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THE MICROSCOPIC STRUCTURE AND GROUP THEORY OF THE INTERACTING BOSON MODEL

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Abstract: The chains of groups used in classifying states of the IBM are compared with the chains used in a composite model with $j = 3/2$ fermion pairs. Many similarities are found, along with differences due to Pauli principle effects in continuum fermion pairs. The classifications are shown to be characterized by several different seniority numbers, which are physically similar but formally different in the two cases because fermion pair and boson pair states used to defined seniority in each model correspond to single bosons and four-fermion clusters respectively in the other model. The SO(6) and SO(5) groups which define boson pair seniorities in the boson sextet model are isomorphic respectively, to SU(4) and Sp(4) which have simple physical interpretations in fermion quartet models.

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1. Introduction. The basic building blocks of the IBM

The Interacting Boson Model (IBM) of Arima and Iachello[1] has had remarkable success in describing many properties of nuclei using states of a basic sextet of bosons having angular momenta L = 0 and L = 2. However those bosons are clearly not fundamental constituents of the nucleus. This recalls the situation in particle physics when octets of mesons and baryons seemed to belong in octet representations of some unknown symmetry group. Attempts to place these octets in the fundamental representation of SU(8) failed, because these hadron octets were not fundamental objects. They are now considered to be composite systems of basic building blocks called quarks, classified in the fundamental representation of an SU(3) group. The meson and baryon octets are classified in the octet representation of this SU(3) group.

This analogy suggests constructing the IBM sextet from more basic building blocks. The SU(6) group which classifies the bosons in the fundamental sextet representation is discarded in the same way as SU(8) in hadron physics. A possible underlying "quark" structure is investigated by examining smaller groups with six dimensional representations. The way such six dimensional representations are constructed from the fundamental representations indicates possible composite structures for the boson sextet. Two candidates, SU(3) and SU(4), have sextet representations suitable for classifying a sextet of bosons having L = 0 and L = 2. Both have interesting physics but the two are incompatible and represent two completely different ways of constructing six boson states with J = 0 and J = 2; either from a pair of fundamental bosons with spin 1 or from a pair of fundamental fermions with spin 3/2. A six dimensional representation arises both for SU(3) in the symmetric combination of two fundamental triplets and in SU(4) in the antisymmetric combination of two fundamental quartets. The SU(3) is the familiar Elliot model[2] whose basic building blocks

or "Elliot quarks" are the harmonic oscillator quanta of the nuclear harmonic oscillator shell model. The SU(4) corresponds to a model recently proposed by Ginocchio[3] whose basic building blocks or "Ginocchio quarks" are fermions of spin 3/2.

Although it seems inconsistent to consider the boson of the IBM as being simultaneously constructed from basic bosons of spin 1 and basic fermions of spin 3/2, both models seem to reproduce certain desirable features of the IBM. The Elliot model uses the L-S coupling scheme while the Ginocchio model uses the jj-coupling scheme. There may be completely different physics in the two domains where LS and jj couplings are good approximations, with both having an independent boson description. The boson model would then provide a smooth interpolation through the intermediate coupling region where neither the LS or jj coupling models are adequate.

In both the SU(3) and SU(4) models no simple SU(6) group occurs just as no SU(8) group is related in any simple way to the SU(3) quark model for hadrons. However, SU(6) plays a rather trivial role in the IBM. The complete set of states of the n boson configuration are classified in a single representation of SU(6); namely, the one denoted by (n,0,0,0,0) and constructed from the totally symmetric combination of n fundamental sextets. In either the SU(3) or SU(4) models, the states constructed by a totally symmetric combination of n sextets are exactly the same and with exactly the same quantum numbers as the corresponding SU(6) representation. However they are reducible representations of SU(3) or SU(4) respectively. The algebraic properties of this reduction are discussed below.

The essential features of the SU(3) model have been well known for a long time since they are essentially the same as the original version of Elliot. However the algebraic structure of the SU(4) model is not generally known and Ginocchio's original presentation refers to this group as SO(6) rather than SU(4). We therefore discuss this SU(4) model in more detail.

2. The SU(4) - SO(8) Model

Let $a^{\dagger}_{mk}$ represent the creation operator for a fermion in a state of angular momentum $j = 3/2$ and projection m on the z axis and with an additional quantum number k labeling another degree of freedom with n_k states. In the Ginocchio model, k is an additional angular momentum variable which is coupled with $j = 3/2$ to give the total angular momentum of the fermion. However for our purposes, the exact physical meaning of the quantum number k is irrelevant as we deal exclusively with sums over k products of operators having the form

$$a^{\dagger}_{m}a^{\dagger}_{m'} \equiv \sum_{k} a^{\dagger}_{mk}a^{\dagger}_{m'\bar{k}} \tag{1a}$$

$$a_{m}a_{m'} \equiv \sum_{k} a_{mk}a_{m'\bar{k}} \tag{1b}$$

and

$$a^{\dagger}_{m}a_{m'} \equiv \sum_{k} a^{\dagger}_{mk}a_{m'k} \tag{1c}$$

where $\bar{k}$ represents the state defined in the particular model to be conjugate to the state k. In the Ginocchio model where k is the projection of an angular momentum, $\bar{k} = -k$. We use the compact notation which drops the index k to define sums over k of bilinear products of these operators.

