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Young-Tableau Methods for Kronecker Products of Representations of the Classical Groups

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ABSTRACT

Diagrammatic methods for decomposing Kronecker products of arbitrary representations of any of the classical groups are presented. For convenience, efficient ways of computing the dimensions and quadratic Casimir's $C_2(R)$ are also given. These methods seem more useful for hand calculations than the method of Schur functions (or characteristic polynomials). An appendix presents the Kronecker products for any two representations of dimension ≤ 100.

INTRODUCTION

The particle physicist looking at the theory of groups is generally interested in certain "practical" questions concerning the representations of the groups. Among these questions are

a) What groups are available?

b) What representations exist for a given group, and what is their nature?

c) Branching rules: How representation R of group G breaks into representations S_i of subgroup H.

d) Kronecker Products: How $R_1 \otimes R_2$ breaks into irreducible representations $S_i \oplus S_2 \oplus ... \oplus S_n$.

e) "Clebsch-Gordan" coefficients for $R_1 \otimes R_2$: This, of course, needs the answer to (d) as a starting point.

There is a tendency to assume that mathematicians have addressed and solved these "practical" questions, yet it is not easy to find answers in the literature. The available groups (in the sense of having finite dimensional representations) are well known: SU(N), SO(N), Sp(N), the five exceptional groups, and products of these groups. Questions (b) and (c) are answered in table form in Patera and Sankoff[1]; but these tables give no insight as to how the representations and branching rules are obtained. A partial table of Kronecker products exists,[2] but it suffers the same flaw, and also omits some important groups, for example, SO(10), and lists no spinors at all. The problem of "Clebsch's" is a most difficult question in practice (although simple in theory once the Kronecker product is understood), and will not be addressed here.

Many physicists are familiar with Young Tableaux methods for finding the dimensions of representations and decomposing Kronecker products in SU(N). This work generalizes these procedures to the groups SO(2N+1), SP(2N), SO(2N) and G_2.

The methods mathematicians describe use "characteristic functions"[3] or Schur functions[2,4] and are both non-intuitive and hard to learn to apply. The Tableau method has the additional advantage that one can check whether $R_1 \otimes R_2$ contains a particular R_3, without having to do the full product.

We have tried to make these rules as simple and "cookbook-like" as possible. Actually drawing out the diagrams is easier than working with lists of numbers, but these diagrams can't appear in the text, so they are represented by a string of numbers in parentheses, with perhaps a symbol ($\uparrow$, $\downarrow$ or $*$) in front, describing the number of boxes in each row. Representations can also be described by the Dynkin numbers, which we put in brackets [ ] ; this notation is standard and is how they appear in reference 1. The notation (abc...n) matches that in reference 2 for non-spinors; we feel our notation for spinors is more convenient for a reason described below.

Our method of getting the dimension of an SU(N) representation may differ from the "product of boxes over product of hooks" rule familiar to some physicists. It is, however, equally easy to apply, and falls into the same pattern as the other groups SO(2N), SO(2N+1), Sp(2N), G_2 and F_4. The six rules for Kroenecker products may look imposing, but rules 1-3 cover all but certain SO(2N) cases, and in any event, these rules are easier to use than to concisely describe.

In the literature,[1] it is advised that the practical way of multiplying two representations is to multiply their dimensions, and look for a set of irreducible representations whose dimensions total that number, resolving ambiguities by using "Dynkin indices" (values of the quadratic Casimir operator). This method works for the few smallest representations, but for larger numbers it becomes fantastically cumbersome and ambiguous (e.g. in SO(7) there is a 35 dimensional representation, a 27 representation, a 7 representation, and the trivial 1 representation, $35 = 27 + 7 + 1$. Their Dynkin indices are[1] 20, 18, 2 and 0 respectively. Thus whenever a 35-dimensional representation appears in $R_1 \otimes R_2$, it could be replaced by 27 + 7 + 1). The rules set forth below are trivial to apply in these low-dimension cases, and are unambiguous in all cases. Dimensionally checking the result is, of course, still useful to prevent errors.

We append to this article a list of decompositions of all products where R_1 and R_2 are both ≤ 100, and up to 210 for SO(10), which is of special interest to grand unification theorists. We omit SU(N), which is easy to decompose using rules 1 and 2. The Sp(2N) products appear in reference 2 and are included here for completeness.

REPRESENTATIONS AND THEIR DIMENSIONS

A representation of a simple group of rank r can unambiguously be specified by a set of r integers corresponding to the r simple roots of the group. For example, in SU(3), [1,0] is the 3, [0,1] the $\overline{3}$, and [1,1] the 8; in SO(10), [10000] is the 10, and [00010], [00001] are the 16 and $\overline{16}$. This is how the representations are listed in ref. 1, and we always will use square brackets and integers not separated by commas when referring to such a specification (we have had no occasion to look at any representation in which a number is more than 9 in this specification scheme).

It is well known to physicists, at least for SU(N), that it is often more convenient to specify representations by "Young Tableaux." In the case of SU(N), the Young Tableau corresponding to $[a_1a_2...a_{n-1}]$ consists of a_{n-1} columns of n-1 boxes, followed on the right by a_{n-2} columns of n-2 boxes...with lastly a_1 "columns" of one box each. Thus in SU(6), for example, [21031] is drawn as shown in Figure 1. We will find it convenient to describe a Tableau by listing in parentheses the number of boxes appearing in each row. [21031] in SU(6) is then written as (75441). This notation should enable the reader to easily draw out any tableau in an example here.

The advantages of using such Tableaux are 3-fold: In terms of the Tableaux, one can compute the dimensionality of the representation, compute Kroenecker products of two representations, and identify the symmetry properties of a representation (two boxes in a row mean two symmetric indices; boxes in a column imply antisymmetric indices). The justification for our particular way of defining Tableaux for SO(N) and Sp(2N) is that we want to preserve the first two properties; the third can't be kept when spinor representations are involved.

For Sp(2N) the tableau is the same as in the SU(N) case. It will be seen, however, that where [a,b,...,z] and [z,...,b,a] are conjugate representations in SU(N), they are not related in Sp(2N).

SO(2N+1) has the property of including spinors. The last number z in [a,b,c,...,z] will determine if the representation is a spinor: if z is odd, it is a spinor. A <u>pair</u> of spinor indices can form vectorlike indices. Thus, if z is even, the tableau will contain z/2 columns of N boxes (as opposed to z such columns in, say, Sp(2N)). If z is odd, the tableau will look the same: there are (z–1)/2 columns of N boxes, and to indicate a spinor is being described, an arrow is added to the notation. For example, as in figure 2, in SO(7), [123] = (↑431). [002] would be (111) while [003] is (↑111) and [001] is (↑000).

SO(2N) also has spinors; it has the added complexity of the last 2 roots referring to spinor indices. Let the representation be [a,b,...,y,z] where $z \geq y$. Then if y + z is odd, it is a spinor indicated by an upward pointing arrow. There are y columns of N-1 boxes, and <u>(z - y)/2</u> (or (z-y-1)/2 in the case of a spinor) columns of N boxes. What is happening is that pairs of one of each type of spinor indices form vector indices of one kind, and then excess pairs of one type of spinor index form other vector indices. Thus, as in figure 3, [00014] in SO(10) becomes (↑22221).

When $y > z$, the conjugate representation is formed: In SO(10), 16 is [00001] and $\overline{16}$ is [00010]. In this case, there are z columns of N-1 boxes, and $(y - z)/2$ (or $(y-z-1)/2$) columns of N boxes. When taking Kroenecker products, it is important to know whether you are doing $R \times R$ or $R \times \bar{R}$, so we distinguish the $y > z$ representations by a down arrow if spinors $\{[00021] = (\downarrow 11110)\}$ or a star if the representation is a non-spinor $\{([10040] = (*32222)\}$.

To find the dimension of a representation $(a_1 a_2 \ldots a_n)$ in a group one follows the following prescription: Add to the a_i (or twice a_i, if the group is SO(N)) some simple set of numbers (again dependent on the group), to get ℓ_i. Form the product of some combination of the ℓ_i, their differences $\Delta_{ij} = \ell_i - \ell_j$ $(i > j)$ and their sum $\varepsilon_{ij} = \ell_i + \ell_j$ $(i > j)$, and divide by a specified denominator, which is the same as the numerator for all the $a_i = 0$. The specifics of this process are given in Table 1.

Table 1

	Group	ℓ_i	Numerator	Denominator
	SU(N)	$\ell_i = a_i + N - i$	$\prod \ell_i \prod \Delta_{ij}$	$1!2!\ldots(N-1)!$
	Sp(2N)	$\ell_i = a_i + N - i$	$\prod \ell_i \prod \Delta_{ij} \prod$	$1!3!\ldots(2N-1)!$
Non-spinor	SO(2N+1)	$\ell_i = 2a_i + 2N+1 - 2i$	$\prod \ell_i \prod \Delta_{ij} \prod \varepsilon_{ij}$	$2^{N(N-1)} 1!3!5!\ldots(2N-1)!$
spinor	SO(2N+1)	$\ell_i = 2a_i + 2N+2 - 2i$	same	same
Non-spinor	SO(2N)	$\ell_i = 2a_i + 2N - 2i$	$\prod \Delta_{ij} \prod \varepsilon_{ij}$	$2^{N(N-1)}(N-1)!1!3!5!\ldots(2N-3)!$
spinor	SO(2N)	$\ell_i = 2a_i + 2N+1 - 2i$	same	same

The process is illustrated for the representations $(\uparrow 1000)$ in SO(8) and SO(9) (figure 4); the numbers down the left side are common to any SO(8) (SO(9)) representation, and the +1's are because this representation is a spinor.

When the group in question is G_2, two integers [p,q] will label the representation, and the dimension can be computed from the Young diagram (which is (p+q, q)) by labelling the side with 1,2 so that $\ell_i = 2 - i + a_i$, and forming the numerator $\prod \ell_i \prod \Delta_{ij} \prod \varepsilon_{ij} \times (2\ell_1 + \ell_2)(2\ell_2 + \ell_1)$. The denominator is 120. Equivalently, the dimension is given by $(p+1)(q+1)(p+q+2)(p+2q+3)(p+3q+4)(2p+3q+5)$ divided by 120.

For representations of F_4, there are non-spinors [a,b,2c,d] and spinors [a,b,2c+1,d]. The diagram for a nonspinor has the form (d+a+2b+3c, a+b+c, b+c, c) and for a spinor ($\uparrow$d+a+2b+3c+1, a+b+c, b+c, c). Equivalently, (w,x,y,z) = [x-y, y-z, 2z(+1 for spinor), a-b-c-d(-1)]. Notice that the first row is always at least as long as the sum of the lengths of the other three rows. To find the dimension of a representation, write 11,5,3,1 down the left side, and add two per box, plus 1 more if it is a spinor. ($\ell_i = 2a_i + 11$, 5, 3 or 1 for i = 1, 2, 3 or 4, +1 for a spinor or 0 for a nonspinor.) Then form the numerator: $\prod \ell_i \prod \Delta_{ij} \prod \varepsilon_{ij} (\ell_1+\ell_2+\ell_3+\ell_4)(\ell_1+\ell_2+\ell_3-\ell_4) \times (\ell_1+\ell_2-\ell_3+\ell_4)(\ell_1+\ell_2-\ell_3-\ell_4)(\ell_1-\ell_2+\ell_3+\ell_4)(\ell_1-\ell_2+\ell_3-\ell_4)(\ell_1-\ell_2-\ell_3+\ell_4)(\ell_1-\ell_2-\ell_3-\ell_4)$. Note that because $a_1 \geq a_2+a_3+a_4$, all of those are positive. The denominator is, as usual, the numerator with $a_1 = a_2 = a_3 = a_4 = 0$, which works out to be $11!9!!25 \times 2^{25}$. Of course, dim [pqrs] can be written as a polynomial (of degree 24) in p,q,r and s, but this is not very illuminating or convenient.

