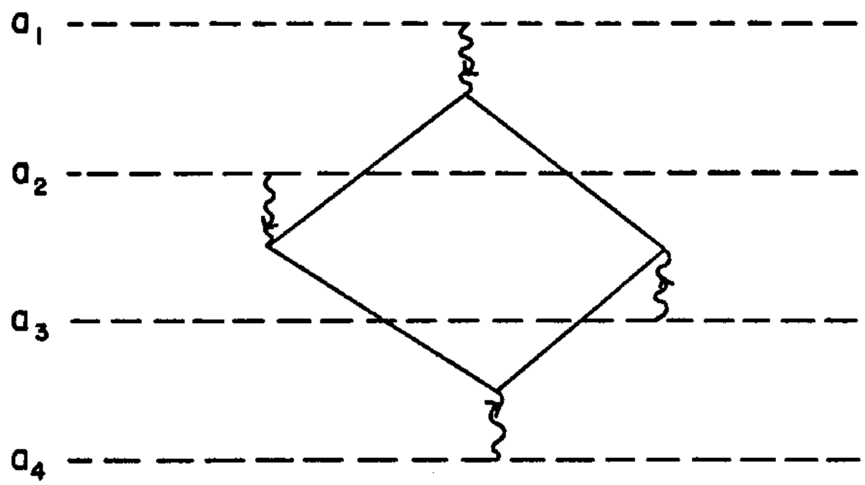
Fig. 5



(a)

(b)

Fig. 6



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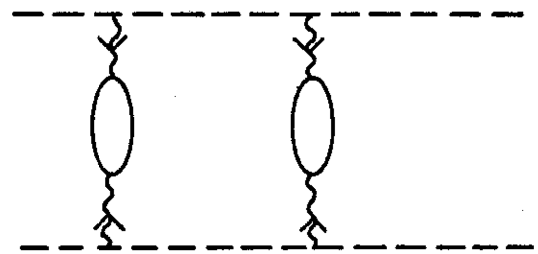

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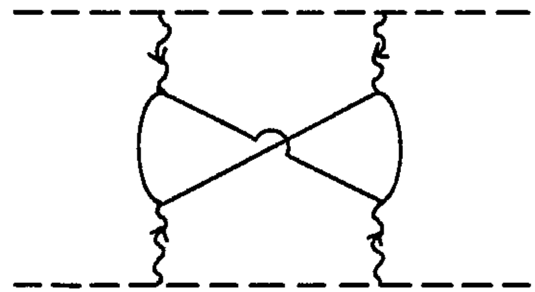


(b)