The set of all operators of the type (1) with all values of m and m' define an SO(8) algebra[4]. The subgroup of the SO(8) algebra generated by all operators of the form (1c) which conserve the number of particles is SU(4), which is isomorphic to SO(6). However in the context of this model based on fermion creation operators in states labeled by an index m which takes on four values, the SU(4) description seems more appropriate.

The operators (1a) create a fermion pair in one of six possible states having exactly the angular momentum quantum numbers of the six bosons in the IBM. However these are not bosons but fermion pairs having these quantum

numbers. States corresponding to those of n bosons in the IBM can be constructed by operating n times on the vacuum with the operators (1a). However these correspond to boson states only when there is no appreciable overlap between fermion pair wave functions, so that the effects of the Pauli principle can be neglected. This requires that the number of pairs be small compared with the number of available fermion states specified by the number of allowed values for the quantum number k. When this condition is met, the states obtained by operating n times with the operators (1a) are in a one to one correspondence with the states of the interacting boson model and can also be classified in representations of SO(8) and the subgroup SU(4). The model also contains fermion states with no analog in the boson model, constructed by operating with single creation operators or pair operations not of the form (1). Such states have recently been investigated in the description of odd-A nuclei[5]. However we do not consider them further here.

3. The correspondence between boson and fermion models

When the number of pairs is not small compared with the number of available fermion states, the correspondence between the fermion pair model and the IBM must break down. States in the IBM which have more bosons than the total number of fermion pair states available in the fermion model, $2n_k$, clearly have no counterpart in the fermion model, as the analogous states created by the product of more than $2n_k$ operators of the form (1a) must vanish. A state created by raising a single operator of the form (1a) with fixed values of m and m' to a power greater than n_k also must vanish, since this operator populates only half of the levels available, those with eigenvalues m and m'. Thus there are states in the IBM which can be placed in one-to-one correspondence with states in the fermion model, like the basic sextet itself, while there are other states in the IBM whose counterparts in the fermion model vanish identically. Since both types of states in the IBM are in the same Hilbert space, linear combinations of these states are good IBM states whose counterparts in the fermion model

constructed by the use of the operators (1a) can be very peculiar. Two orthogonal linear combinations of IBM states one with a counterpart in the fermion model and one whose counterpart vanishes give the same state when created by the same linear combinations of the operators (1a). Thus the analogs in the fermion model of an orthonormal set of boson states may not be an orthonormal set.

The problem of exhibiting the nature of the correspondence between the two models thus becomes one of choosing a particular basis in the IBM which has only two kinds of states, those whose counterparts in the fermion model vanish identically, and an orthogonal set of states whose counterparts in the fermion model are likewise a set of orthogonal states. There should be no pairs of states in the IBM in this basis which are orthogonal and have nonvanishing analogs in the fermion model which are not orthogonal. We shall see below how such bases can be constructed.

The key to the construction of this basis is the group structure. Two states classified in irreducible representations of some group but with different quantum numbers must be orthogonal. If the basic sextet is classified according to a given group in both models, all the representations of this group which appear in the symmetric product of n sextets are uniquely defined by the algebra, independent of the particular model. In the IBM all the representations exist, since they are constructed explicitly by combining the products of n boson creation operators and using the Clebsch-Gordan coefficients of the group. In the fermion model the same procedure leads to the same representations, except that some states may vanish identically. However, if any state vanishes, all of the states in the same irreducible representation must vanish, because they can be transformed into one another by the group. Thus any complete specification of states of the IBM labeled by groups with counterparts in the fermion model defines a basis whose counterparts in the fermion model must be an orthogonal set of states. The normalization problem is complicated and not considered as it is not essential to the discussion. Any orthogonal basis can be normalized. We therefore examine the subgroups of SO(8) and compare them with the chains of subgroups commonly used in the classification of the IBM.

4. The subgroups of SO(8)

A subgroup Sp(4) of SU(4) can be defined which gives the usual seniority for a shell with $j = 3/2$. The six states of two fermions created by the operation of the operators (1a) on the vacuum are split by Sp(4) into two irreducible representations, one of seniority zero having angular momentum $J = 0$ and one of seniority 2 having $J = 2$. These are analogous to the s and d bosons of the IBM.

Another subgroup of SO(8) is the conventional quasi-spin group whose generators create and annihilate pairs of particles coupled to angular momentum zero.

$$S_+ = \sum_{m>0} (-1)^{(3/2)-m} a_m^\dagger a_{-m}^\dagger \tag{2a}$$

$$S_- = \sum_{m>0} (-1)^{(3/2)-m} a_{-m} a_m \tag{2b}$$

$$S_0 = \frac{1}{4} \sum_m [a_m^\dagger a_m - a_m a_m^\dagger] \tag{2c}$$

This is really an Sp(2) group which happens to be isomorphic to ordinary spin. These generators (2) do not commute with all the generators of SU(4) but do commute with the Sp(4) subgroup of SU(4) mentioned above, and the direct product of these groups Sp(2) x Sp(4) is a subgroup of SO(8). We can thus define two chains of subgroups

$$SO(8) \supset SU(4) \supset Sp(4) \tag{3a}$$

and

$$SO(8) \supset Sp(2) \times Sp(4) \supset Sp(4) \ . \tag{3b}$$

In a single j shell the irreducible representations of Sp(4) are characterized by two quantum numbers, the number of particles and the seniority number. The two chains (3) pick out these two quantum numbers in both possible orders. The subgroup SU(4) in Eq. (3a) picks out sets of states which all have the same number of particles but include different seniorities, while the subgroup Sp(2) x Sp(4) in Eq. (3b) picks out sets of states all having the same seniority but different numbers of particles.