KROENECKER PRODUCTS

When taking the Kroenecker product of two representations, arrange the less complicated representation on the right, and label it with an "a" in each box in the first row, b's in the second row, etc. Then follow the rules set down below. Note that one can check the result by seeing whether the sum of the dimensions of the results is equal to the product of the dimensions of the representations being multiplied. Also, it is sometimes easier to use the following trick than to multiply out explicitly: Say you need $R_1 \otimes R_2$, and you know that $S_1 \otimes S_2 = R_2+T_1+T_2+\ldots$, where the S_i and T_i are all much simpler than R_2. Then one may write $R_2 = S_1 \otimes S_2 - T_1 - T_2 \ldots$ and do $(R_1 \otimes S_1) \otimes S_2$, subtracting the results of $R_1 \otimes T_1 + R_2 \otimes T_2 + \ldots$ This trick will be illustrated below.

Two techniques were utilized in deriving these rules. Careful manipulations of tensors and group invariants can indicate the procedure when there are no spinor indices (or implied spinor indices). When spinors are present, it is possible to use the trick described above to determine what the product is; one can then carefully note for the general cases which representations will remain after subtracting $R_1 \otimes T_1 + R_2 \otimes T_2 + \ldots$ This procedure could in principle have become prohibitively cumbersome, but any combination rules simple enough to be practical to apply are also relatively easy to derive in this way.

<u>Rule 1</u>: Adding one box: One tacks a single box onto the end of any row (including a row of 0 length) in all ways so as to leave a correct tableau for the particular group (no row longer than the one above, and the number of rows not exceeding the rank of the group). For example (see figure 5) in SU(4), (110) $\otimes$ (100) contains (210) and (111). For notational convenience, we will write the operation of appending an "a" box in the nth position of the kth row as $\{a \to n, k\}$. Thus in this example, we have $\{a \to 2, 1\}$ and $\{a \to 1, 3\}$.

<u>Rule 1a</u>: In SU(N) you can add the box to the nth row (the rank is n-1) and cancel that whole column. This corresponds to contracting n indices via an epsilon symbol. Thus in SU(3), (1,1) $\otimes$ (1,0) contains (0,0) via $\{a \to 1, 3$ (elim. col. 1)$\}$.

<u>Rule 1b</u>: In SO(2N), SO(2N+1) or Sp(2N), one may also use the added box to cancel a box in the existing tableau. For example, in SO(10), (11000) $\otimes$ (10000) contains, via $\{a \to 1, 2$ cancel$\}$, (10000). This corresponds to contraction with δ_{ab}, an invariant in SO(N), or with Ω_{ab}, the antisymmetric invariant in Sp(2N).

<u>Rule 1c</u>: In SO(2N) or SO(2N+1), if R_1 is a spinor and does not contain any N-box column, you may also use the added box to simply flip the direction of the

spinor-indicating arrow. In SO(2N+1) this means simply "absorbing" the box in the spinor arrow. For example, in SO(8), (↑200) ⊗ (100) contains, via {a → ↑}, (↓200). In SO(7), (↑200) ⊗ (100) contains (↑200).

Rule 1d: Only in SO(2N+1), one may also "merge" the added box with the last box in the Nth row. Thus in SO(7), (222) ⊗ (100) contains, via {a → 2, 3 merge}, (222).

Rule 1e: In SO(2N), when adding a box at the Nth row in the 1st column {a → 1, N}, both (abc...1) and (*abc...1) appear in the result. For example, in

SO(10), (11110) is [00011] and (10000) is [10000]. In their product, since they are both self-conjugate (under exchange of the last two Dynkin numbers) you would get both [00002] and [00020], that is, both (11111) and (*11111).

<u>Rule 2</u>: Adding more than one box. Label the boxes in the top row "a," the next row "b," etc. Add each box one by one, always in a one-box-permissible way, the top row first, then the 2nd row, etc., and such that reading from right to left and then up to down, the number of "a" boxes encountered is always $\geq$ the number of "b'"s $\geq$ number of "c'"s, and so on. Two representations are distinct if the a,b,c... labelling differs. For instance, in SU(4), (210) $\otimes$ (210) contains both a (321) from $\{a \rightarrow 3, 1;\ a \rightarrow 2, 2,\ b \rightarrow 1, 3\}$ <u>and</u> a (321) from $\{a \rightarrow 3, 1;\ a \rightarrow 1, 3;\ b \rightarrow 2, 2\}$. Also, no two a's (or b's or c's...) may appear in different rows of the same column. Such a representation would be both symmetric and antisymmetric in those two indices. Rules 1, 1a and 2 fully cover the case of SU(N).

<u>Rule 2a</u>: In Sp(2N), SO(2N+1) or SO(2N), you may use a box from R_2 to replace a box in R, that was previously cancelled. Thus (200) $\times$ (110) contains (200) via $\{a \rightarrow 2, 1 \text{ cancels};\ b \rightarrow 2, 1\}$. For the purposes of rule 2, these would count as an "a" and a "b" simultaneously.

<u>Rule 2b</u>: Rarely, when applying rule 2a using boxes of two different rows, it will be found that rule 2 is satisfied (the right-to-left and up-to-down part) whether the labelling of the re-added box is ab or ba. For example, in Sp(6), (110) $\otimes$ (210) contains, via $\{a \rightarrow 2, 1;\ a \rightarrow 1, 2 \text{ cancel};\ b \rightarrow 1, 2\}$ the representation (210). As can be seen from figure 6, rule 2b applies here. In this case, two (210)'s appear in the result.

<u>Rule 2c</u>: A box may never cancel a previously added box. This operation would correspond to taking a trace (or symplectic trace by contracting with Ω_{ab}) over two indices which both appear in R_2, but R_2 is irreducible, so the operation gives zero.

Rule 2d: In Sp(2N), two boxes from different rows may not cancel and re-add a box in the Nth row. Thus in Sp(8), $(1111) \otimes (1100)$ does not contain (1111).

Rule 2e: Up to one "a," one "b"... may be absorbed by a spinor line on SO(2N) or SO(2N+1).

Rule 2f: When cancelling boxes, you may anticipate future cancellations. An example of this should explain: In SO(9), $(2110) \otimes (1100)$ contains (2000) via $\{a \to 1$, 2 cancels; $b \to 1$, 3 cancels$\}$ (see figure 7). When a box is subsequently re-added, if an Nth row box was cancelled in Sp(2N) rule 2d applies and the tableau should not appear in the result. In figure 8, $(1111) \otimes (1110)$ is done in Sp(8). Note that (1110) does not appear in the answer.

Rules 1 thru 2f fully cover Sp(2N). When spinors or representations with N-1 or more rows in SO(2N) or SO(2N+1) appear, one must also apply the following rules:

Rule 3: (Rules 3-3d apply to SO(2N+1)) No two boxes from the same row may merge together. For example, in SO(7), $(100) \otimes (111)$ contains (111) via $\{a \to 2, 1;$ $b \to 3, 1;$ $c \to 3, 1$ merge$\}$. But $(110) \otimes (200)$ does not contain (111) via $\{a \to 3, 1;$ $a \to 3, 1$ merge$\}$.

Rule 3a: "Merging" boxes, in actuality, is adding the boxes in the N+1st row, and using the 2N+1-index epsilon symbol SO(2N+1) invariant to reduce the column of X boxes to 2N+1-X boxes. It is necessary to use this more cumbersome point of view when R_2 contains a column of N boxes. For example, in SO(7), $(111) \otimes (111)$ contains (100) via $\{a \to 4, 1;$ $b \to 5, 1;$ $c \to 6, 1$ contract $\varepsilon\}$, which would not be obtained by any combination of merging and cancelling.

Rule 3b: When R_1 is a spinor, boxes absorbed by the spinor line as per rule 1c count as being put in the N+1st row for the purposes of the right to left and up to down part of rule 2. This also applies to SO(2N).

Rule 3c: When R_2 (but not R_1) is a spinor, the resulting representations are all spinors and are formed by multiplying as if R_2 was a non-spinor, adding the spinor line, and, for each result, removing zero or one box per row. Figure 9 illustrates this for SO(7) (110) × (↑100): The (210) in (110) × (100), for instance leads to (↑210), (↑110), (↑200) and (↑100). When applying this rule, sometimes one gets repeated diagrams. In this case, discard one of each group of identical diagrams. In the example shown, of the 3 (↑100)'s, only two are kept. This rule will also apply to SO(2N) when R_1 has less than N rows.

Rule 3d: When both R_1 and R_2 are spinors, the answers will, of course, be non-spinors. Multiply the non-spinor parts, and then for each result, add zero or one box per row in all possible ways. This is illustrated in figure 10 which does (↑100) × ↑(100) in SO(7). Note that the repeated diagrams are all counted. Applying rule 3d, one can immediately see that (↑0000...) × (↑0000...) = (0000...) + (1000...) + (1100...) + ... + (11...11).

Rules 1-3d fully cover SO(2N+1). Rules 4 and 5 apply to SO(2N).

Rule 4: When at least one of R_1 and R_2 is a nonspinor and contains no column of N rows, use rules 4a thru 4f.

Rule 4a: When a box is being added to the Nth row, 1st column it can stand for either [...,0,2] or [...,2,0]. Thus the representation is counted twice, (a,b,c...) and (*a,b,c...). This is a generalization of rule 1e.

Rule 4b: When R_1 is a spinor, or already has N-box columns, the doubling in rule 4a does not apply. For example, in SO(10), (↑11110) ⊗ (10000) = (↑11100) + (↑21110) + (11110) + (↑11111): There is not also (↓11111).

Rule 4c: When cancelling and re-adding in the (N,1) position, this doubling does not apply. Thus, in SO(10), (11111) ⊗ (11000) contains (11111) but not (*11111).

Rule 4d: The spinor line in R_1 changes direction once for each box "absorbed" in it.

Rule 4e: When R_1 and R_2 have long enough columns, one may also form columns of M > N boxes and use the 2N index epsilon symbol to create 2N-M box columns. This result is not distinct from what would be gotten by cancellations only, without re-adds. Figure 11 shows how this works: In SO(8), $(1110) \otimes (1110)$ contains (1100) via $\{a \to 2, 1 \text{ cancels}; b \to 2, 1; c \to 3, 1 \text{ cancels}\}$ and another (1100) via $\{a \to 4, 1; b \to 5, 1; c \to 6, 1 \text{ epsilon}\}$. Yet in $(1111) \otimes (1110)$, the (1000) from $\{a \to 2, 1 \text{ cancels}; b \to 3, 1 \text{ cancels}; c \to 4, 1 \text{ cancels}\}$ is the only one counted; $\{a \to 5, 1; b \to 6, 1; c \to 7, 1 \text{ epsilon}\}$ is the same (1000).