Fig. 8

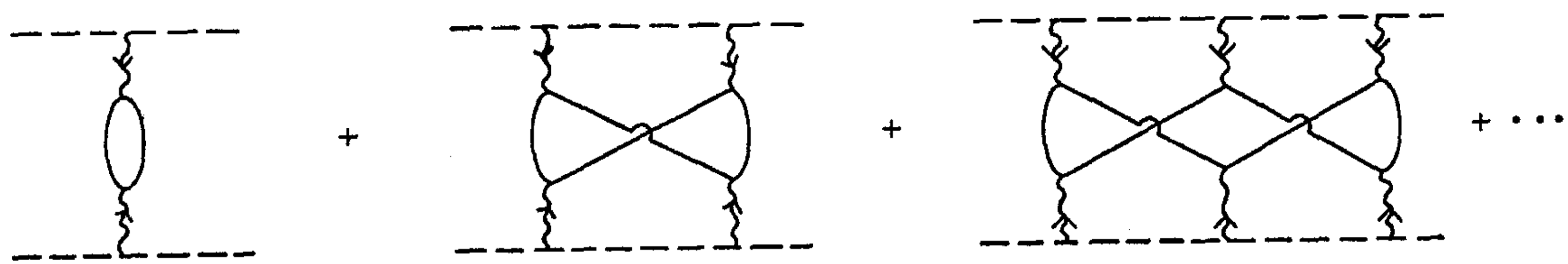


Fig. 9

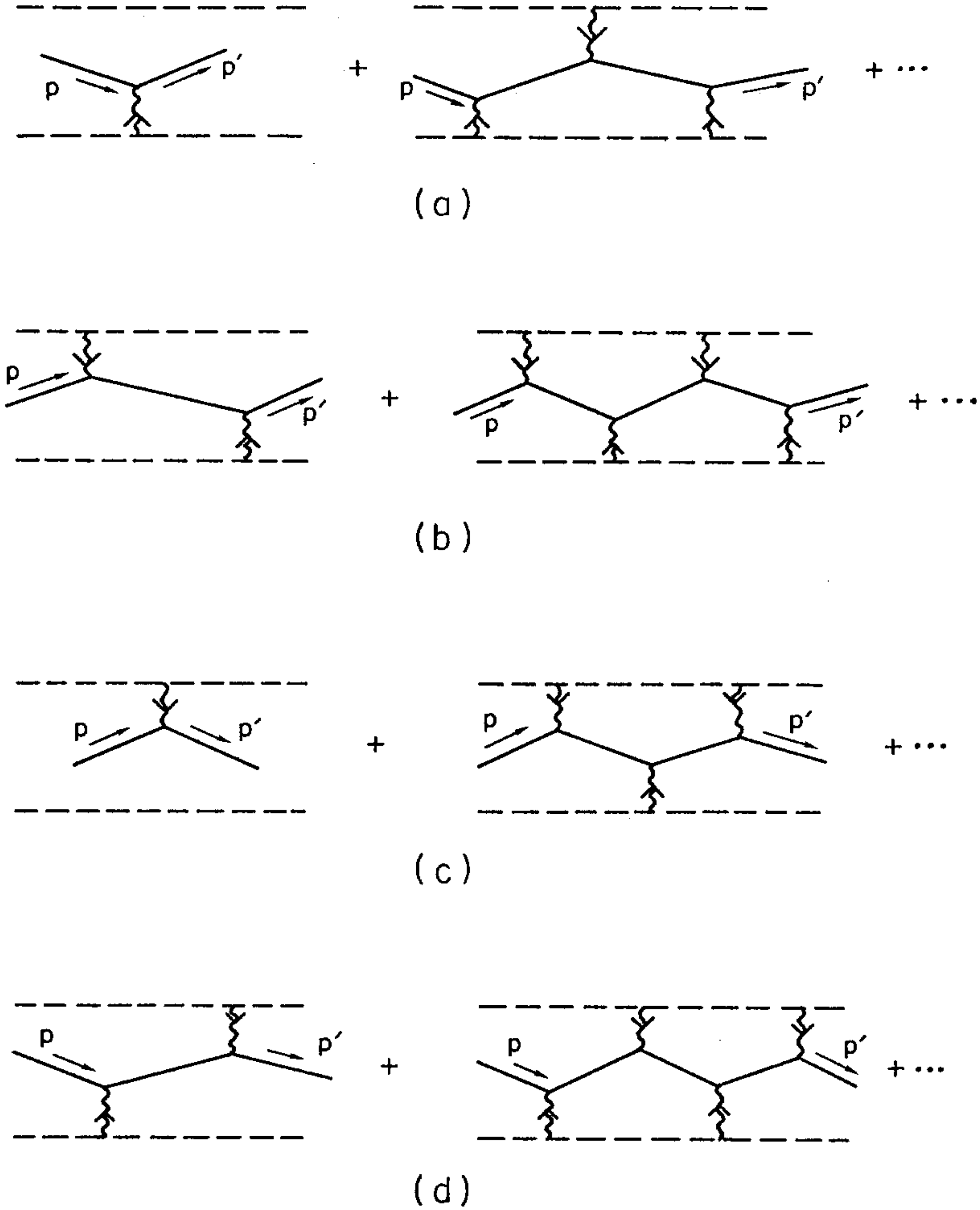
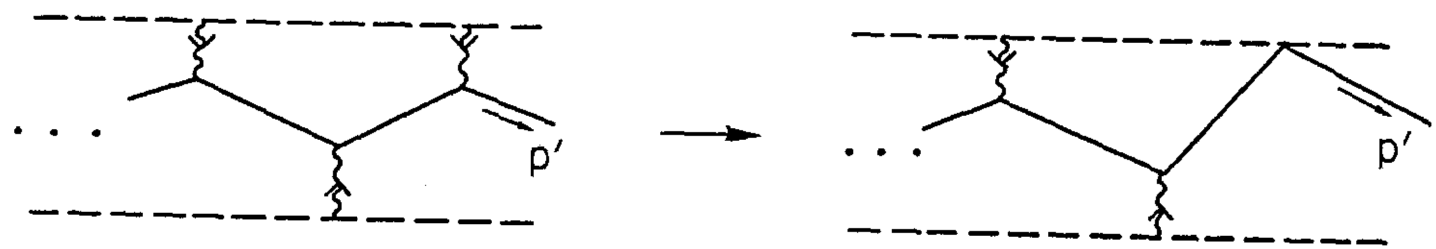


Fig. 10



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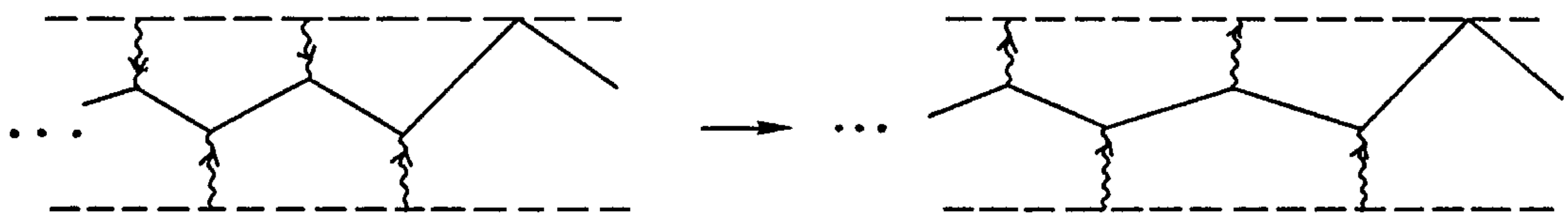