In the single j shell, the seniority number can be defined either from the Casimir operator of Sp(4) or from the Casimir operator of quasispin and the two definitions are equivalent. In the generalized Ginocchio model this equivalence no longer holds. Seniority numbers can be defined using either the Sp(4) group or the quasispin group but the two are no longer the same. This is most easily seen by examining the properties of the operator

$$A^{\dagger} = \sum_{m,m'} a^{\dagger}_{m} a^{\dagger}_{m'} a^{\dagger}_{-m'} a^{\dagger}_{-m} \tag{4}$$

This operator is clearly a scalar in SU(4) as it is the scalar product of the SU(4) sextet of operators (1a) with itself. For the single j shell this operator is trivial since it creates particles in all the states in the shell and has a nonvanishing matrix element only between the empty shell and the completely filled shell. However, once the additional degree of freedom defined by the quantum number k is included the operator (4) becomes nontrivial and adds a four particle cluster which is an SU(4) singlet to any state. Since the operator is a scalar in SU(4), it is also a scalar in the subgroup Sp(4) and therefore does not change the quantum numbers defined by these two groups. However the operator (4) does not commute with the quasispin operators (2) and is not a quasispin scalar. It is also not a simple function of the quasispin generators. Thus the operator (4) can change quasispin and therefore the seniority number defined by quasispin whereas it cannot change the seniority

number defined by the group Sp(4). On the other hand, the quasispin operator (2a) which adds a pair of particles to the system is a quasispin generator and a scalar under Sp(4) but not under SU(4). Thus the quasispin operator (2a) adds two particles to a system without changing the seniority as defined either by the quasispin or by Sp(4) but it can change the SU(4) quantum numbers.

We now note that it is possible to add four particles to a given state in two different ways without changing the seniority number defined by Sp(4), namely by operating either with the operator (4) or by operating twice with the quasispin raising operator (2a). The first procedure preserves SU(4) but changes quasispin whereas the second preserves quasispin and changes SU(4). We are thus led to the conclusion that the same irreducible representations of Sp(4) must appear more than once in a given representation of SO(8) when there are more than four particles and that the two chains of subgroups (3) choose two different bases to distinguish between these equivalent representations of Sp(4). The physical meaning of these two different chains becomes evident below when we discuss the correspondence with the boson model.

5. Three kinds of seniority

The concept of seniority is the key to the understanding of the physical basis of the chains of subgroups (3) and the corresponding chains in the IBM. Consider a state of n bosons or fermion pairs. A seniority classification for such states characterized by a seniority number v picks out states in which the n particles are divided into two groups. There are v active or "senior" particles and n-v "spectator" particles usually grouped in pairs coupled to zero angular momentum. In the IBM there are three completely different ways of defining a seniority classification and a seniority number.

1. <u>The number of d bosons</u>. States of n bosons can be defined which have a definite number n_d of d-bosons and $(n-n_d)$ s-bosons. The number n_d can be considered a kind of seniority in the fermion model where the s-boson is really a fermion pair coupled to angular momentum zero.

2. Boson-pair seniority. The operator (4) and its analog in the IBM can be considered as an operator which creates a pair of fermion pairs or bosons in a state of J=0. A seniority number v_B can be defined as the number of particles in a state of n bosons or fermion pairs containing $n-v_B$ boson pairs in this special state (4) and v_B unpaired bosons. Note that the pair state with J=0 is defined to be completely symmetric in all of the six states of the sextet. It is therefore a linear combination of a state of two s-bosons and a state of two d-bosons coupled to L=0. Such a state is not an eigenstate of the number of s-bosons nor of the number of d-bosons. Thus the two kinds of seniority defined by the quantum numbers n_d and v_B do not commute and define different bases for the states.

3. d-boson pair seniority. A seniority number v_d can be defined as the number of d-bosons which are not in pairs coupled to L=0. The remaining $n-v_d$ bosons are then either s-bosons or d-boson pairs coupled to L=0. Note that the d boson seniority number v_d is compatible with either the number of d-bosons n_d or the boson pair seniority v_B defined above although n_d and v_B are mutually incompatible.

These three seniority numbers form the basis for the chains of subgroups (3) and the corresponding chains in the IBM. In these chains there are three groups whose irreducible representations are characterized by the three seniority quantum numbers n_d, v_B and v_d. We first show this in the IBM and then return to the chains (3) and see the correspondence and the analogs of these three seniorities.