Rule 4f: When R_2 (but not R_1) is a spinor, and R_1 has no N-box columns, do the multiplication just as in SO(2N+1), using rule 3c. When an odd number of boxes was eliminated, the resulting spinor arrow is in the opposite direction from that of R_2 if R_2 had no N-box columns.

This covers SO(2N) except when a circumstance peculiar to SO(2N) occurs: If both R_1 and R_2 are not self-conjugate (they are spinors or have N rows), then it makes a difference whether you multiply R_1 by R_2 or by its conjugate. For instance, in SO(10), it is well known that $16 \otimes \overline{16} = 1 + 45 + 210$, while $16 \otimes 16 = 10 + 120 + 126$. This phenomenon, covered in rule 5, is distinct from that of rule 1e, although the underlying reason for both is the two distinct (yet isomorphic) types of spinors available.

Rule 5: $(\uparrow 000...00) \otimes (\uparrow 000...00) = (11...111111) + (11...111100) + (11...110000) + ...$, while $(\uparrow 0000...00) \otimes (\downarrow 000...00) = (11...11110) + (11...11000) + (11...100000) + ...$ This is illustrated for SO(10): $16 \otimes \overline{16}$ in figure 12. It is interesting to note that in SO(4N), a spinor $\otimes$ itself contains the 1 representation, and a spinor $\otimes$ its conjugate contains the "vector" 4N, while in SO(4N+2), spinor $\otimes$ $\overline{\text{spinor}}$ contains 1, while spinor $\otimes$ spinor contains the vector. This pattern is easy to verify for SO(4) (isomorphic to SU(2) × SU(2): $(\uparrow 00) \to \{½, 0\}$ and

$(\downarrow 00) \to \{0,½\}$ so $\uparrow \times \uparrow = \{1,0\} + \{0,0\} = (11) + (00)$ while $\uparrow \times \downarrow = (10)$) and for SO(6) (isomorphic to SU(4): $\uparrow \to 4$, $\downarrow \to \overline{4}$, so $\uparrow \times \uparrow$ contains a 6, the vector in SO(6), while $\uparrow \times \downarrow$ has a 1). It is also obvious in SO(8), the first non-trivial SO(2N), because of the symmetry of SO(8) which says [abcd] and [cbad] (and [abdc] = $\overline{[abcd]}$, of course) are isomorphic. Thus [1000] looks like [0001]: the vector and spinor 8_v, 8_s are as alike as 8_s and $\overline{8}_s$. So, when taking 8s × 8s, this can't contain 8_v, because if it did, the symmetry would tell you that $8_v \times 8_v$ contains $\overline{8}_s$, a contradiction since two non-spinors can't produce a spinor. This symmetry property is useful for doing SO(8) Kroenecker products. For instance, (1111) × (*1111) is [0002] × [0020] which is related to [0002] × [2000] or (1111) × (2000), an easier product to take. For this reason, SO(8) is also a good "laboratory" for seeing how complicated SO(2N) representations multiply; for the remainder of rule 5, SO(6) and SO(4), where the results are easy to derive in a different way, can also be used to illustrate the various rules.

Rule 5a: When R_2 is an elementary spinor, [000...01] = ($\uparrow$000...) or [00...10] = ($\downarrow$00...) and R_1 is a non-spinor with N rows (if R_1 has fewer rows, see rule 3c). For the sake of illustration, we will assume R_1 is of the type [ab...yz], z > y. If y > z, then one can still use these rules by multiplying the conjugates of R_1 and R_2, and conjugating the answer. E.g. [0002] ⊗ [0001] = [0003] + [0021] + [0101] + [1010] + [0001] in SO(8), so [0020] ⊗ [0010] = [0030] + [0012] + [0110] + [1001] + [0010]. The rule is to eliminate zero or one box per row in R_1 in all possible ways such that an even (odd) number of boxes are eliminated if R_2 is $\uparrow$ ($\downarrow$). Assign a $\uparrow$ spinor orientation to each of the results. (The z in R_1, which must be at least y+2 since R_1 is a non-spinor, dominates even if R_2 is $\downarrow$.)

Rule 5b: When R_1 is a spinor and R_2 is an elementary spinor, add zero or one box per row in R_1 in all possible ways such that the total number of boxes added is of the same parity as N (N+1) if the two spinor arrows are in the same (opposite) direction. The results are non-spinors, and if R_1 = (↓abc...) they all have $y \geq z$; if R_1 = (↑ab...) they all have $z \geq y$. This rule is shown in figure 13, wherein $560 \otimes \overline{16}$ is done for SO(10). The 560 is [01001] or (↑11000); the $\overline{16}$ is (↓00000) or [00010]. The result is (11000) + (11110) + (21100) + (22000) + (22200) + (22110) + (21111). $\overline{560} \times 16$ would contain (*21111) = [10020] rather than (21111).

Rules 5c-5g will cover the cases where R_2 is a non-elementary spinor <u>and</u> R_1 contains N rows. It will be more practical, however, to treat these cases by writing R_2 as ↑ × S - various smaller spinors. For example, in SO(8), (2111) × (↑1100) = (2111) × {(↑0000) × (1100) - (↓1000) - (↑0000)} = (2111) × {(1100) × (↑0000) - (1000) × (↓0000) + (↑0000) - (↑0000)}. Each of the two resulting triple products is easy to do. Figure 14 shows (1111) ⊗ (↑1000) in SO(8) graphically, to illustrate how simple the tableau method makes things.

<u>Rule 5c</u>: When R_1 is a non-spinor of N rows and R_2 is a spinor containing less than N rows: Combine the non-spinor parts normally, and eliminate one box each from an even number of rows (if R_2 is ↑; an odd number if ↓). All the resulting representations are ↑, except if in combining the non-spinor part, you get a representation with less than N rows (the bottom boxes were all killed). In that case, for that representation, the results are all ↓, and the number of boxes eliminated is of the other parity (odd if R_2 is ↑). This process is illustrated in figure 15, in which in SO(8), (1111) × ↑(1000) is done directly. (1111) × (1000) = (2111) + (1110), (2111) → (↑2111) + (↑2100) + (↑1110) + (↑1000), (1110) → (↓1100) + (↓0000). After doing this, certain representations may have to be discarded, as outlined in 5d and 5e.

<u>Rule 5d</u>: When a representation can be arrived at in two or more ways, by elimination of boxes in two or more different representations, discard one of those

representations. For example, in SO(8) (4321) × (↑1000), (5321) → (↑4311) + others, (4421) → (↑4311) + others, but only one of these (↑4311)'s appears in the result.

<u>Rule 5e</u>: When all the eliminated boxes are the boxes coming from R_2, eliminate one of that kind of representation, even if there appear no others. For example, in SO(8), (1111) × (↓1000): (1111) × (1000) contains, via {a→ 1, 2}, (2111). (2111) → (↑1111) by eliminating the box at the end of row 1, which came from R_2. Thus (1111) × (↓1000) <u>does</u> <u>not</u> contain this (↑1111). On the other hand (2111) × (↓1000) can form (↑2111) in two such ways: (3111) → (↑2111) and (2211) → (↑2111), so one of these is discarded, and one appears in the answer.

<u>Rule 5f</u>: When R_2 is a spinor with N rows and R_1 is a spinor with less than N rows: Multiply the non-spinor parts. To each result add an odd or even number of boxes in all possible ways, adding up to one per row. The number of boxes added is of the same parity as N if R_1 and R_2 are both ↑ (or both ↓) and the opposite parity otherwise. (Here is another case in which SO(4N+2) differs from SO(4N).) Then eliminate one of each set of duplicated representations as in 5d, and one of each type of representation formed by adding, in the 2nd step, boxes that were eliminated in the 1st (as in rule 5e). Also eliminate one of each type of representation wherein just one entire column was added in the two steps.

<u>Rule 5g</u>: Multiplying two N-row representations is best done via the procedure in rule 5c. However, when both R_1 and R_2 are of the form [00...02m] or [00...2m], a simple pattern emerges: start with the representation formed by adding [00...2m] (R_1) to [00...2m] or [00...2m,0] (R_2), to get [00...2m,2m] or [00...4m] and eliminate pairs of vertically touching boxes. This is illustrated in figure 16 for [00002] × [00002] of SO(10): (11111) × (11111) = (22222) + (22211) + (22200) + (21111) + (21100) + (20000). This process is applicable to any R_1 and R_2 of the form [000...K] or [00...K0], with

the 1st M columns ($M = |K/2\,(R_1) - K/2\,(R_2)|$) untouched by the elimination process: [00004] $\times$ [00020] in SO(10) is shown in figure 17: (22222) $\times$ (*11111) = (33331) + (33221) + (22221) + (33111) + (22111) + (11111).

<u>Rule 6</u>: Products of representations of G_2: the simplest diagrammatic means of multiplying two representations of G_2 relies on the fact that SU(3) is a subgroup of G_2. The procedure entails 4 steps: 1) Break R_1 and R_2 down into their SU(3) content $(S_{11} + S_{12} + S_{13}...) \times (S_{21} + S_{22} + S_{23}...)$. 2) Multiply these SU(3) representations to get $(S_{11} \times S_{21}) + (S_{11} \times S_{22}) + ... = T_1 + T_2 + ...$ 3) Choose a particular T_i to be part of the content of some G_L representation. 4) From the set $\{T_i\}$, eliminate the content of that representation. Repeat steps 3 and 4 until no representations are left over. Steps 1-4 are each explained in some detail in rules 6a-6d.

<u>Rule 6a</u>: To get the SU(3) content of a G_2 representation: write the symbols p, q and o, one in each box in the Young diagram, in all ways such that within a row, the p's precede the q's which precede the o's, and within a column, p is above q, which is above o. Both boxes in one column may not contain the same symbol. Each of these arrangements becomes one SU(3) representation, [p,q] with p = the number of p's appearing in the labelled tableau, and q = the number of q's. This process is illustrated in figure 18 for $7 = [10] \rightarrow 3 + \bar{3} + 1$ and in figure 19 for [11] (64 dimensions in G_2) $\rightarrow 15 + 6 + \overline{15} + 8 + 8 + 3 + \bar{6} + \bar{3}$.

<u>Rule 6b</u>: Of course, SU(3) Kroenecker products are easy to do. But a further factor of two in time spent can be saved if one makes use of the fact that the G_2 representation always contains both R and $\bar{R}$ of SU(3).

<u>Rule 6c</u>: The G_2 representation to eliminate first is found by picking the remaining SU(3) representation with the longest first row (when two have equal first rows, the longest second row). This is the T_i of step 3. Call this

representation (a+b, b) or [a,b] of SU(3) (b will always be greater than a); then the G_2 representation to eliminate this is (b,a). The reason this representation is chosen is that the minimal G_2 representation that contains (a+b, b) is (b, a). (b, a) decomposes into SU(3) representations (p+q, q) with $p+q \leq a+b$ and if $p+q = a+b$, $q \leq b$. Thus by eliminating any of the other representations, one would never get a G_2 representation that includes T_i. So eventually, one will have to use (b, a) in G_2. This representation, however, will decompose into others of the remaining representations, so one should eliminate those first. For example, when doing 14×7 in G_2, one has the SU(3) representation (32), 3 (21)'s, as well as others. If you tried to cover the (21)'s first by using 3 (11)'s of G_2 {(11) in G_2 contains (21) in SU(3)}, then later, when taking care of the (32), you would find you still have to eliminate two more (21)'s {which are no longer available}. But if you start with the (32), this dilemma can't occur.