Fig. 12

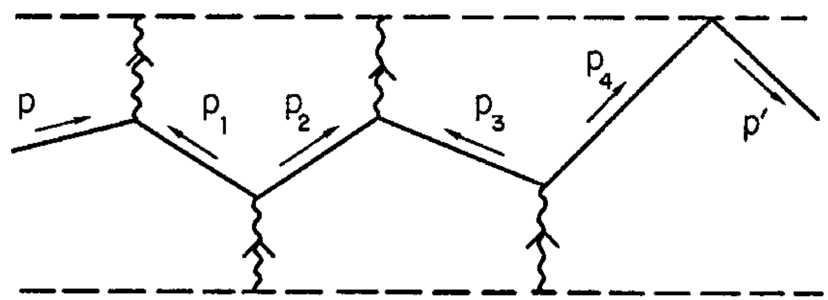

Fig. 13

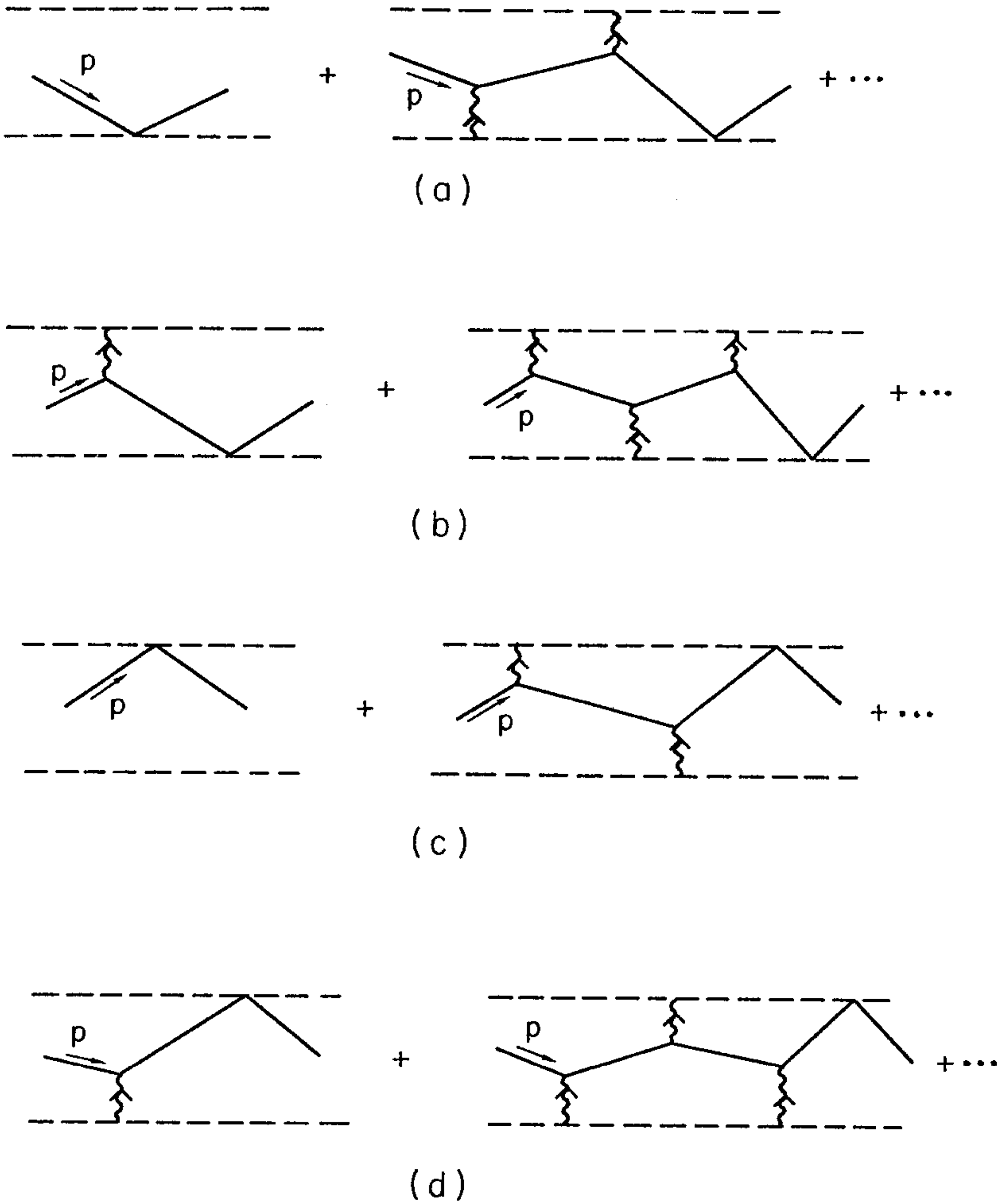