6. Seniorities in the IBM

The IBM begins with the SU(6) group whose totally symmetric representations include all possible states of n bosons. For our purposes it is convenient to begin with U(6) which includes the total number operator. This is trivial in the IBM but is of interest in the fermion models where one considers quasi-spin operators (2) which change the number of particles. We now write the two chains of subgroups of U(6) commonly used in the IBM and note the similarities and differences between them and the chains for the fermion model (3).

$$\begin{array}{ccccccc} U(6) & \supset & U(5)\times U(1) & \supset & SO(5)\times U(1)\times U(1) & \equiv & Sp(4)\times U(1)\times U(1) \\ n & & n_d \quad n & & v_d \quad n_d \quad n & & \end{array} \tag{5a}$$

and

$$\begin{array}{ccccccc} U(6) & \supset & SO(6)\times U(1) & \equiv & SU(4)\times U(1) & \supset & SO(5)\times U(1) \equiv Sp(4)\times U(1) \\ n & & v_B \quad n & & v_B \quad n & & v_B \quad v_d \quad n \end{array} \tag{5b}$$

These chains have a direct interpretation in terms of the three seniorities defined above. The separation of the boson sextet into five d bosons and one s boson appears as the decomposition of the U(6) group which acts on the whole sextet into a U(5) which acts only on d bosons and a U(1) which acts only on s bosons. The U(5) classification in the chain (5a) thus defines the seniority number n_d. Seniority for boson pairs is defined by orthogonal groups. For the case of n boson states classified according to SU(n), the subgroup SO(n) defines a seniority classification. Thus the SO(6) group in the chain (5b) defines the boson seniority number v_B, and the SO(5) group in both chains (5) define the d-boson seniority v_d. The existence of two chains characterized by the appearance of the incompatible subgroups U(5) and SO(6) of U(6) reflects the incompatibility of the two kinds of seniority defined by n_d and v_B. The appearance of the SO(5) subgroup in both chains

reflects the compatibility of the seniority defined by v_d with either n_d or v_B. The way in which each group picks a seniority number is indicated by labeling each group with the relevant quantum numbers in Eq. (5).

7. The correspondence between the two sets of chains

We now examine the correspondence between the chains (3) and (5). The groups SU(6) and SU(5) in the boson chain (5a) have no counterparts in the Ginocchio model and the groups SO(8) and Sp(2) in the Ginocchio chains (3) have no counterpart in the boson model. This lack of correspondence is due to the fundamental difference between bosons and fermions. The generators of SU(6) and of SU(5) are products of a boson creation operator and a boson annihilation operator. The analogs of these operators in the Ginocchio model are products of fermion pair creation operators (1a) and annihilation operators (1b). These operators can be defined with a normalization so that their action on the basic sextet of two fermion states is exactly the same as the operation of the SU(6) and SU(5) generators on the boson sextet in the IBM. However these fermion operators do not form a closed set under commutation and do not define a Lie algebra. This reflects the basic physical difference between the two models.

There is a one to one correspondence between the six states of the IBM and the two fermion states obtained by operating on a vacuum with the operators (1a). However the correspondence is not so clear between two boson states and four fermion states obtained by operating on the vacuum with two operators of the form (1a). All the four fermions in this four-fermion state are equivalent and there is no unique prescription for separating them into two pairs which could be considered as bosons. For example, the product of two operators of the form (1a) creating fermions in states of J=2 might be considered as a state of two d-bosons with no s-bosons present. But the quasi-spin lowering operator (2b) which annihilates s-bosons would not necessarily

annihilate this state because there is a piece of the wave function in which a fermion from one of the d-bosons and a fermion from the other is coupled to J=0 and behaves like an s-boson.

Similarly the difference between bosons and fermions is responsible for the absence in the IBM of the SO(8) group and the quasi-spin Sp(2) group of the Ginocchio model. This is most clearly seen by noting that raising the pair creation operators (1a) and (2a) to a power greater than $2n_k$ must give zero as they would create more fermions than the number of states available. In the boson case no such limit exists and operators creating s or d bosons can be raised to an arbitrary power. The analogs in the boson case of the quasi-spin raising and lowering operators (2a) and (2b) are the s boson creation and annihilation operators. These generate the noncompact Euclidean group in one dimension and have infinite dimensional representations.

8. Boson pair seniority in the two models

In view of the difference between the bosons and fermions it is at first sight rather surprising that there is a clear one-to-one correspondence between the SO(6) group arising in the boson chain of subgroups (5b) and the SU(4) group of the fermion chain (3a). The reason for this correspondence can be seen by examining the operator (4) in the boson and fermion cases. We have seen that this operator is a scalar in SU(4). Thus the SU(4) group can be characterized as generating linear transformations on the sextet representation in a manner which leaves this particular scalar product of two sextets (4) invariant. As a result of the isomorphism between SU(4) and SO(6) these can also be considered to be a group of linear transformations in a six dimensional real vector space.

The SO(6) group in the boson model defines the v_B classification. It generates linear transformations on the basic sextet which leave invariant a particular scalar product of two creation operator sextets defined by the boson analog of Eq. (4). This operator creates the special L=0 state which is the "spectator" state in the v_B classification. Note that there are two states of a boson pair with angular momentum zero, one with two s-bosons and one with two d-bosons. The particular state defined by the SO(6) algebra is constructed symmetrically from all six boson states and is therefore a linear combination of the s boson and d boson states which is 1/6 s boson and 5/6 d boson. The seniority number v_B defined by the SO(6) group thus counts not only bosons having no L=0 pairs but also L=0 boson pairs in the state orthogonal to this particular state.