To illustrate this procedure, [01] × [20] (or 14 × 27) is done below: Step 1: G_2[01] → [11] + [10] + [01], G_2[20] → [20] + [11] + [10] + [02] + [01] + [00]. Step 2: The product of the 1st set with the 2nd set is {([31] + [12] + [20] + [01]) + ([30] + [21]) + ([21] + [10])} + {([22] + [30] + [03] + [11] + [00]) + ([21] + [02] + [10]) + ([12] + [20] + [01])} + {([21] + [02] + [10]) + ([01] + [20]) + ([11] + [00])} + {([13] + [21] + [02] + [10]) + ([12] + [01]) + ([03] + [12])} + {([13] + [20] + [01]) + ([10] + [02]) + ([11] + [00])} + {[11] + [10] + [01]}. Step 3: The 1st representation to eliminate is [13] because 4 is the biggest sum, and among the representations with a sum of 4, [13] has the biggest 2nd number, i.e. [13] = (43) while [22] = (42). [13] → (31)G_2 = [21] G_2. Step 4: [21]G_2 → [31] + [30] + [22] + [21] + [21] + [20] + [13] + [12] + [12] + [11] + [11] + [10] + [03] + [02] + [01]. Of course the [13] (and [31]) are what we picked [21]G_2 for. Cancelling these 15 representations, we return to step 3: the [31], [13] and, by accident, the only

[22] have all been eliminated; next is [03] = (33), since one of them is left. [03] $\to$ (30)G_2 = [30] G_2 $\to$[30]+ [21] + [20] + [12] + [11]+[10] +[03] + [02] + [01] + [00]. Eliminate these 10, and repeat step 3. Completing the process, we find that, in G_2, [20] $\times$ [01] = [21] + [30] + [11] + [20] +[01] + [10], or 27 $\times$ 14 = 189 + 77 + 64 + 27 + 14 + 7 in terms of dimensions.

It is not as easy to formulate rules for the other exceptional groups. (F_4 representations may be decomposed into SO(9) representations[2] and multiplied as in rule 6, but the simple decomposition rule we have put forth for G_2 is absent in this case.)

AN APPLICATION IN GRAND UNIFIED THEORIES

It is currently popular to postulate that the grand unified group is some large group (usually SU(N) or SO(2N) with $N \geq 5$). The symmetry is then broken in some series of steps down to SU(3) $\times$ SU(2) $\times$ U(1). It has been proposed[4] that instead of elementary Higgs, the scalars should be composites made of two fermions. Dimopoulos and Susskind[5] give a "rule" for determining which fermions might condense out: Say the fermions are in representations $R_1, R_2,\ldots,R_n$. Then the condensate will form in the "most attractive channel" (this assumes that one gluon exchange is the important process). To determine the relative attractiveness of channels $R_i + R_j \to S$ (where $S \in R_i \otimes R_j$), one compares $C_2(S) - C_2(R_i) - C_2(R_j)$. Thus to apply the Maximally Attractive Channel (MAC) prescription, one needs to know $R_i \otimes R_j$ and the quadratic Casimir's for the relevant representations. The rules for getting $R_i \otimes R_j$ are given above. Since these large groups don't appear in reference 1, it would be useful to find a way to compute $C_2(R)$. Fortunately, $C_2(R)$ is, for the classical groups, a polynomial of degree 2 in the indices [a_n] which describe the representation. For SU(N+1), with R = [$a_1a_2a_3\ldots a_N$]

$$C_2(R) = \left(\sum_{k=1}^{N} k(N-k+1)(a_k^2 + (N+1)a_k) \right.$$

$$\left. + 2 \sum_{i=2}^{N} \sum_{j<i} j(N-i+1)a_i a_j \right) / 2(N+1) \quad .$$

When R is the defining (N + 1) dimensional representation, $C_2(R)$ is normalized to $N(N+2)/2(N+1)$. Thus for SU(3), $C_2(R) = \frac{4}{12}(a_1^2 + 3a_1 + a_2^2 + 3a_2 + a_1 a_2)$, contrary to the formula given in Reference 1. For SO(2N), with $R = [a_1 a_2 \ldots a_{N-2} yz]$

$$C_2(R) = \left(\sum_{k=1}^{N-2} (ka_k^2 + [2kN - k(k+1)]a_k) \right.$$

$$+ \frac{1}{4}(Ny^2 + (2N-2)Ny + Nz^2 + (2N-2)Nz)$$

$$\left. + 2 \sum_{i=2}^{N-2} \sum_{j<1} j a_i a_j + \sum_{k=1}^{N-2} ka_k(y+z) + (\tfrac{N}{2} - 1)yz \right) / (2N^2 - N) \quad .$$

For Sp(2N), with $R = [a_1 a_2 \ldots a_N]$,

$$C_2(R) = \left(\sum_{k=1}^{N} k[a_k^2 + (2N+1-k)a_k] \right.$$

$$\left. + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} 2ia_i a_j \right) / 2N(2N+1) \quad .$$

And for SO(2N + 1), with $R = [a_1 a_2 \ldots a_{N-1} z]$,

$$= \left(\sum_{k=1}^{N-1} k\,[a_k^2 + (2N - k)a_k] \right.$$

$$+ \ (Nz^2 + 2N^2z)/4 + \sum_{k=1}^{N-1} ka_k z$$

$$\left. + \sum_{i=1}^{N-2} \ \sum_{j=i+1}^{N-1} 2ia_ia_j \right) /N(2N+1) \qquad .$$

The Dynkin index (ℓ in reference 1) is given by $\ell(R) = C_2(R) \times \mathrm{Dim}(R)$. These values of $C_2(R)$ are normalized such that C_2 of an elementary spinor = C_2 of the defining (2N-dimensional) representation of Sp(2N) = 1. For the purposes of MAC analysis, this normalization can be specified arbitrarily.

In principle, one would also like to have at hand rules for dimensions, Kroenecker products, and $C_2(R)$ for the exceptional groups. However, the procedures would certainly contain enough different cases and exceptions that they would not shed much light on matters, and would be sufficiently cumbersome to preclude any thought of hand calculations. In these cases, the use of Schur functions is required.

APPENDIX: KROENECKER PRODUCTS

For compactness, the representations are referred to by their dimension, when no ambiguity results and the dimension is less than 1000 so that the representation appears in reference 1.

SO(7)

$7 \times 7 = 1 + 21 + 27$

$7 \times 8 = 8 + 48$

$7 \times 21 = 7 + 35 + 105$

$7 \times 27 = 7 + 77 + 105$

$7 \times 35 = 21 + 35 + 189$

$7 \times 48 = 8 + 48 + 112 + 168_s$ $\qquad 112 = [011]$; $112' = [003]$

$7 \times 77 = 27 + 182 + 330$ $\qquad 168_s = [201]$; $168 = [020]$

$8 \times 8 = 1 + 7 + 21 + 35$

$8 \times 21 = 8 + 48 + 112$

$8 \times 27 = 48 + 168_s$

$8 \times 35 = 8 + 48 + 112 + 112'$

$8 \times 48 = 7 + 21 + 27 + 35 + 105 + 189$

$8 \times 77 = 168_s + 448$

$21 \times 21 = 1 + 21 + 27 + 35 + 168 + 189$

$21 \times 27 = 21 + 27 + 189 + 330$

$21 \times 35 = 7 + 21 + 35 + 105 + 189 + 378$

$21 \times 48 = 8 + 48 + 48 + 112 + 112' + 168_s + 512$

$21 \times 77 = 77 + 105 + 616 + 819$

$$27 \times 27 = 1 + 21 + 27 + 168 + 182 + 330$$

$$27 \times 35 = 35 + 105 + 189 + 616$$

$$27 \times 48 = 8 + 48 + 112 + 168_s + 448 + 512$$

$$27 \times 77 = 7 + 77 + 105 + 378 + 693 + 819$$

$$35 \times 35 = 1 + 7 + 21 + 27 + 35 + 105 + 168 + 189 + 294 + 378$$

$$35 \times 48 = 8 + 48 + 48 + 112 + 112 + 112' + 168_s + 512 + 560$$

$$35 \times 77 = 189 + 330 + 616 + [302] = 1560$$

$$48 \times 48 = 1 + 7 + 21 + 21 + 27 + 35 + 35 + 77 + 105 + 105 + 168 + 189 + 189 + 330 + 378 + 616$$

$$48 \times 77 = 48 + 168_s + 448 + 512 + [401] = 1008 + [211] = 1512$$

$$77 \times 77 = 1 + 21 + 27 + 168 + 330 + 714 + 825 + [410] = 1750 + [600] = 1911$$

<u>SO(9)</u>

$$9 \times 9 = 1 + 36 + 44$$

$$9 \times 16 = 16 + 128$$

$$9 \times 36 = 9 + 84 + 231$$

$$9 \times 44 = 9 + 156 + 231$$

$$9 \times 84 = 36 + 126 + 594$$

$$16 \times 16 = 1 + 9 + 36 + 84 + 126$$

$$16 \times 36 = 16 + 128 + 432$$

$$16 \times 44 = 128 + 576$$

$$16 \times 84 = 16 + 128 + 432 + 768$$

$$36 \times 36 = 1 + 44 + 126 + 495 + 594$$

$$36 \times 44 = 36 + 44 + 594 + 910$$

$$36 \times 84 = 9 + 84 + 126 + 231 + 924 + [0110] = 1650$$

$44 \times 44 = 1 + 36 + 44 + 450 + 495 + 910$

$44 \times 84 = 84 + 231 + 924 + [2010] = 2457$

$84 \times 84 = 1 + 44 + 84 + 126 + 495 + 594 + 924 + [0020] = 1980 + [0102] = 2772$

SO(11)

$11 \times 11 = 1 + 55 + 65$

$11 \times 32 = 32 + 320$

$11 \times 55 = 11 + 165 + 429$

$11 \times 65 = 11 + 275 + 429$

$32 \times 32 = 1 + 11 + 55 + 165 + 330 + 462$

$32 \times 55 = 32 + 320 + [01001] = 1408$

$32 \times 65 = 320 + [20001] = 1760$

$55 \times 55 = 1 + 55 + 65 + 330 + [02000] = 1144 + [10100] = 1430$

$55 \times 65 = 55 + 65 + 1430 + [21000] = 2025$

$65 \times 55 = 1 + 55 + 65 + 935 + 1144 + 2025$

SO(13)