Fig. 14

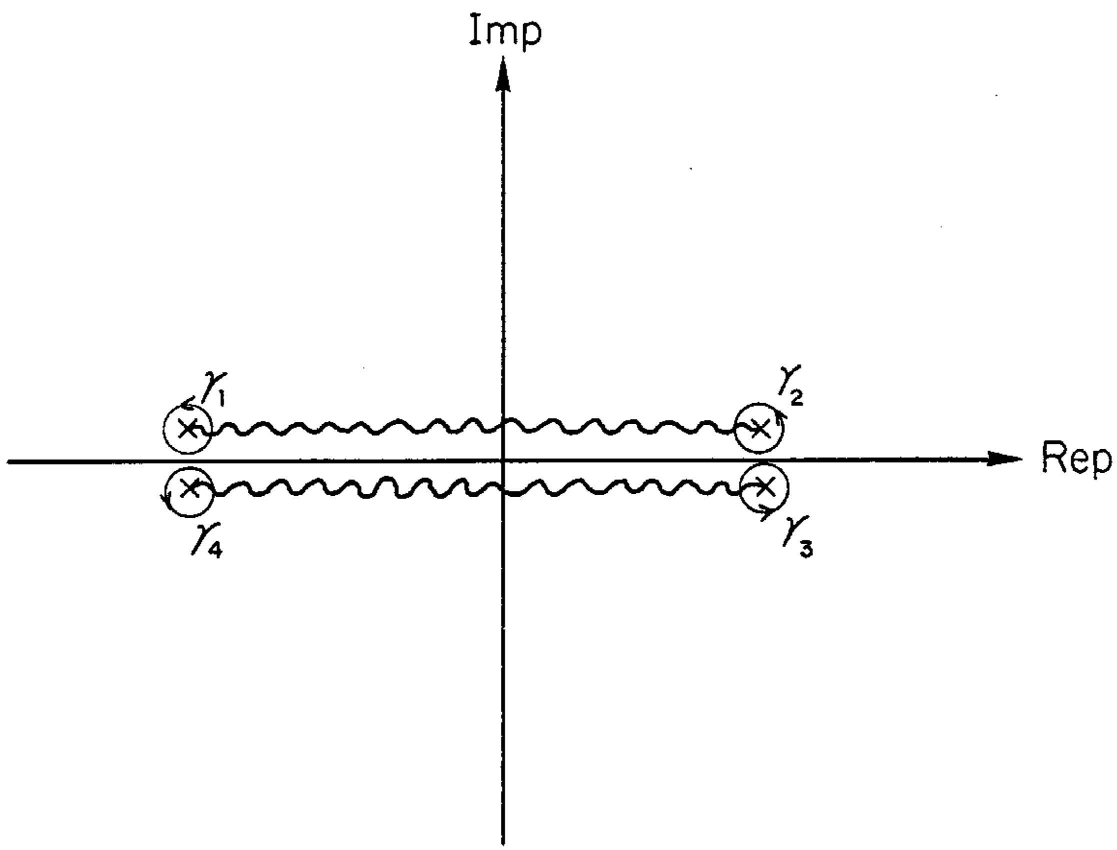

Fig. 15

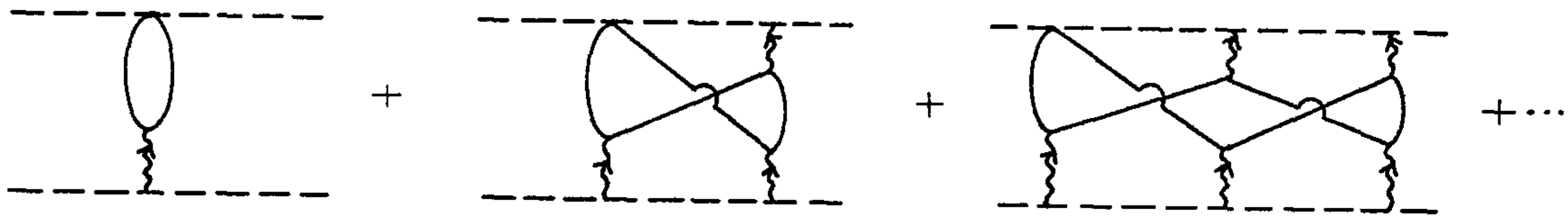

Fig. 16