We have thus defined two Lie algebras, a boson algebra acting on the basic boson sextet and a fermion algebra whose action on the fermion pair sextet creation operators (1a) is exactly the same as the action of the boson algebra on the boson sextet. Thus given any state in the IBM constructed from products of boson creation operators combined in a way which transforms under SO(6) like an irreducible representation of the algebra, we can construct an analogous state in the fermion model by replacing each boson creation operator by the corresponding fermion pair creation operator. This fermion state may vanish as it may contain products of identical fermion creation operators. However if it does not vanish then its transformation properties under the group SU(4) are completely determined by the transformation properties of the sextet of pair creation operators (1a). Fermion states created in this way which do not vanish then have exactly the same classification under SU(4) as the corresponding boson states under SO(6).

Each nonvanishing representation of SU(4) defines a set of orthonormal states exactly analogous to the states of the corresponding

representations in the boson model. If any fermion state constructed this way vanishes then the entire SU(4) representation in which this state is classified must also vanish. We thus see the following general features of the correspondence between the generalized Ginocchio model and the IBM. When the SO(6) classification is used in the IBM a finite set of irreducible representations of SO(6) which appear in the IBM are in one-to-one correspondence with states in the generalized Ginocchio model which have the identical classification under SU(4). The remaining SO(6) representations in the IBM have no analog in the fermion model because they involve fermion states excluded by the Pauli principle. The only open problem for specific cases is to determine which of the SO(6) representations appearing in the IBM have counterparts in the fermion model and which do not. This is easily seen by examining the representations of these groups arising in both models.

The totally symmetric states of n bosons are classified in the single SU(6) representation denoted by (n,0,0,0,0). When this is reduced according to the chain (5b) it contains all the representations of SO(6) = SU(4) denoted by $(0,v_B,0)$, where $v_B = n-2t$ and $t = 0,1...[n/2]$. These are states containing both s and d bosons classified according to the boson-pair seniority v_B defined above by the boson analog of the operator (4).

In the fermion pair model the relevant representations of SO(8) are those denoted by (0,0,0,m), where $m = n_k$ is the number of pairs in a half-filled shell. This representation of SO(8) contains the SU(4)×U(1) representations denoted by $(0,v_B,0)(n)$ where $v_B = 0,1...m$, $n = v_B+2t$, and $t = 0,1...m-v_B$. The quantum number v_B defined in this way is directly analogous to the boson pair seniority v_B in the IBM. States with a given value of v_B are seen to occur for values of n symmetrically located about the middle of the shell and varying in steps of 2 from v_B to $2m-v_B$. For a fixed value of $n \leq m$ the representations of SU(4) which appear are just those denoted by $(0,v_B,0)$ where $v_B = n-2t$ and

$t = 0,1...[n/2]$ as in the boson case. Thus all the states in the IBM have non-vanishing counterparts in the fermion pair model for $n \leq n_k$. For $n > n_k$, the fermion states are determined by particle-hole symmetry and the SO(6) representations in the IBM having $v_B > 2n_k - n$ do not occur in the fermion model.

9. The correspondence between states in the two models

We now examine the correspondence between specific states arising in the Ginocchio model and in the IBM. The set of $4n_k$ fermion states can be considered as one large shell. This shell is half filled when n_k pairs of particles are in the states. The states obtained by operating on the vacuum with pair creation operators (1a) are in one-to-one correspondence with states obtained by operating with the boson creation operators of the IBM until the middle of this shell is reached; i.e. until there are n_k pairs or bosons present. At this point the fermion character of the Ginocchio model becomes evident and it is no longer possible to obtain all of the states of the IBM. When the number of particles is greater than $2n_k$ the simplest description is in terms of creating holes in a completely filled shell with the states completely determined by particle-hole symmetry from the ones already obtained by having a number of bosons less than or equal to n_k.

It is particularly interesting that the fermion character of the Ginocchio model has no effect at all in the one-to-one correspondence between the states of the Ginocchio model and the IBM as long as the number of particles is less than in a half-filled shell. This can be seen most easily by using the SU(4) classification and the boson pair seniority v_B. The boson sextet is represented in the SU(4) algebra as the antisymmetric combination of two fundamental quartets denoted by (0,2,0) with the Young diagram shown in fig. 1.

States of two bosons are classified in the IBM in the 21 dimensional representation of SU(6). In SU(4) the symmetric product of two sextets gives a reducible representation of SU(4) which can be broken down into a singlet and a 20-plet denoted by (0,4,0) with the Young diagram shown in fig. 2. In general the states of n bosons which are formed by the symmetric product of n sextets form a reducible representation of SU(4) which contains all the representations having the form $(0,v_B,0)$ appearing previously in the product of n-2 sextets with $v_B = n-2-2t$ and $t = 0,1...[(n-2)/2]$ and one new representation with $v_B = n$ and the Young diagram shown in fig. 3.