$13 \times 13 = 1 + 78 + 90$

$13 \times 64 = 64 + 768$

$13 \times 78 = 13 + 286 + 715 \qquad (715 = [110000];\ 715' = [000100])$

$13 \times 90 = 13 + 442 + 715$

$64 \times 64 = 1 + 13 + 78 + 286 + 715 + [000010] = 1287 + [000002] = 1716$

$64 \times 78 = 64 + 768 + [010001] = 4160$

$64 \times 90 = 768 + [200001] = 4992$

$78 \times 78 = 1 + 78 + 90 + 715' + [020000] = 2275 + [101000] = 3925$

$78 \times 90 = 78 + 90 + [210000] = 2927 + 3925$

$90 \times 90 = 1 + 78 + 90 + 2275 + [400000] = 2629 + 2927$

<u>SO(2N + 1) N > 6</u>

$(2N + 1) \otimes (2N + 1) = 1 \oplus N(2N + 1) \oplus (2N^2 + 3N)$

<u>SO(8)</u>

$$\begin{cases} 8 \times 8 = 1 + 28 + 35 & (R_1 = R_2) \\ 8 \times 8 = 8 + 56 & (R_1 \neq R_2) \end{cases}$$

$8 \times 28 = 8 + 56 + 160$

$$\begin{cases} 8 \times 35 = 8 + 112 + 160 \\ 8 \times 35 = 56 + 224 \end{cases}$$

$$\begin{cases} 8 \times 56 = 28 + 35 + 35 + 350 \\ 8 \times 56 = 8 + 56 + 160 + 224 \end{cases}$$

$28 \times 28 = 1 + 28 + 35 + 35 + 35 + 300 + 350$

$28 \times 35 = 28 + 35 + 350 + 560$

$28 \times 56 = 8 + 56 + 56 + 160 + 224 + 224 + 840$

$$\begin{cases} 35 \times 35 = 1 + 28 + 35 + 294 + 300 + 567 \\ 35 \times 35 = 35 + 350 + 840 \end{cases}$$

$$\begin{cases} 35 \times 56 = 8 + 56 + 160 + 224 + 672 + 840 \\ 35 \times 56 = 56 + 160 + 224 + 224 + [2011] = 1296 \text{ (or } [1021] \text{ or } [1012]) \end{cases}$$

$$\begin{cases} 56 \times 56 = 1 + 28 + 28 + 35 + 35 + 35 + 300 + 350 + 350 + 567 + 567 + 840 \\ 56 \times 56 = 8 + 56 + 56 + 112 + 160 + 160 + 224 + 224 + 840 + 1296 \end{cases}$$

SO(10): Representations appearing here are:

1 = [00000]	1050 = [10002]	5280 = [10003]
10 = [10000]	1200 = [00101]	5940 = [01011]
16 = [00001]	1386 = [21000]	6930 = [00013]
45 = [01000]	1440 = [00012]	7644 = [03000]
54 = [20000]	1728 = [10011]	7920 = [40001]
120 = [00200]	1782 = [50000]	8085 = [20011]
126 = [00002]	2640 = [30001]	8800 = [10101]
144 = [10001]	2772 = [00004]	8910 = [00022]
210 = [00011]	2970 = [01100]	10560 = [00111]
210' = [30000]	3696 = [01002]	11088 = [10012]
320 = [11000]	3696_s = [11001]	12870 = [41000]
560 = [01001]	4125 = [00200]	14784 = [30100]
660 = [40000]	4290 = [60000]	15120 = [21001]
672 = [00003]	4312 = [20100]	16380 = [22000]
720 = [20001]	4410 = [12000]	17325 = [30002]
770 = [02000]	4608 = [31000]	21860 = [30011]
945 = [10100]	4950 = [20002]	

$10 \times 10 = 1 + 45 + 54$

$10 \times 16 = 16 + 144$

$10 \times 45 = 10 + 120 + 320$

$10 \times 54 = 10 + 210' + 320$

$10 \times 120 = 45 + 210 + 945$

$10 \times 144 = 16 + 144 + 560 + 720$

$10 \times 210 = 120 + 126 + 126 + 1728$

$10 \times 210' = 54 + 660 + 1386$

$$\left\{\begin{array}{ll} 16 \times 16 = 1 + 45 + 210 & ([00001] \times [00010]) \\ 16 \times 16 = 10 + 120 + 126 & ([00001] \times [00001]) \end{array}\right.$$

$$16 \times 45 = 16 + 144 + 560$$

$$16 \times 54 = 144 + 720$$

$$16 \times 120 = 16 + 144 + 560 + 1200$$

$$\left\{\begin{array}{ll} 16 \times 126 = 144 + 672 + 1200 & ([00001] \times [00002]) \\ 16 \times 126 = 16 + 560 + 1440 & ([00001] \times [00020]) \end{array}\right.$$

$$\left\{\begin{array}{ll} 16 \times 144 = 10 + 120 + 126 + 320 + 1728 & ([00001] \times [10010]) \\ 16 \times 144 = 45 + 54 + 210 + 945 + 1050 & ([00001] \times [10001]) \end{array}\right.$$

$$16 \times 210 = 16 + 144 + 560 + 1200 + 1440$$

$$16 \times 210' = 720 + 2640$$

$$45 \times 45 = 1 + 45 + 54 + 210 + 770 + 945$$

$$45 \times 54 = 45 + 54 + 945 + 1386$$

$$45 \times 120 = 10 + 120 + 126 + 320 + 1728 + 2970$$

$$45 \times 126 = 120 + 126 + 1728 + 3696$$

$$45 \times 144 = 16 + 144 + 144 + 560 + 720 + 1200 + 3696_s$$

$$45 \times 210 = 45 + 210 + 210 + 945 + 1050 + 1050 + 5940$$

$$45 \times 210' = 210' + 320 + 4312 + 4608$$

$$54 \times 54 = 1 + 45 + 54 + 660 + 770 + 1386$$

$$54 \times 120 = 120 + 320 + 1728 + 4312$$

$$54 \times 126 = 126 + 1728 + 4950$$

$$54 \times 144 = 16 + 144 + 560 + 720 + 2640 + 3696_s$$

$$54 \times 210 = 210 + 945 + 1050 + 1050 + 8085$$

$$54 \times 210' = 10 + 210' + 320 + 1782 + 4410 + 4608$$

$$120 \times 120 = 1 + 45 + 54 + 210 + 210 + 770 + 945 + 1050 + 1050 + 4125 + 5940$$

$$120 \times 126 = 45 + 210 + 945 + 1050 + 5940 + 6930$$

$$120 \times 144 = 16 + 144 + 144 + 560 + 560 + 720 + 1200 + 1440 + 3696_s + 8800$$

$$\begin{aligned} 120 \times 210 = {} & 10 + 120 + 120 + 126 + 126 + 320 + 1728 + 1728 + 2970 + \\ & 3696 + 3696 + 10560 \end{aligned}$$

$$120 \times 210' = 945 + 1386 + 8085 + 14784$$

$$\left\{ \begin{array}{l} 126 \times 126 = 1 + 45 + 210 + 770 + 5940 + 8910 \\ 126 \times 126 = 54 + 945 + 1050 + 2772 + 4125 + 6930 \end{array} \right.$$

$$\left\{ \begin{array}{ll} 126 \times 144 = 16 + 144 + 560 + 1200 + 1440 + 3696_s + 11088 & ([00002] \times [1001 \\ 126 \times 144 = 144 + 560 + 720 + 1200 + 1440 + 5280 + 8800 & ([00002] \times [1000 \end{array} \right.$$

$$126 \times 210 = 10 + 120 + 126 + 320 + 1728 + 2970 + 3696 + 6930 + 10560$$

$$126 \times 210' = 1050 + 8085 + 17325$$

$$\left\{ \begin{array}{l} 144 \times 144 = 1 + 45 + 45 + 54 + 210 + 210 + 770 + 945 + 945 + 1050 + 1050 + \\ \qquad\qquad 1386 + 5940 + 8085 \qquad ([10001] \times [10001]) \\ 144 \times 144 = 10 + 120 + 120 + 126 + 126 + 210 + 320 + 320 + 1728 + 1728 + \\ \qquad\qquad 2970 + 4312 + 4950 + 3696 \qquad ([10001] \times [10010]) \end{array} \right.$$

$$\begin{aligned} 144 \times 210 = {} & 16 + 144 + 144 + 560 + 560 + 642 + 720 + 1200 + 1200 + 1440 + \\ & 3696_s + 8800 + 11088 \end{aligned}$$

$$144 \times 210' = 144 + 720 + 2640 + 3696_s + 7920 + 15120$$

$$\begin{aligned} 210 \times 210 = {} & 1 + 45 + 45 + 54 + 210 + 210 + 770 + 945 + 945 + 1050 + 1050 + \\ & 4125 + 5940 + 5940 + 6930 + 6930 + 8910 \end{aligned}$$

$$\left\{ \begin{array}{l} 210 \times 210' = 1728 + 4312 + 4950 + 4950 + 28160 \\ 210 \times 210' = 1 + 45 + 54 + 770 + 1386 + 4290 + 7644 + 12870 + 16380 \end{array} \right.$$

SO(12)

$12 \times 12 = 1 + 66 + 77$

$12 \times 32 = 32 + 352_s \qquad (352_s = [100001])$

$12 \times 66 = 12 + 220 + 560$

$12 \times 77 = 12 + 352 + 560 \qquad (352 = [300000])$

$32 \times 32 = 1 + 66 + 462 + 495$

$32 \times 32 = 12 + 220 + 792$

$32 \times 66 = 32 + 352_s + [010001] = 1728$

$32 \times 77 = 352_s + [200001] = 2112$

$66 \times 66 = 1 + 66 + 77 + 495 + [020000] = 1638 + [101000] = 2079$

$66 \times 77 = 66 + 77 + 2079 + [210000] = 2860$

$77 \times 77 = 1 + 66 + 77 + [400000] = 1287 + 1638 + 2860$

SO(14)

$14 \times 14 = 1 + 91 + 104$

$14 \times 64 = 64 + 832$

$14 \times 91 = 14 + 304 + 896$

$64 \times 64 = 1 + 91 + [0001000] = 1001 + [0000011] = 3003$

$64 \times 64 = 14 + 364 + [0000002] = 1716 + [0000100] = 2002$

$64 \times 91 = 64 + 832 + [0100001] = 4928$

$91 \times 91 = 1 + 91 + 104 + 1001 + [0200000] = 2240 + [1010000] = 4844$

SO(2N) N > 7

$$(2N) \otimes (2N) = (1) \oplus (2N^2 - N) \oplus (2N^2 + N - 1)$$

Sp(6)