Thus each SU(4) representation can be labeled with the v_B seniority number corresponding to the value of n for which it first appears. The seniority number v_B is identical to the boson-pair seniority v_B defined by the reduction of SU(6) to SO(6) in the IBM.

After the middle of the shell is reached, no new larger representations of SU(4) occur. This is seen by noting that the nth power of a creation operator (1a) exists and is nonvanishing for $n \leq n_k$, but must vanish for $n > n_k$ because each term includes the square of a fermion creation operator. Thus for $n > n_k$ no new representations of SU(4) occur, and the old ones are completely determined by particle-hole symmetry from the cases where $n < n_k$.

The exact nature of the correspondence between the states of the Ginocchio model and the IBM can be further clarified by examining the state having the largest value of L_z for a given number n of bosons or fermion pairs. This is $L_z = 2n$ in the IBM. The corresponding state in the Ginocchio model is

$$\psi(L_z=2n) = (a^{\dagger}_{3/2}a^{\dagger}_{1/2})^n|0\rangle \tag{6}$$

where the operator on the right hand side is defined by Eq. (1a). This state (6) exists for all values of $n < n_k$, but vanishes for $n > n_k$ since the square of at least one fermion creation operator then appears in every term on the right hand side. Thus the state of maximum L_z for a given n in the IBM has an existing corresponding state in the Ginocchio model for all values of n up to and including a half-filled shell.

We now examine the classification of this state (5) under the various groups in the chains (3) and (5). In the IBM this state has only d bosons and no s bosons, and there are no d boson pairs coupled to L=0, since the state has the maximum value of L_z allowed for the n boson system. It is therefore classified in the representation of $SO(6) \equiv SU(4)$ which corresponds to "boson-pair seniority" $v_B = n$. The SU(4) representation is (0,n,0); namely the largest representation that occurs in the symmetric product of n sextets.

We thus see that for each value of $n \leqslant n_k$ there is a one-to-one correspondence between the states classified in the largest SU(4) representation in the two models. States not classified in this largest SU(4) representation have a boson pair seniority v_B smaller than n, and can be constructed by adding SU(4) singlet boson pairs to states of maximum seniority at lower values of n. The same construction is possible in the Ginocchio model by operating on a state of the form (5) with the operator (4) which creates two fermion pairs in an SU(4) singlet state. Thus we can define

$$\psi(n,v_B,L_z=2v_B) = (A^{\dagger})^{(n-v_B)/2}(a^{\dagger}_{3/2}a^{\dagger}_{1/2})^{v_B}|0\rangle \tag{7}$$

This state of n pairs classified in the SU(4) representation corresponding to boson-pair seniority v_B exists for values of n from v_B to $2n_k-v_B$ and therefore for all values less than or equal to n_k. Thus we have shown that all the SU(4) representations appearing in the IBM for a given value of $n < n_k$ exist also in the Ginocchio model and are constructed by the prescription (7) to get the state with the highest value of L_z in the SU(4) supermultiplet.

10. The SO(5) = Sp(4) groups in the two models

Once the correspondence between the SO(6) boson pair seniority classification in the IBM and the SU(4) classification in the Ginocchio model is established it is clear that a similar correspondence exists for the SO(5) ≡ Sp(4) subgroup of SO(6) ≡ SU(4). We first consider the chain (5b). We have seen that the representation (0,0,0,0,n) of SU(6) contains the representations of SO(6) = SU(4) denoted by $(0,v_B,0)$, where $v_B = n-2t$ and $t = 0,1...[n/2]$. Each of these representations contains the representations of SO(5) ≡ Sp(4) denoted by $(0,v_d)$, where $v_d = 0,1...n-2t$. The representation $(0,v_d)$ of Sp(4) thus occurs with a multiplicity $[\frac{n-v_d}{2}] + 1$ in the representation (n,0,0,0,0) of SU(6). The same reduction holds for the same SU(4) representations appearing in the fermion pair model.

The SO(5) subgroup of SU(6) acts only on the d bosons and defines a seniority number in the conventional manner; namely the number of d bosons having no pairs coupled to L=0. The general state in this classification then contains a certain number of d bosons having no pairs coupled to L=0, a certain number of d boson pairs coupled to L=0 and a certain number of additional s bosons. The SO(5) group thus defines transformations acting on the basic sextet and leaving invariant the creation operator for an s boson as well as the creation operator for a pair of bosons in the particular state analogous to Eq. (4). This is seen to be directly analogous to the Sp(4) group in the Ginocchio model

which generates linear transformations among the states of the basic fermion pair sextet and leaves invariant the quasispin creation operator (2a) as well as the four-particle SU(4) scalar operator defined by Eq. (4).

The reduction of the representation (0,0,0,m) of SO(8) according to the chain (3b) shows that the representation $(0,v_d)$ of Sp(4) occurs with values of the Sp(2) quasispin $S = m-v-2r$, where $r = 0,1...[\frac{m-v_d}{2}]$. We can now see the correspondence between this fermion chain (3b) and the boson chain (5a). If the quasispin operators (2) are interpreted as creation and annihilation operators for s bosons, a state with quasispin S classified in the representation (0,0,0,m) of SO(8) contains a number of "d bosons" given by $n_d = m-S$. For fixed values of S, m and $n \leq m$, the representations $(0,v_d)$ of Sp(4) occur in the representations (0,0,0,m) of SO(8) for $v_d = m-S-2r = n_d-2r$, where $r = 0,1...[q/2]$. These are just the representations of Sp(4) which appear in the representation (n,0,0,0) of SU(5). Thus the number of "d bosons" defined by the quasispin S in the SO(8) fermion model is the same as the number of d bosons defined by the SU(5) subgroup of SU(6) in the IBM for all values of $n < m$.

11. The number of d-bosons in both models

We now consider the chain (5a). The representation (n,0,0,0,0) of SU(6) contains all the SU(5) representations denoted by $(n_d,0,0,0)$, where $n_d = 0,1...n$ and is just the number of d bosons. These representations of SU(5) contain the representations of Sp(4) $\equiv$ SO(5) denoted by $(0,v_d)$ where $v_d = n_d-2r$ and $r = 0,1...[(n_d-v_d)/2]$. These contain states with (n_d-v_d) d-boson pairs coupled to L=0, $(n-n_d)$ s-bosons and v_d d-bosons not in L=0 pairs. The representation $(0,v_d)$ thus is seen to occur in the SU(6) representation (n,0,0,0,0) with a multiplicity $[\frac{n-v_d}{2}] + 1$ since it appears for all values of $r \leq [\frac{n-v_d}{2}]$ and $n_d = v+2r$. This agrees with the previous result from the chain (5b). The states of d boson-pair seniority v_d occur in the n boson system for states having v_d d-bosons, v_d+2 d-bosons, etc. up to n-1 or n d-bosons depending

upon whether $n-v_d$ is odd or even. The states with $v_d = n$ and $v_d = n-1$ occur only once. The states with $v_d = n-2$ and $n-3$ occur twice, the states with $v_d = n-4$ and $n-5$ occur three times, etc.

12. The three seniorities in the two models

We are now in a position to see the basic physics underlying the correspondence between the two models and the classifications. The three different seniority numbers n_d, v_B and v_d can be defined in the two models. Some of them are described by the same group in both models, others are not. The boson pair seniority number v_B which defines the number of bosons in a state which are not in the particular state defined by Eq. (4) is defined in both models by the $(0,v_B,0)$ representation of $SO(6) \equiv SU(4)$. Also the d-boson pair seniority number v_d which is the number of particles that are neither s bosons nor boson pairs coupled to L=0 in the particular state (4) is defined by both models by the $(0,v_d)$ representation of $SO(5) \equiv Sp(4)$. But the number n_d of d bosons cannot be defined in an analogous way in both cases. In the boson case, n_d is defined by the $(n_d,0,0,0)$ representation in the SU(5) classification. In the fermion model, the Sp(2) quasispin algebra (2) defines a quantum number S which can be related to $n_d \equiv m-S$; i.e. the number of fermion pairs which are not in the "s boson" state defined by the creation operator (2a). There is a one-to-one correspondence between these two separations into s and d bosons as far as counting the states is concerned, but the two groups do not have counterparts in the other model.

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FIGURE CAPTIONS

Fig. 1 Young diagram for (0,2,0).

Fig. 2 Young diagram for (0,4,0).

Fig. 3 Young diagram for $(0,v_B,0)$.