$6 \times 6 = 1 + 14_a + 21 \qquad (14_a = [010] \;;\; 14_b = [001])$

$6 \times 14_a = 14_b + 64 + 6$

$6 \times 14_b = 14_a + 70$

$6 \times 21 = 6 + 56 + 64$

$6 \times 56 = 21 + 126' + 189 \qquad (126' = [400] \;;\; 126 = [011])$

$6 \times 64 = 14_a + 21 + 70 + 90 + 189$

$6 \times 70 = 14_b + 64 + 126 + 216$

$6 \times 84 = 126 + 378$

$6 \times 90 = 64 + 126 + 357$

$14_a \times 14_a = 1 + 14_a + 21 + 70 + 90$

$14_a \times 14_b = 6 + 64 + 126$

$14_a \times 21 = 14_a + 21 + 70 + 189$

$14_a \times 56 = 56 + 64 + 216 + 448$

$14_a \times 64 = 6 + 14_b + 56 + 64 + 64 + 126 + 216 + 350$

$14_a \times 70 = 14_a + 21 + 70 + 84 + 90 + 189 + 512$

$14_a \times 84 = 70 + 512 + 594$

$14_a \times 90 = 14_a + 70 + 90 + 189 + 385 + 512$

$14_b \times 14_a = 1 + 21 + 84 + 90$

$14_b \times 21 = 14_b + 64 + 216$

$14_b \times 56 = 70 + 189 + 525$

$14_b \times 64 = 14_a + 21 + 70 + 90 + 189 + 512$

$14_b \times 70 = 6 + 56 + 64 + 126 + 350 + 378$

$14_b \times 84 = 14_b + 216 + 330 + 616$

$14_b \times 90 = 14_b + 64 + 216 + 350 + 616$

$21 \times 21 = 1 + 14_a + 21 + 90 + 126' + 189$

$21 \times 56 = 6 + 56 + 64 + 252 + 350 + 448$

$21 \times 64 = 6 + 14_b + 56 + 64 + 64 + 126 + 216 + 350 + 448$

$21 \times 70 = 14 + 70 + 70 + 90 + 189 + 512 + 525$

$21 \times 84 = 84 + 90 + 512 + [202] = 1078$

$21 \times 90 = 21 + 70 + 84 + 90 + 189 + 512 + 924_a \qquad (924_a = [220];\ 924_b = [410])$

$56 \times 56 = 1 + 12 + 21 + 90 + 126' + 189 + 385 + 462 + 924_a + 924_b$

$56 \times 64 = 14a + 21 + 70 + 90 + 512 + 126' + 189 + 189 + 525 + 924_a + 924_b$

$56 \times 70 = 14a + 64 + 126 + 216 + 216 + 350 + 448 + [401] = 1100 + [211] = 1386$

$56 \times 84 = 126 + 350 + 378 + 1386 + [302] = 2464$

$56 \times 90 = 56 + 64 + 126 + 216 + 350 + 378 + 448 + 1386 + [320] = 2016$

$64 \times 64 = 1 + 14a + 14a + 21 + 21 + 70 + 70 + 70 + 84 + 90 + 90 + 189 + 189 +$
$189 + 385 + 512 + 512 + 525 + 924_a$

$64 \times 70 = 6 + 14_b + 56 + 64 + 64 + 64 + 126 + 126 + 216 + 216 + 350 + 350 +$
$378 + 448 + 616 + 1386$

$64 \times 84 = 64 + 126 + 216 + 616 + 350 + 378 + 1386 + [112] = 2240$

$64 \times 90 = 6 + 14a + 64 + 64 + 126 + 126 + 216 + 216 + 350 + 350 + 378 +$
$448 + 616 + [130] = 1344 + 1386$

$70 \times 70 = 1 + 14a + 21 + 21 + 70 + 84 + 90 + 126' + 189 + 189 + 385 + 512 +$
$512 + 594 + 924 + [202] = 1078$

$70 \times 84 = 14_a + 70 + 189 + 385 + 512 + 525 + 594 + [103] = 1386' + [121] = 2205$

$70 \times 90 = 14_a + 21 + 70 + 70 + 90 + 189 + 189 + 395 + 512 + 512 + 594 +$
$924_a + [121] = 2205$

$$84 \times 84 = 1 + 21 + 84 + 90 + 126' + 924_a + [004] = 1001 + 1078 + [040] = 1274 + [022] = 2457$$

$$84 \times 90 = 21 + 84 + 90 + 189 + 512 + 924_a + 1078 + 2205 + 2457$$

$$90 \times 90 = 1 + 14_a + 21 + 70 + 84 + 126' + 90 + 90 + 169 + 385 + 512 + 512 + 525 + 924_a + 1078 + 1274 + 2205$$

Sp(8)

$$8 \times 8 = 1 + 27 + 36$$

$$8 \times 27 = 8 + 48 + 160$$

$$8 \times 36 = 8 + 120 + 160$$

$$8 \times 42 = 48 + 228$$

$$27 \times 27 = 1 + 27 + 36 + 42 + 308 + 315$$

$$27 \times 36 = 27 + 36 + 315 + 594$$

$$27 \times 42 = 27 + 315 + 792_a \qquad (792_a = [0110];\ \ 792_b = [0101])$$

$$27 \times 48 = 8 + 48 + 160 + 288 + 792_a$$

$$36 \times 36 = 1 + 27 + 36 + 308 + 330 + 594$$

$$36 \times 42 = 42 + 315 + [2001] = 1155$$

$$36 \times 48 = 48 + 160 + 288 + [2010] = 1232$$

$$42 \times 42 = 1 + 36 + 308 + 594 + 825$$

$$42 \times 48 = 8 + 160 + 792_a + [0011] = 1056$$

$$48 \times 48 = 1 + 27 + 36 + 308 + 315 + 792_b + 825$$

<u>Sp(10)</u>

$10 \times 10 = 1 + 44 + 55$

$10 \times 44 = 10 + 110 + 320$

$10 \times 55 = 10 + 220 + 320$

$44 \times 44 = 1 + 44 + 55 + 65 + 780 + 891$

$44 \times 55 = 44 + 55 + 891 + [21000] = 1430$

$55 \times 55 = 1 + 44 + 55 + 715 + 780 + 1430$

<u>Sp(12)</u>

$12 \times 12 = 1 + 65 + 78$

$12 \times 65 = 12 + 208 + 560$

$12 \times 78 = 12 + 364 + 560$

$65 \times 65 = 1 + 65 + 78 + 429 + [020000] = 1650 + [120000] = 2001$

$65 \times 78 = 65 + 78 + 2002 + [210000] = 2925$

$78 \times 78 = 1 + 65 + 78 + [400000] = 1365 + 1650 + 2925$

<u>Sp(14)</u>

$14 \times 14 = 1 + 90 + 105$

$14 \times 90 = 14 + 350 + 896$

$90 \times 90 = 1 + 105 + 90 + 910 + [0200000] = 3094 + [1010000] = 3900$

<u>Sp(2N) N > 7</u>

$(2N) \times (2N) = 1 + (N(2N - 1) - 1) + (N(2N + 1))$

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FIGURE CAPTIONS

Fig. 1: [21031] in SU(6) can be written as (75441).

Fig. 2: (a) [123] = ($\uparrow$431) in SO(7). (b) [002] = (111). (c) [003] = ($\uparrow$111). (d) [001] = ($\uparrow$000).

Fig. 3: [00014] = ($\uparrow$22221) in SO(10).

Fig. 4: (a) ($\uparrow$1000) in SO(8) has dimension $56 = \frac{14\cdot 12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 4\cdot 6\cdot 8\cdot 2\cdot 4\cdot 2}{2^{12}3!1!3!5!}$. (b) ($\uparrow$1000) in SO(9) has dimension $128 = \frac{10\cdot 6\cdot 4\cdot 2\cdot 16\cdot 14\cdot 12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 6\cdot 8\cdot 2\cdot 4\cdot 2}{2^{12}1!3!5!7!}$.

Fig. 5: In SU(4), (110) $\otimes$ (100) = (210) + (111).

Fig. 6: In Sp(6), (110) $\otimes$ (210) contains two (210)'s.

Fig. 7: (2110) $\otimes$ (1100) in SO(9) contains (2000).

Fig. 8: In Sp(8), (1111) $\times$ (1110) does not contain (1110).

Fig. 9: (110) $\times$ ($\uparrow$100) in SO(7) = ($\uparrow$210) + ($\uparrow$110) + ($\uparrow$200) + ($\uparrow$100) + ($\uparrow$111) + ($\uparrow$000) + $\uparrow$(100).

Fig. 10: ($\uparrow$100) $\otimes$ ($\uparrow$100) in SO(7) = (110) + (210) + (111) + (220) + (211) + (221) + (200) + (300) + (210) + (310) + (211) + (311) + (000) + (100) + (110) + (111).

Fig. 11: In SO(8) (1110) $\otimes$ (1110) contains two (1100)'s.

Fig. 12: 16 $\times$ $\overline{16}$ in SO(8).

Fig. 13: ($\uparrow$11000) $\otimes$ $\downarrow$(00000) in SO(10) = (11000) + (11110) + (21100) = (22000) + (22200) + (22110) + (2111).

Fig. 14: In SO(8), (1111) $\otimes$ ($\uparrow$1000) = (1111) $\otimes$ $\{$(1000) $\otimes$ ($\uparrow$0000) - $\downarrow$(0000)$\}$ = ($\uparrow$2111) + $\uparrow$(1110) + ($\uparrow$2100) + ($\uparrow$1000) + ($\uparrow$1100) + ($\uparrow$0000).

Fig. 15: Doing (1111) $\otimes$ ($\uparrow$1000) using rule 5c.

Fig. 16: [00002] $\otimes$ [00002] (126 $\times$ 126) in SO(10) = (22222) + (22211) + (22200) + (21111) + (21100) + (20000): 126 $\times$ 126 = 2772 + 6930 + 4125 + 1050 + 945 + 54.

Fig. 17: (22222) × (*11111) = (33331) + (33221) + (22221) + (33111) + (22111) + (11111). Note that the first column remains untouched.

Fig. 18: [10] = (10) in G_2 = [10] = (10) + [01] = (11) + [00] = (00) in SU(3).

Fig. 19: [11] = (21) in G_2 contains (31) + (20) + (32) + (21) + (21) + (10) + (22) + (11) of SU(3).

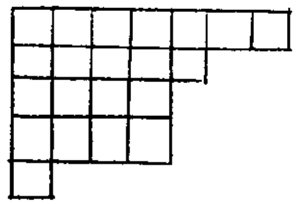
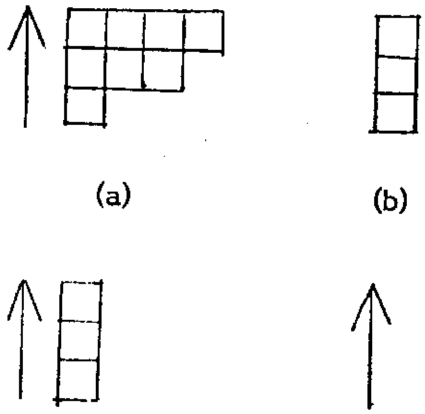


Fig. 1



(a) (b)

(c) (d)

Fig. 2

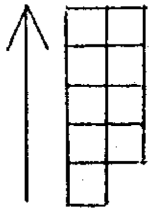


Fig. 3

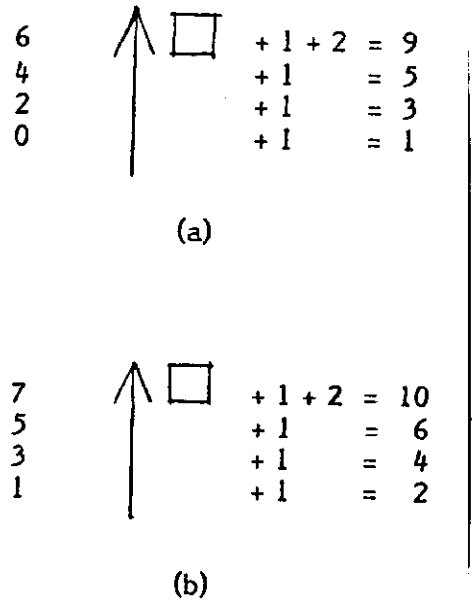


Fig. 4

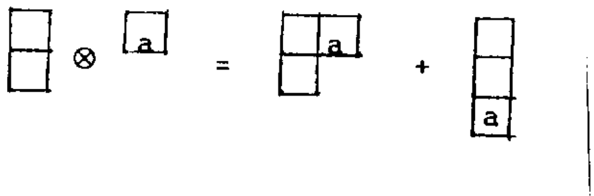


Fig. 5

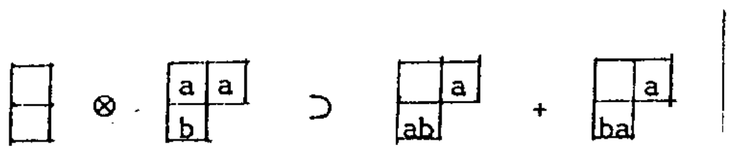


Fig. 6

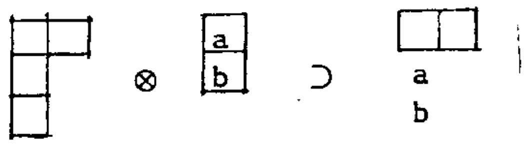


Fig. 7

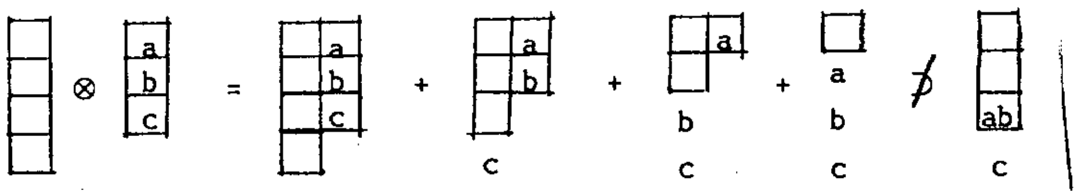


Fig. 8

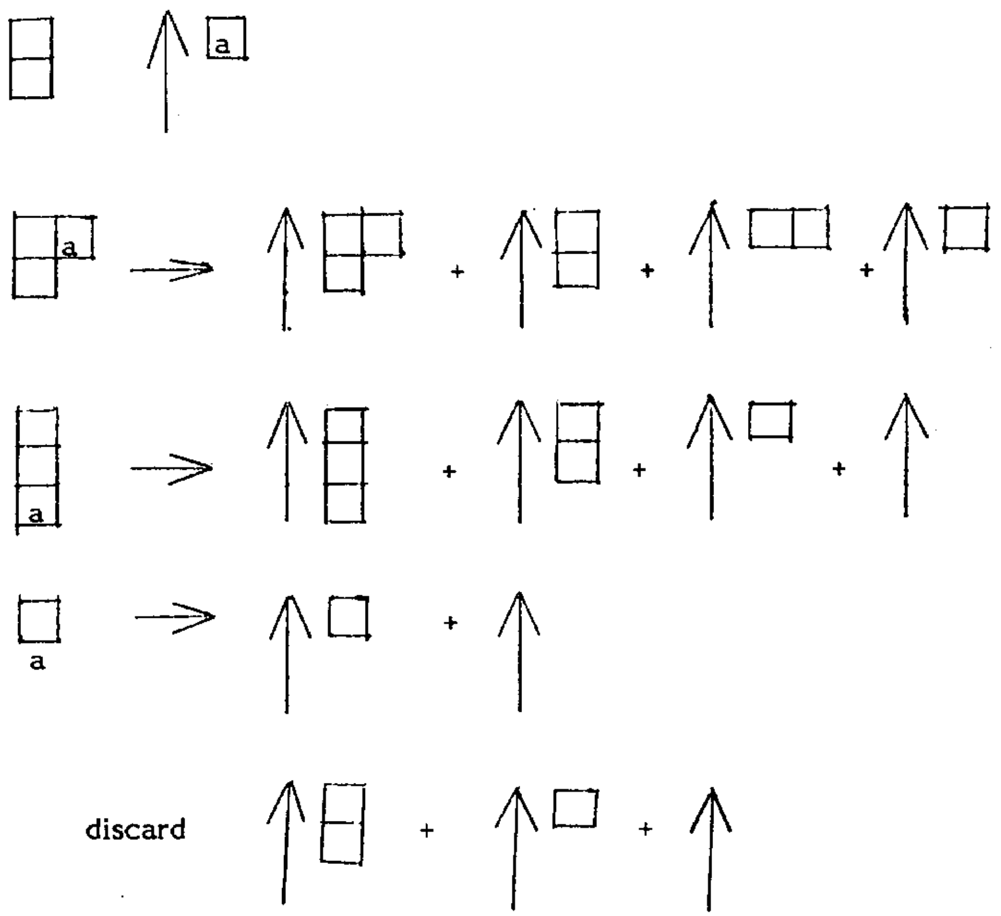


Fig. 9

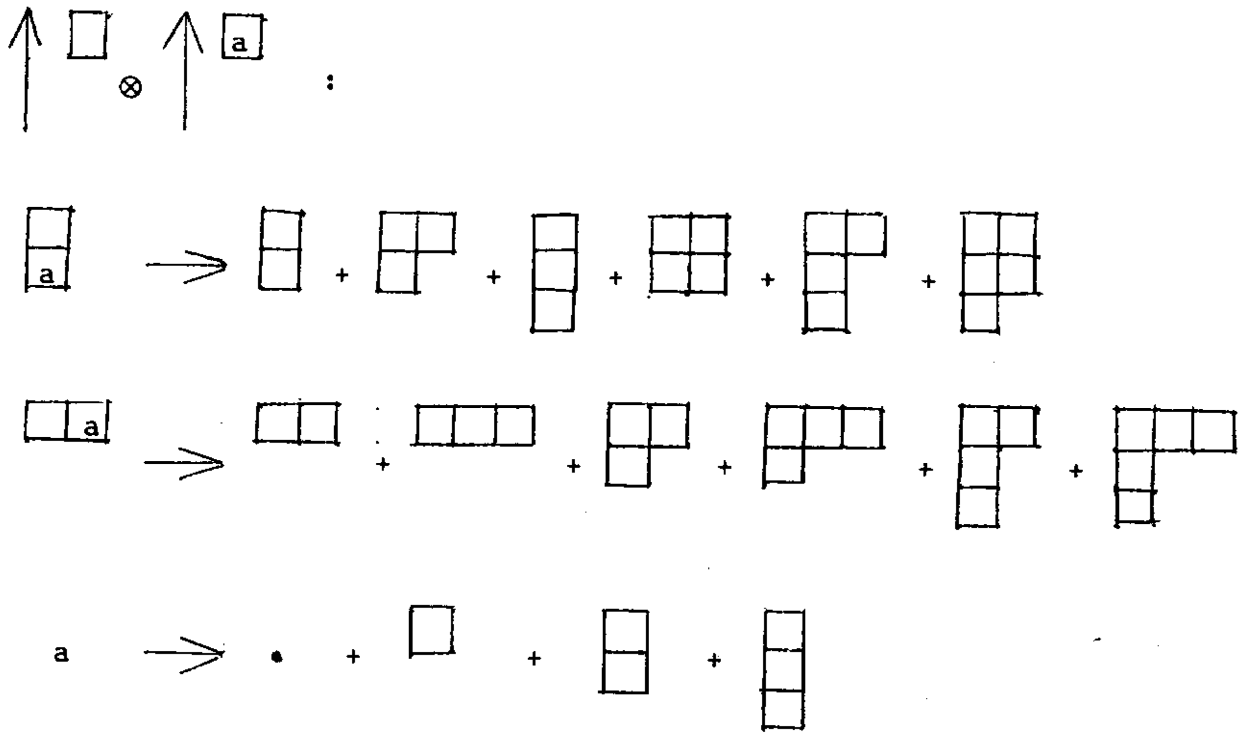


Fig. 10

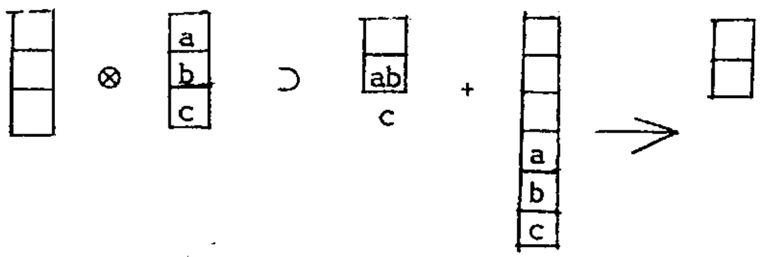