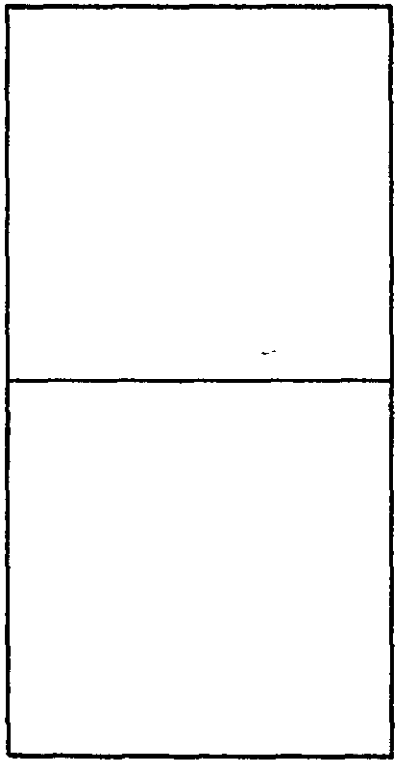

Figure 1

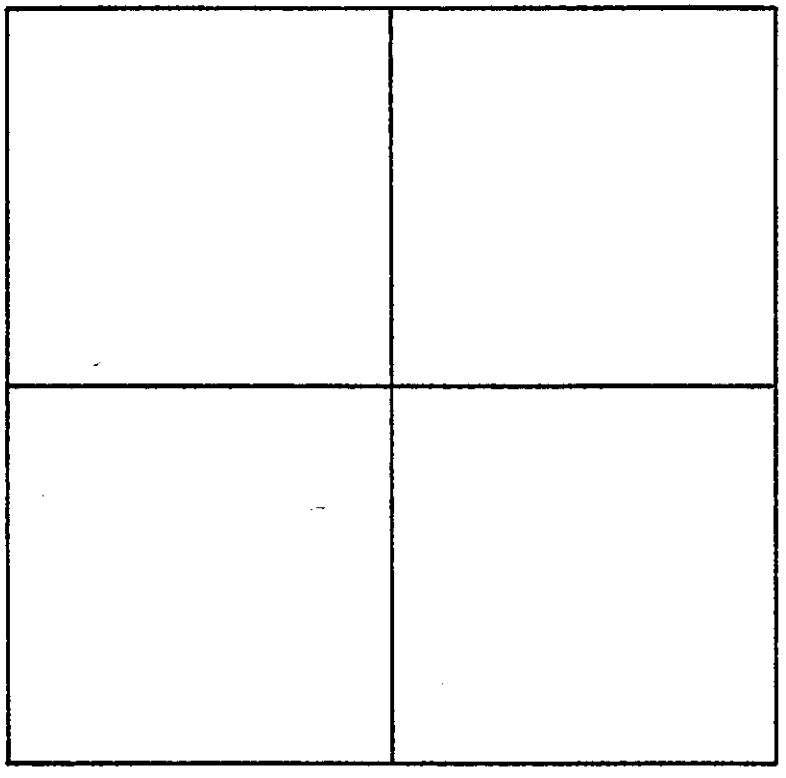

Figure 2

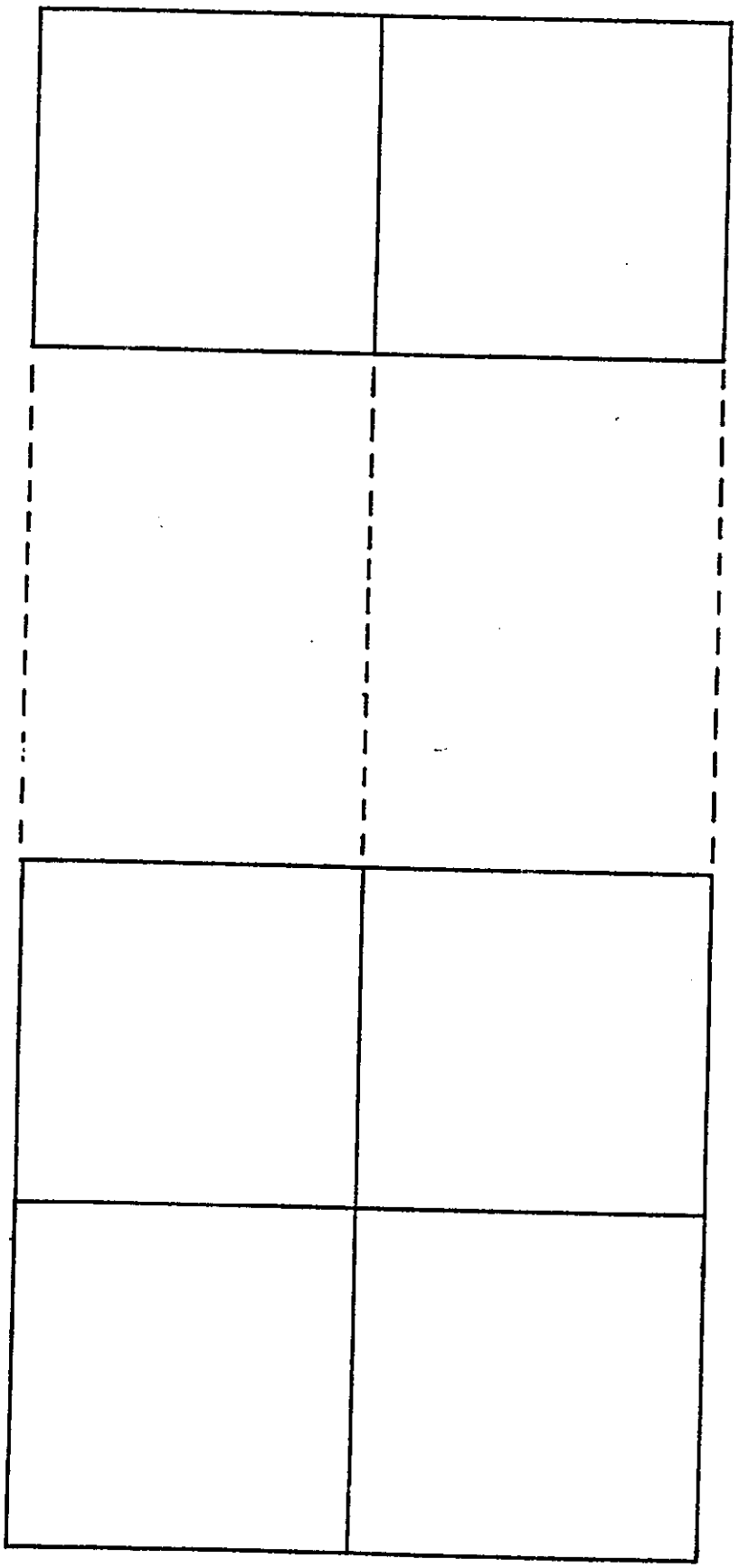

Figure 3