Fig. 11

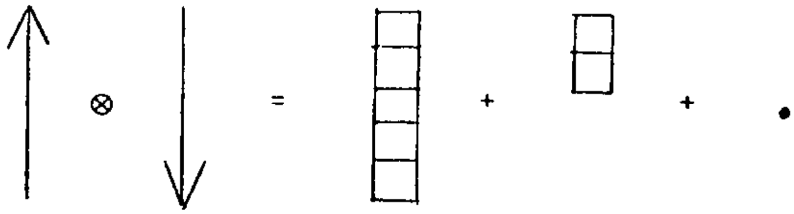


Fig. 12

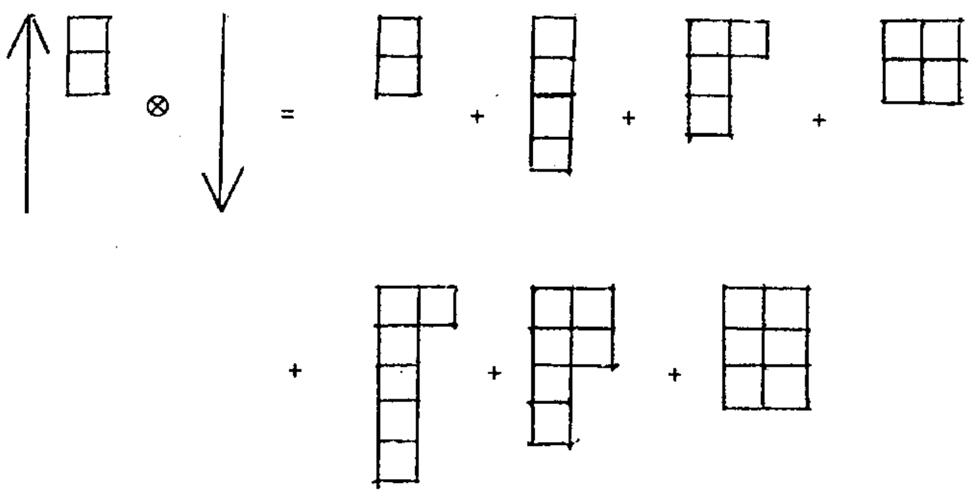


Fig. 13

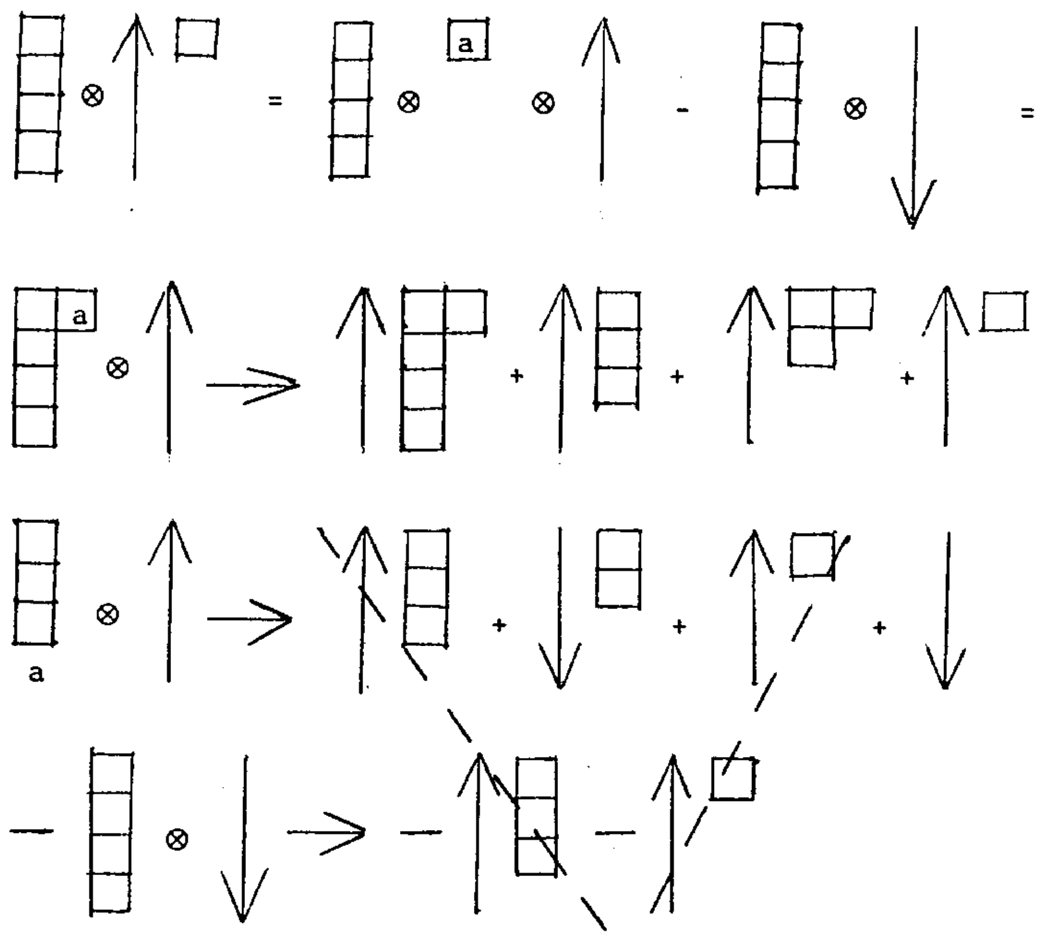


Fig. 14

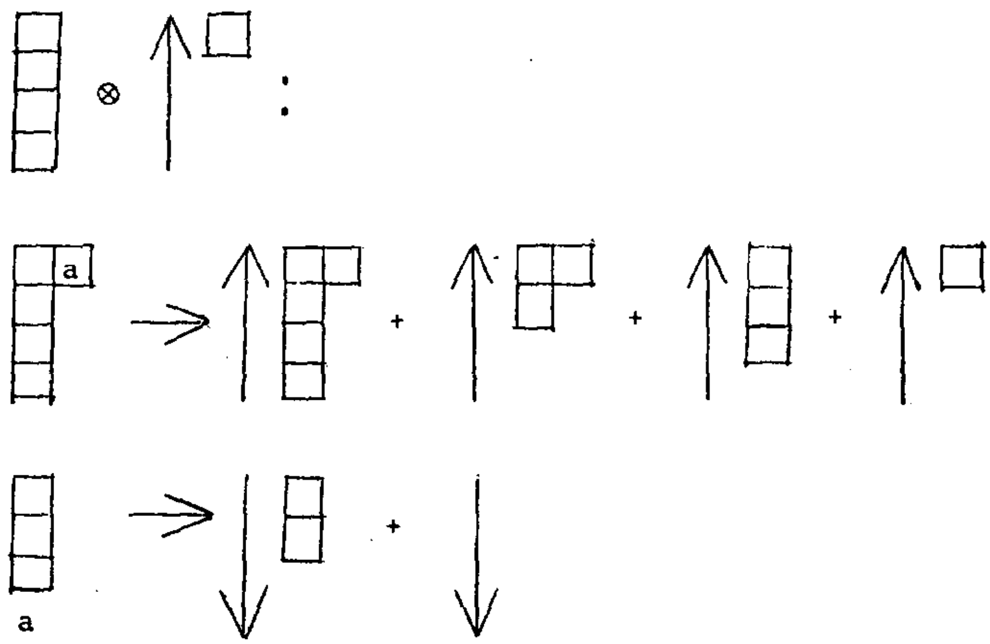


Fig. 15

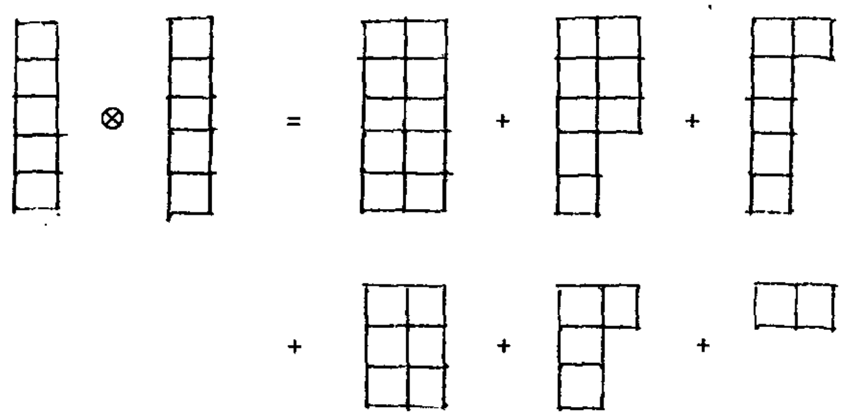


Fig. 16

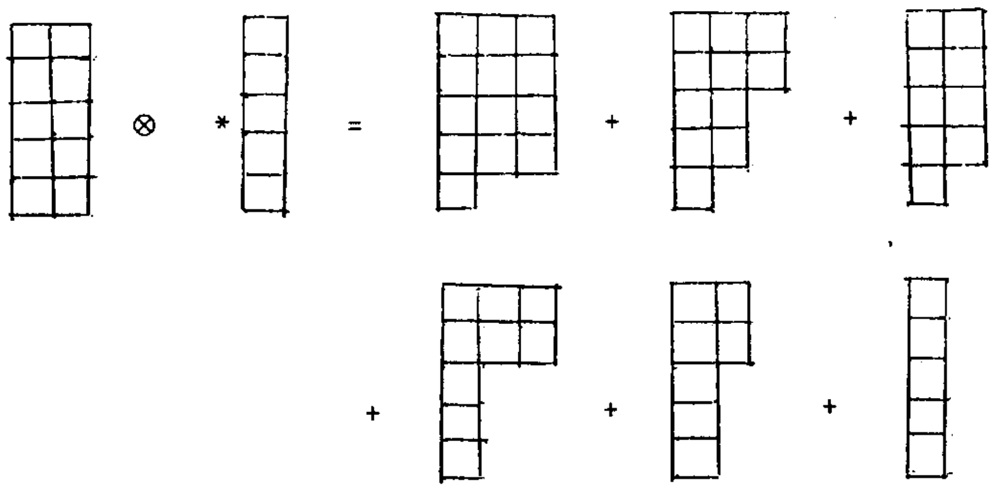


Fig. 17

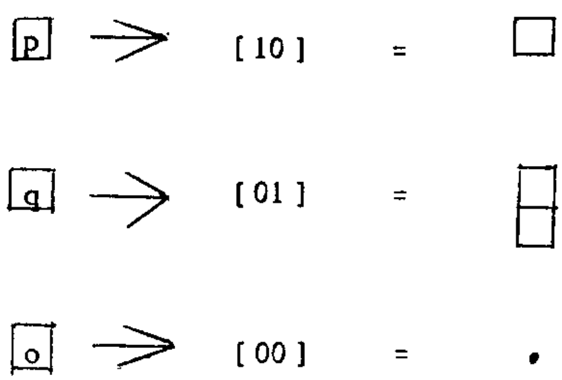


Fig. 18

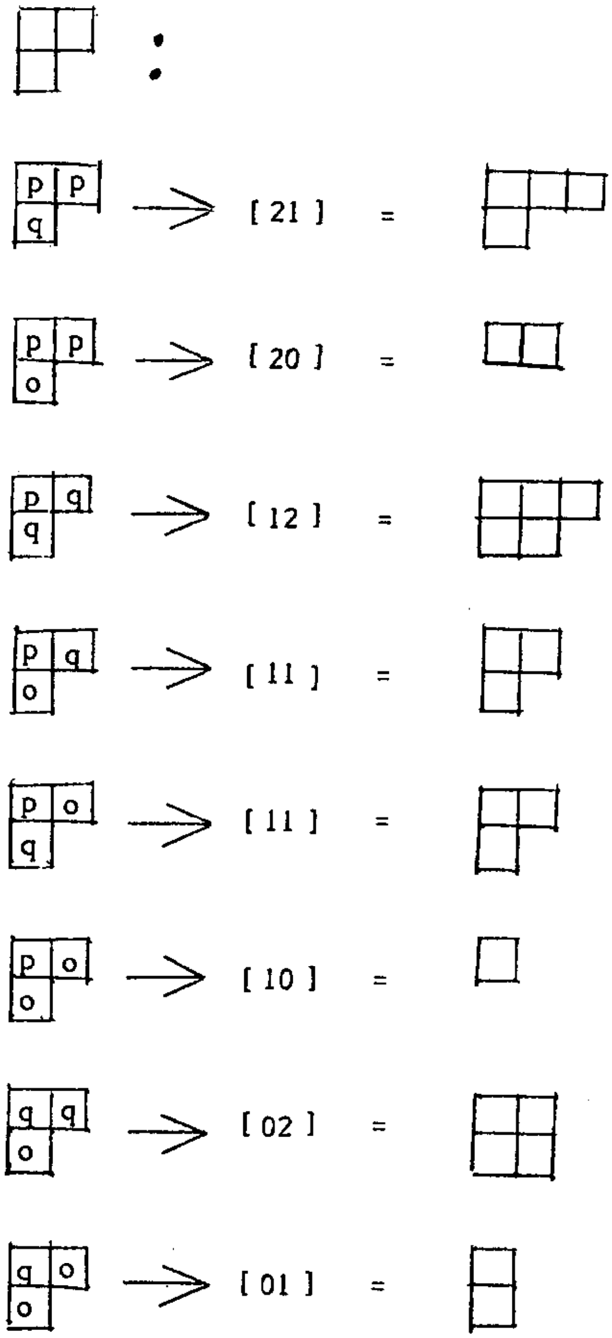


Fig. 19